COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 6

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) Let B and C be rings and let $A = B \times C$ be its cartesian product which is a ring with the obvious operations. Prove that B and C are ring of fractions of A.

2) Let S be a multiplicative set of A and let I be an ideal of A. Prove that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$.

3) Let A be a ring and let S be a multiplicative set. Prove that if A is Noetherian then so is $S^{-1}A$. Prove the same for Artinian.

4) Let $A \subset B$ be an extension of rings and let S be a multiplicative set of A.
 a) Prove that if B is integral over A, so is $S^{-1}B$ over $S^{-1}A$
 b) Prove that is $\tilde{A}$ is the integral closure of A in B, then $S^{-1}\tilde{A}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

5) Let $A = \mathbf{Z}[\sqrt{-5}]$ and let $I = (2, 1 + \sqrt{-5})$. Prove that:
 a) I is a prime ideal.
 b) If $\mathfrak{p}$ is a prime ideal that does not contain 2, then $IA_{\mathfrak{p}} = A_{\mathfrak{p}}$.
 c) If $\mathfrak{p}$ is a prime ideal that contains 2, then $\mathfrak{P} = I$ and $IA_{\mathfrak{p}} = (1 + \sqrt{-5})A_{\mathfrak{p}}$.
 d) For each prime ideal $\mathfrak{p}$ of A, the ideals $IA_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ are isomorphic as $A_{\mathfrak{p}}$–modules, but I and A are not isomorphic as A–modules.

6) Let $A = \mathbf{k}[X, Y, Z]/(X^3 - YZ, Y^2 - XZ, Z^2 - X^2Y)$, let $\mathfrak{p} = (\bar{X} - 1, \bar{Y} - 1, \bar{Z} - 1)$ and let $\mathfrak{q} = (\overline{X}, \overline{Y}, \overline{Z})$.
 a) Find a generator for the maximal ideal of $A_{\mathfrak{p}}$.
 b) Can the maximal ideal of $A_{\mathfrak{q}}$ be generated by one element?

7) Consider the ring $\mathbf{k}[X, Y, Z]/(XY)$ and let R be its localization at the prime ideal $(\overline{X}, \overline{Z})$. Prove that the maximal ideal of R is principal and that R is isomorphic to $\mathbf{k}[Y, Z]_{(Z)}$.

8) Let A be a UFD, let $f \in A$ be an irreducible element of A and let $\mathfrak{p} = (f)$. Prove that there exists an element $t \in A_{\mathfrak{p}}$ such that for any $0 \neq x \in A_{\mathfrak{p}}$, there exists $n \in \mathbf{N} \cup \{0\}$ and a unit u of $A_{\mathfrak{p}}$ such that $x = u \cdot t^n$.

9) Let $\mathfrak{p} = (X_1, \ldots, X_n)$. Prove the existence of a natural monomorphism

$$\mathbf{k}[X_1, \ldots, X_n]_{\mathfrak{p}} \hookrightarrow \mathbf{k}[[X_1, \ldots, X_n]].$$

10) Let $f \in A$.
 a) Prove that $\mathrm{Spec}(A_f)$ can be identified with the complement of $\mathscr{V}(f)$ in $\mathrm{Spec}A$.
 b) Let A be the ring of functions of a variety V of $\mathbf{A}^n_{\mathbf{k}}$ (that is, $A = \mathbf{k}[X_1, \ldots, X_n]/I(V)$) and let $f \in A$. Prove that A_f is isomorphic to the ring of functions of a variety V_f of $\mathbf{A}^{n+1}_{\mathbf{k}}$, which is, in a natural way, in one–to–one correspondence with $V \smallsetminus V(f)$.
 c) Prove that if A is a finitely generated $\mathbf{k}$–algebra, so is A_f.

11) Let V be an irreducible subvariety of $\mathbf{A}^n_{\mathbf{k}}$ and let $\mathfrak{p} = I(V)$ be the ideal of V in $A = \mathbf{k}[X_1, \ldots, X_n]$.
 a) Let $f/g \in \mathbf{A}_{\mathfrak{p}}$. Prove the existence of a Zariski open set U of $\mathbf{A}^n_{\mathbf{k}}$ such that $U \cap V$ is a dense Zariski open set of V and f/g gives a well–defined function on U. Prove that the class of f/g in $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ induces a well–defined function on $U \cap V$.
 b) Show that an element of $\mathrm{Frac}(A/\mathfrak{p})$ also induces a well–defined function on some dense Zariski open set of V.
 c) Give an algebraic and a geometric explanation of why the geometric interpretations of the elements of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and the elements of $\mathrm{Frac}(A/\mathfrak{p})$ seem to be the same.

Definition. We say that a property P for a ring A (respectively, for a module M; respectively, for a homomorphism Φ) is a *local property* if the following happens: A (respectively, M; respectively Φ) satisfies P if and only if $A_\mathfrak{p}$ (respectively $M_\mathfrak{p}$; respectively $S^{-1}\Phi = \Phi_\mathfrak{p}$ ($S = A \smallsetminus \mathfrak{p}$)) satifies P, for all prime ideals $\mathfrak{p}$ of A.

The remaining exercises provide some examples of local properties.

12) Let M be an A–module. Prove that the following are equivalent:
- i) $M = 0$;
- ii) $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$ of A;
- iii) $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ of A.

13) Let M and N be modules over A and let $\Phi : M \longrightarrow N$ be a homomorphism of A–modules. Prove that the following are equivalent:
- i) Φ is injective;
- ii) $\Phi_\mathfrak{p} : M_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is injective for all prime ideals $\mathfrak{p}$ of A;
- iii) $\Phi_\mathfrak{m} : M_\mathfrak{m} \longrightarrow N_\mathfrak{m}$ is injective for all maximal ideals $\mathfrak{m}$ of A.

14) Let M and N be modules over A and let $\Phi : M \longrightarrow N$ be a homomorphism of A–modules. Prove that the following are equivalent:
- i) Φ is surjective;
- ii) $\Phi_\mathfrak{p} : M_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is surjective for all prime ideals $\mathfrak{p}$ of A;
- iii) $\Phi_\mathfrak{m} : M_\mathfrak{m} \longrightarrow N_\mathfrak{m}$ is surjective for all maximal ideals $\mathfrak{m}$ of A.

15) Let A be an integral domain. Prove that the following are equivalent:
- i) A is normal;
- ii) $A_\mathfrak{p}$ is a normal domain for all prime ideals $\mathfrak{p}$ of A;
- iii) $A_\mathfrak{m}$ is a normal domain for all maximal ideals $\mathfrak{m}$ of A.

Hint: consider the inclusion of A in its normalization and use exercises 6.4, 6.13 and 6.14.

16) Let $n \in N$ and let $A = \mathbf{k}[X, Y]/(X^nY - 1)$. Prove that A is a normal domain.

17) Write the rings that appear in Exercises 1.22 b) and 2.7 as localizations of $\mathbf{k}[X, Y]$ at suitable prime ideals.