COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 4

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) Let A be a UFD and let K be its field of fractions. Prove that if $b \in K \setminus A$, then b is not integral over A (this means that A is a *normal domain*).

2) Prove that the real number $\alpha = (1+\sqrt{3})/2$ is not integral over $\mathbf{Z}$.

3) Let A be a UFD, let K be its field of fractions, let $K \subset L$ a field extension and let $\alpha \in L$ be algebraic over K. Prove that α is integral over A if and only if the coefficients of its minimal polynomial over K are in A.

4) Let $A \subset B$ an integral extension. Find an integral dependence relation over A for each $f \in B$ in the following cases:

a) $A = \mathbf{k}[X^2]$, $B = \mathbf{k}[X]$ and $f \in B$.

b) $A = \mathbf{Z}$, $B = \mathbf{Z}[\tau]$, $\tau = (1+\sqrt{5})/2$ (the golden ratio) and $f \in B$.

c) $A = \mathbf{Z}$ and $f = \sqrt{3} + \frac{1+\sqrt{5}}{2}$; $f = \sqrt{3} \cdot \frac{1+\sqrt{5}}{2}$.

d) Consider $y, z \in B$ satisfying $y^3 + a_2y^2 + a_1y + a_0 = 0$ and $z^3 + a'_2z^2 + a'_1z + a'_0 = 0$, $a_i, a'_j \in A$. Let $f = y + z$; $f = y \cdot z$.

5) Let d be a square–free integer (that is, without multiple factors). Prove that the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$ is $\mathbf{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1 \pmod 4$ and $\mathbf{Z}[\sqrt{d}]$ otherwise (these are the *rings of integers* of the *number fields* of degree 2).

6) Let $A = \mathbf{k}[X,Y]/(Y^2 - X^3)$.

a) Prove that A is isomorphic to the subring $\mathbf{k}[T^2, T^3]$ of $\mathbf{k}[T]$.

b) Prove that A is not a UFD (hint: prove that $\overline{X}$ and $\overline{Y}$ are non associated irreducible elements).

c) Prove that A and $\mathbf{k}[T]$ have the same field of fractions and that the normalization of A is $\mathbf{k}[T]$.

7) Think of $\mathbf{k}[X,Y]/(Y^2 - X^2 - X^3)$ as a subring of $\mathbf{k}[T]$ by identifying the class of the polynomial $f(X,Y)$ with $f(T^2-1, T^3-T)$. Prove that $\mathbf{k}[X,Y]/(Y^2-X^2-X^3)$ and $\mathbf{k}[T]$ have the same field of fractions, and that the normalization of $\mathbf{k}[X,Y]/(Y^2 - X^2 - X^3)$ is $\mathbf{k}[T]$.

8) Find the normalization of the following domains:

a) $\mathbf{k}[X,Y]/(X^2Y - XY^2 - (X+Y)^4)$.

b) $\mathbf{k}[X,Y]/(Y^2 - 4X^3 - 6XY - 3X^2 - 4Y)$.

c) $\mathbf{k}[X,Y]/(Y^3 - X^5)$.

9) Let $A = \mathbf{k}[X]$, let $f \in A$ and let $B = \mathbf{k}[X,Y]/(Y^2 - f)$.

a) Find the necessary and sufficient condition that f must satisfy in order for B to be an integral domain.

Assume from now on that B is an integral domain.

b) Let K be the field of fractions of B, that is, let $K = \mathbf{k}(X)(\sqrt{f})$. For each $\alpha = \alpha_1 + \alpha_2\sqrt{f} \in K$ ($\alpha_1, \alpha_2 \in \mathbf{k}(X)$), find a polynomial $h(T) \in A[T]$ such that $h(\alpha) = 0$.

Assume from now on that the characteristic of $\mathbf{k}$ is different from 2.

c) Prove that B is normal if and only if f is square–free.

d) If B is not normal, find the normalization of B.

e) Can you extend the results proved in c) and d) to a larger class of domains?

10) Prove that $\mathbf{C}[X,Y,Z]/(X^2+Y^2-Z^2)$ is not a unique factorization domain but is a normal domain nevertheless.

11) Let $\mathbf{k}$ be an arbitrary field. Consider $A = \mathbf{k}[X,Y]/(XY-1)$. Let $\epsilon \in \mathbf{k}$, $\epsilon \neq 0$. Let $X' = X - \epsilon Y$. Prove that $\mathbf{k}[X'] \subset A$ is a finite extension. Give a geometric interpretation of this fact and of the fact that $\mathbf{k}[X] \subset A$ is not a finite extension.

12) Let $\mathbf{k}$ be an arbitrary field and let $B = \mathbf{k}[X,Y,Z]/(X^2-Y^3-1, XZ-1)$. Find $\alpha, \beta \in \mathbf{k}$ such that B is integral over $A = \mathbf{k}[X+\alpha Y+\beta Z]$ and give, for these α, β, a family of generators of B as an A–module.

13) Let $\mathbf{k}$ be a finite field. Find an example of $f \in \mathbf{k}[X,Y]$ such that the ring $B = \mathbf{k}[X,Y]/(f)$ is not a finite module over $A_\alpha = \mathbf{k}[X-\alpha Y]$, for any $\alpha \in \mathbf{k}$.

14) Let $A \subset B$ be a ring extension with B finite over A.

a) Let $\mathfrak{p} \subset A$ be a prime ideal. Prove that there exists a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$.

b) In the situation of part a), prove that $\mathfrak{p}$ is maximal if and only if $\mathfrak{q}$ is maximal.

c) If $A = \mathbf{k}$ is a field, prove that B possesses finitely many prime ideals.

15) Let G be a finite group of automorphisms of the ring A and let $A^G = \{a \in A | \sigma(a) = a$ for all $\sigma \in G\}$ be the subring consisting of those elements which are invariant by G. Prove that A is integral over A^G.

16) Let A be a normal domain, let K be its field of fractions and let L be a finite, normal and separable field extension of K. Let G be the Galois group of L over K and let B be the integral closure of A in L. Prove that $\sigma(B) = B$ for all $\sigma \in G$ and that $B^G = A$.