COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 3

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) What rings among the following are Noetherian?

a) The ring of rational functions on z without poles on the unit circle $|z| = 1$.

b) The ring of power series in z with positive convergence radius.

c) The ring of power series in z with infinite convergence radius.

d) The ring of polynomials in z whose first k derivatives vanish at the origin, with k a given natural number.

e) The ring of polynomials in z, w all whose partial derivatives with respect to w vanish along the line $z = 0$.

In all cases the field of coefficients is $\mathbf{C}$.

2) Let A be a Noetherian ring. Prove that if M is a finite A–module, then so is $\mathrm{End}(M)$.

3) Let $I_1, \ldots, I_k$ be ideals such that A/I_i is a Noetherian ring. Prove that $\bigoplus A/I_i$ is a Noetherian A–module and deduce that if $\bigcap I_i = 0$, the A is also a Noetherian ring.

4) Prove that if A is a Noetherian ring and M is a finite A–module, then there exists an exact sequence of A–modules $A^q \longrightarrow A^p \longrightarrow M \longrightarrow 0$; that is, M is finitely generated and finitely *presented* over A. Deduce the existence of an exact sequence

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow M \longrightarrow 0,$$

not necessarily of finite lenght, where the F_i are free modules of finite rank.

5) Let A be a Noetherian ring. Prove that any surjective ring homomorphism $\varphi : A \longrightarrow A$ is an isomorphism.

6) Let M be an A–module. We define the *annhilator* of M as $\mathrm{ann}M = \{a \in A | am = 0 \text{ for all } m \in M\}$.

a) Check that $\mathrm{ann}M$ is an ideal of A.

b) Prove that if M is a Noetherian A–module, then $A/\mathrm{ann}M$ is a Noetherian ring. Is this result still true if we replace Noetherian by Artinian?

7) If $A[X]$ is Noetherian, is A necessarily Noetherian?

8) Prove that if A is a Noetherian ring, then so is the ring of formal power series $A[[X]]$.

9) Let A be a non Noetherian ring and let Σ be the set of ideals of A that are not finitely generated. Prove that Σ has maximal elements and that the maximal elements of Σ are prime ideals. Deduce from this this theorem by I. S. Cohen: A ring such that all of its prime ideals are finitely generated is Noetherian.

10) Let $0 \to L \to M \to N \to 0$ be an exact sequence of A–modules.

a) Prove that M is Artinian if and only if L and N are.

b) Prove that the direct sum of finitely many Artinian A–modules Artinians is an Artinian A–module.

c) Prove that if A is Artinian and M is a finite A–module, the M is Artinian.

11) Let $\mathbf{k} \subset A$ be a subring of A (that is, A is a $\mathbf{k}$–*algebra*). Prove that if A is a finite dimensional $\mathbf{k}$–vector space (that is, if A is *finite* $\mathbf{k}$–*algebra*), then A is an Artinian ring.

12) a) Prove that fields are the only Artinian integral domains.

b) Prove that in an Artinian ring, every prime ideal is a maximal ideal.

13) a) Prove that, up to isomorphism (of $\mathbf{C}$–algebras), there are only two finite $\mathbf{C}$–algebras of dimension 2 (as vector space) over $\mathbf{C}$.

b) Prove that, up to isomorphism (of $\mathbf{R}$–algebras), there are only three finite $\mathbf{R}$–algebras of dimension 2 (as vector space) over $\mathbf{R}$.

c) Give examples of four $\mathbf{C}$–algebras, non isomorphic as $\mathbf{C}$–algebras, of dimension 3 (as vector spaces) over $\mathbf{C}$. Say which of them are local rings.

d) Let $A_1 = \mathbf{C}[X,Y]/(X^2,Y^2)$ and $A_2 = \mathbf{C}[X,Y]/(X^3,XY,Y^2)$. Check that A_1 and A_2 have dimension 4 as $\mathbf{C}$–vector spaces and decide if they are isomorphic as $\mathbf{C}$–algebras.

Remark: Exercise 11 implies that all the algebras in this exercise are Artinian rings.

14) Prove that if A is an Artinian ring, then $\sqrt{(0)}$ is nilpotent, that is, there is $k \in \mathbf{N}$ such that $(\sqrt{(0)})^k = 0$. Conclude that if $(A,\mathfrak{m})$ is an Artinian local ring, then $\mathfrak{m}$ is nilpotent.

15) Consider the ring A.

a) Prove that if the ideal $(0) \subset A$ is a product of maximal ideals (non necessarily distinct) $\mathfrak{m}_1, \ldots \mathfrak{m}_k$, then A is a Noetherian ring if and only if A is an Artinian ring. Hint: consider the family of short exact sequences

$$0 \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_i \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i) \longrightarrow 0$$

of A–modules and the fact that $\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i)$ is a vector space over $A/\mathfrak{m}_{i-1}$.

b) Prove that if A is an Artinian ring, then A has finitely many maximal ideals (i.e., A is *semilocal*). Hint: argue by contradiction, using the definition of Artinian ring and studying what happens when $I_1 \cdots I_k \subset \mathfrak{p}$, for $I_1, \ldots, I_k$ ideals and $\mathfrak{p}$ prime ideal.

c) Conclude that if A is an Artinian ring, then it is also a Noetherian ring. Hint: use 3.14 and 3.15.b).

16) Let $(A,\mathfrak{m})$ be local Artinian ring and let $\mathbf{k} = A/\mathfrak{m}$ be its residue filed. Prove that the following are equivalent:

i) each ideal of A is principal;

ii) the ideal $\mathfrak{m}$ is principal;

iii) the $\mathbf{k}$–vector space $\mathfrak{m}/\mathfrak{m}^2$ has dimension less than or equal to 1.