COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 2

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) Let A be an integral domain and let K be its field of fractions. Let $f \in A$ such that $f \neq 0$ and $f \notin U(A)$. Prove that $A[1/f]$ is not finitely generated as a module over A.

2) Consider the ideal $I = (X, Y)$ of $A = \mathbf{k}[X, Y]$. Prove that I is not a free A–module.

3) Let $A = \mathbf{k}[X, Y, Z]$ and let $M \subset A^4$ be the A–module consisting of the elements (f_1, f_2, f_3, f_4) of A^4 such that $f_1XY + f_2XZ + f_3YZ + f_4Z^2 = 0$.

a) Prove that $\{v_1 = (Z, -Y, 0, 0), v_2 = (0, Y, -X, 0), v_3 = (0, -Z, 0, X), v_4 = (0, 0, -Z, Y)\}$ is a family of generators for M but not a basis of M.

b) Prove that the module of relations among v_1, v_2, v_3 and v_4 is free of rank 1.

4) Let $(A, \mathfrak{m})$ be a local ring, let $\mathbf{k} = A/\mathfrak{m}$, let M be a finite A–module and let $s_1, \dots, s_n \in M$. Let $\overline{s}_1, \dots, \overline{s}_n \in \overline{M} = M/\mathfrak{m}M$ be the classes of $s_1, \dots, s_n$.

a) Prove that $\overline{s}_1, \dots, \overline{s}_n$ span the $\mathbf{k}$–vector space $\overline{M}$ if and only if $s_1, \dots, s_n$ generate M.

b) Prove that $\{\overline{s}_1, \dots, \overline{s}_n\}$ is a basis of the $\mathbf{k}$–vector space $\overline{M}$ if and only if $\{s_1, \dots, s_n\}$ is an irredundant family of generators for M.

5) Let A be the ring of Exercise 1.22.b), let $\mathfrak{m} = (X, Y) \subset A$ and let $f = f_1 + f'$ and $g = g_1 + g'$, where $f_1, g_1 \in \mathbf{k}[X, Y]$ are homogeneous linear forms, linearly independent over $\mathbf{k}$ and $f', g' \in \mathbf{k}[X, Y]$ are polynomials of order > 1. Prove that f and g generate $\mathfrak{m}$.

6) Prove that A^n and A^m are isomorphic as A–modules if and only $n = m$, when

a) A is a local ring;

b) A is an arbitrary ring.

7) Let $A = \{g/h | g, h \in \mathbf{R}[X, Y] \text{ and } h(1, 0) \neq 0\}$. Prove that the elements (X, Y) and (Y, X) of A^2 generate A^2 as an A–module.

8) Let A be a UFD, let $x, y \in A$ be elements with no common factors and let $\varphi : A \oplus A \longrightarrow A$ be the homomorphism given by $\varphi((1, 0)) = x$ and $\varphi((0, 1)) = y$.

a) Compute the image of φ and prove that the kernel of φ is generated by $(-y, x)$.

b) Let $I = (x, y)$. Conclude that the following short exact sequence of A–modules

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} A \oplus A \xrightarrow{(x \quad y)} I \longrightarrow 0$$

is exact (this exact sequence is called the *Koszul complex* of x and y).

c) Let $A = \mathbf{k}[X, Y]$ and let $x = X$ and $y = Y$; conclude that the following sequence

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} A \oplus A \xrightarrow{(X \quad Y)} A \xrightarrow{\pi} \mathbf{k} \longrightarrow 0$$

is exact.

d) Let $A = \mathbf{C}[X, Y]$ and let $I = (X^3 - Y, X^2 - 1) \subset \mathbf{C}[X, Y]$; prove that A/I fits in a exact sequence like this:

$$0 \longrightarrow A \longrightarrow A \oplus A \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

9) Let $A = \mathbf{k}[X, Y, Z]$ and let $\mathfrak{m} = (X, Y, Z)$. Let us consider the surjective homomorphism $\alpha : A^3 \longrightarrow \mathfrak{m}$ given by $(f, g, h) \mapsto fX + gY + hZ$.

a) Prove that the kernel of α is generated by three elements $u, v, w \in A^3$.

b) Let $\beta : A^3 \longrightarrow \ker \alpha$ be the surjective homomorphism given by $(f', g', h') \mapsto f'u + g'v + h'w$. Find the kernel of β.

c) Conclude the existence of an exact sequence

$$0 \longrightarrow A \longrightarrow A^3 \xrightarrow{\beta} A^3 \xrightarrow{\alpha} A \longrightarrow \mathbf{k} \longrightarrow 0,$$

(the Koszul complex of X, Y and Z).

10) Let $A = \mathbf{k}[X, Y, Z]$.

a) Prove that the kernel of the homomorphism $A^3 \to A$ given by $(f, g, h) \mapsto (Z^2 - X^2Y)f + (X^3 - YZ)g + (Y^2 - XZ)h$ is free and that $\{(X, Y, Z), (Y, Z, X^2)\}$ is a basis for it.

b) Use part a) to build a free resolution (that is, an exact sequence of free A–modules) of the ring of polynomial functions on the curve C of part d) of Exercise 1.8.

11) Let

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be a short exact sequence of A–modules. Prove that there exists an isomorphism $M \simeq L \oplus N$ under which α is given by $l \mapsto (l, 0)$ and β by $(l, n) \mapsto n$, if either

a) there exists a section of β, that is, a homomorphism $s : N \longrightarrow M$ such that $\beta \circ s = \mathrm{id}_N$; or

b) there exists a retraction of α, that is, a homomorphism $r : M \longrightarrow L$ such that $r \circ \alpha = \mathrm{id}_L$.

12) Let A be a ring.

a) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be a short exact sequence of modules over A. Prove that if L and N are finite over A, so is M.

b) Let $M_1, M_2 \subset M$ be two submodules. Prove that if $M_1 + M_2$ and $M_1 \cap M_2$ are finite modules over A, so are M_1 and M_2.

13) Let M be an A–module. We define the dual module of M as the set consisting of all A–module homomorphisms from M to A.

a) Prove that the dual module of M has a natural structure of A–module.

b) If $M = \bigoplus_{l \in \Lambda} M_l$, express the dual module of M in terms of the dual modules of M_l.

14) Find the dual module of $\mathbf{Q}$ as a $\mathbf{Z}$–module.

15) Find the dual module of $\mathbf{k}[X, Y]/(XY - 1)$ as a module over $\mathbf{k}[X]$. Deduce that $\mathbf{k}[X, Y]/(XY - 1)$ is not a free $\mathbf{k}[X]$–module.