COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 5

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ be distinct maximal ideals of A.

a) Prove that for any $1 \leq i \leq s$, $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_i$ and $\mathfrak{m}_{i+1}$ are strongly coprime. Conclude that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s = \mathfrak{m}_1 \cdots \mathfrak{m}_s$.

b) Prove that for any $k \in \mathbf{N}$, $(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s)^k = \mathfrak{m}_1^k \cdots \mathfrak{m}_s^k = \mathfrak{m}_1^k \cap \cdots \cap \mathfrak{m}_s^k$.

c) Prove that $A/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_s)^k$ is isomorphic to $A/\mathfrak{m}_1^k \times \cdots \times A/\mathfrak{m}_s^k$.
Hint: use Exercise 1.3.

2) Let A be a finitely generated $\mathbf{k}$–algebra. Prove that if A is a Artinian local ring, then A is a finite $\mathbf{k}$–algebra. Hint: use short exact sequences similar to those of Exercise 3.15, a), Exercise 3.14 and Weak Nullstellensatz.

3) Let A be a finitely generated $\mathbf{k}$–algebra. Prove that the following are equivalent:

i) A is a finite $\mathbf{k}$–algebra;

ii) A is an Artinian ring.

To give a proof, follow these steps:

a) Exercise 3.11 proves i) implies ii).

To prove ii) implies i) you may reduce the general case to the case proven in Exercise 5.2 by following these steps:

b) By Exercise 3.15 b) we know that the $\mathfrak{m}$–Spec$A$ is a finite set $\{\mathfrak{m}_1, \dots, \mathfrak{m}_s\}$. Use 3.12 b), 3.14 and 5.1 to prove the existence of $k \in \mathbf{N}$ such that $A \simeq A/\mathfrak{m}_1^k \times \cdots \times A/\mathfrak{m}_s^k$.

c) Prove that $A/\mathfrak{m}_i^k$ is an Artinian local ring. Then use Exercise 5.2 to conclude that ii) implies i).

d) Is the result true if A is not a finitely generated $\mathbf{k}$–algebra?

4) Let I be an ideal of $\mathbf{k}[X_1, \dots, X_n]$.

a) Assume that $\mathbf{k}$ is an algebraically closed field. Prove that $V(I)$ is a finite subset of $\mathbf{A}^n_{\mathbf{k}}$ if and only if $\mathbf{k}[X_1, \dots, X_n]/I$ is a finite $\mathbf{k}$–algebra. Hint: for the "only if" part it might help you to use (after proving it) the following fact: for any ideal J of a Noetherian ring there exists $m \in \mathbf{N}$ such that $(\sqrt{J})^m \subset J$; then use Exercise 5.1.

b) What kind of result can you get if $\mathbf{k}$ is an arbitraty field?

5) Let V and W be two algebraic varieties of $\mathbf{A}^n_{\mathbf{k}}$ and $\mathbf{A}^m_{\mathbf{k}}$ respectively. Recall that a *morphism* F from V to W is a map $F : V \longrightarrow W$ such that there exist $F_1, \dots, F_m \in \mathbf{k}[X_1, \dots, X_n]$ satisfying $F(x) = (F_1(x), \dots, F_m(x))$, for all $x \in V$. Let $I(V)$ and $I(W)$ the ideals of V and W and let $A(V) = \mathbf{k}[X_1, \dots, X_n]/I(V)$ and $A(W) = \mathbf{k}[Y_1, \dots, Y_m]/I(W)$ be its coordinate rings.

a) Prove the existence of a one–to–one correspondence between the set of morphisms from V to W and the set of $\mathbf{k}$–algebra homomorphisms between $A(W)$ and $A(V)$. Hint: First set up a map from the set of morphisms between $\mathbf{A}^n_{\mathbf{k}}$ and $\mathbf{A}^m_{\mathbf{k}}$ to the set of $\mathbf{k}$–algebra homomorphisms between $\mathbf{k}[Y_1, \dots, Y_m]$ and $\mathbf{k}[X_1, \dots, X_n]$; then carry on the construction to $V, W, A(W)$ and $A(V)$ and show you get a one–to–one correspondence.

b) Decide if this statement is true or false: There exists a one–to–one correspondence between the set of morphisms from $\mathbf{A}^n_{\mathbf{k}}$ to $\mathbf{A}^m_{\mathbf{k}}$ and the set of $\mathbf{k}$–algebra homomorphisms between $\mathbf{k}[Y_1, \dots, Y_m]$ and $\mathbf{k}[X_1, \dots, X_n]$. Hint: argue separatedly when $\mathbf{k}$ is a finite field and when $\mathbf{k}$ is an infinite field.

c) Explain why the answer to b) does not contradict the statement proven in a). Hint: What is the ideal of $\mathbf{A}^n_{\mathbf{k}}$?

6) Let V and W be two algebraic varieties of $\mathbf{A}^n_{\mathbf{k}}$ and $\mathbf{A}^m_{\mathbf{k}}$. Let $F : V \longrightarrow W$ be a morphism and let $F^\sharp : A(W) \longrightarrow A(V)$ be the $\mathbf{k}$–algebra homomorphism induced by F (see 5.5).

a) Prove that if F is surjective, then $F^\sharp$ is injective. Is the converse true?

b) Assume that $F^\sharp$ is injective and $\mathbf{k}$ is algebraically closed. Prove that if the extension given by $F^\sharp$ is finite, then F is surjective and it has *finite fibers* (that is, for every $y \in W$, $F^{-1}(y)$ is a finite set). Is the finiteness of the extension a necessary condition for F to be surjective? Is it a necessary condition for F to have finite fibers?

7) Prove the following properties of I and V (respectively, $\mathscr{I}$ and $\mathscr{V}$) for $\mathbf{k}[X_1,\dots,X_n]$ and $\mathbf{A}^n_{\mathbf{k}}$ (respectively, for A and Spec$A$):

a) If $I \subset J$ are ideals, then $V(J) \subset V(I)$.
b) If $S \subset T$ are subsets, then $I(T) \subset I(S)$.
c) If I_λ, $\lambda \in \Lambda$ are ideals, then $\bigcap_\lambda V(I_\lambda) = V(\bigcup_\lambda I_\lambda) = V(\sum_\lambda I_\lambda)$.
d) If I_1 and I_2 are ideals, then $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 \cdot I_2)$.
e) If I is an ideal, then $V(I) = V(\sqrt{I})$.
f) For any subset X, $V(I(X))$ is the closure of X in the Zariski topology. In particular, if $\mathfrak{p} \in$ Spec$A$, conclude that $\mathscr{V}(\mathfrak{p})$ is the Zariski closure of the point $\mathfrak{p} \in$ Spec$A$. What are the *closed* points of Spec$A$?
g) If $D(f)$ denotes the complement of $V(f)$, then the sets $D(f)$ form a basis for the Zariski topology.

8) Let $A_1 = \mathbf{R}[X,Y]/(Y^2 - X)$ and let $A_2 = \mathbf{R}[X,Y]/(X^2+Y^2-1)$. Let $f = X - a$, $a \in \mathbf{R}$, and $g = X^2 + 1$. Find $\mathscr{V}(\overline{f})$ and $\mathscr{V}(\overline{g})$, both in Spec$A_1$ and SpecA_2.

9) For each prime number $p \in \mathbf{Z}$, find $V(p)$ in Spec$(\mathbf{Z}[i])$.

10) Find a finite set of generators for the ideals of the following affine algebraic varieties:

a) $V(X_1,\dots,X_r) \subset \mathbf{A}^n_{\mathbf{k}}$
b) $V(X,Z) \cup V(Y,Z) \subset \mathbf{A}^3_{\mathbf{k}}$
c) $V(X,Z) \cup V(Y,Z-1) \subset \mathbf{A}^3_{\mathbf{k}}$.

11) Describe the irreducible components of $V(J) \subset \mathbf{A}^3_{\mathbf{k}}$ for each of these ideals J of $\mathbf{k}[X,Y,Z]$:

a) $(Y^2 - X^4, X^2 - 2X^3 - X^2Y + 2XY + Y^2 - Y)$;
b) $(XY + YZ + XZ, XYZ)$;
c) $((X-Z)(X-Y)(X-2Z), X^2 - Y^2Z)$.

Find if possible an element $f \in I(V(J))$ such that $f \notin J$.

12) Let $\mathbf{k} \subset K$ be a normal and separable field extension whose Galois group is $G = \mathrm{Gal}(K/\mathbf{k})$. Let J be an ideal of $\mathbf{k}[X_1,\dots,X_n]$. A *$K$–valued point* of $V(J)$ is a point $(a_1,\dots,a_n) \in \mathbf{A}^n_K$ such that $f(a_1,\dots,a_n) = 0$ for all $f \in J$. Prove that two K–valued points, $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$, of $V(J)$ correspond to the same maximal ideal of $\mathbf{k}[X_1,\dots,X_n]$ if and only if there exists an element $\sigma \in G$ such that $(a_1,\dots,a_n) = (\sigma(b_1),\dots,\sigma(b_n))$.

13) Let A be a Noetherian ring. Prove that a closed set X of Spec$A$ is irreducible if and only if $\mathscr{I}(X)$ is a prime ideal.

14) Let A be a Noetherian ring.

a) Prove that Spec$A$ with the Zariski topology is a Noetherian topological space. Conclude that any closed set $\mathscr{V}$ of Spec$A$ has a unique irredundant decomposition $\mathscr{V} = \mathscr{V}_1 \cup \dots \cup \mathscr{V}_s$. The $\mathscr{V}_i$ are the *irreducible components* of $\mathscr{V}$.
b) Let I be an ideal of A and let $\mathscr{V} = \mathscr{V}(I)$. Prove that an irreducible component of $\mathscr{V}$ is the same as a maximal irreducible closed subset of $\mathscr{V}$. Conclude that the irreducible components of $\mathscr{V}$ are in one–to–one correspondence with the prime ideals which are minimal among those containing I.

15) Let $\mathbf{k}$ be an algebraically closed field, let I be an ideal of $\mathbf{k}[X_1,\dots,X_n]$ and let $V = V(I)$. Prove that an irreducible component of V is the same as a maximal irreducible closed subvariety of V. Conclude that the irreducible components of V are in one–to–one correspondence with the prime ideals minimal among those containing I.

16) Let $\varphi : B \longrightarrow A$ be a ring homomorphism.

a) Prove that the map $\hat{\varphi}$ from Spec$A$ to Spec$B$ that sends every $\mathfrak{p} \in$ Spec$A$ to $\varphi^{-1}(\mathfrak{p})$ is a well–defined continuous map with respect to the Zariski topology.
b) Let $\mathbf{k}$ be an algebraically closed field, let V and W be two algebraic varieties over $\mathbf{k}$ and let A and B be their rings of functions. Identify V with the maximal spectrum of A. Show that the restriction of $\hat{\varphi}$ to V is a morphism between V and W.