COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 7

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) Let A be the coordinate ring of an algebraic variety V of $\mathbf{A}^n_{\mathbf{k}}$ and let M be a finite module over A. Let $x = (x_1, \dots, x_n) \in V$ and let $\mathfrak{m}$ be the ideal of x in A. For any $h \in M$, let $h(x)$ be the class of h in $M/\mathfrak{m}M$ (recall that $M/\mathfrak{m}M$ is a vector space over $A/\mathfrak{m} = \mathbf{k}$; recall also that $A/\mathfrak{m}$ and $A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$ are isomorphic, and so are $M/\mathfrak{m}M$ and $M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m}$ as $\mathbf{k}$–vector spaces). Let $h_1, \dots, h_l \in M$ such that $h_1(x), \dots, h_l(x)$ form a system of generators for $M/\mathfrak{m}M$.

a) Prove the existence of $f \in \mathbf{k}[X_1, \dots, X_n]$, $f(x) \neq 0$, such that $h_1, \dots, h_l$ generate M_f as an A_f–module.
Hint: use Nakayama's lemma; after that, observe that $\mathfrak{m}$ is not in the support of the cokernel of the module homomorphism $A^n \longrightarrow M$ given by $h_1, \dots, h_l$; you will also need to use that localization preserves exact sequences and Exercise 6.14.

b) Conclude that $U = V \smallsetminus V(f)$ is a Zariski open neighborhood of x in V with the following property: for any $y \in U$, $h_1(y), \dots, h_l(y)$ form a system of generators for $M/\mathfrak{n}M$, where $\mathfrak{n}$ is the ideal of y in A.

- **Geometric interpretation of Exercise 7.1:** Exercise 7.1 means that if the "restriction" of $h_1, \dots, h_l$ to x generate $M/\mathfrak{m}M$, then there is a "basic" (see Exercise 5.7, g)) open neighborhood U of x such that "the restriction" of M to U and to any point $y \in U$ is also "generated" by $h_1, \dots, h_l$. Otherwise said, a system of generators of $M/\mathfrak{m}M$ "extends" to a system of generators of the "restriction" of M to a suitable open neighborhood U of x. Observe also that if V is irreducible, then U is a non–empty, dense open set of V.

- **Remark:** The statement of Exercise 7.1 can be generalize to a finite module over an arbitrary ring A, Spec$A$ playing the role of the variety V

2) Let V, A, M, x, $\mathfrak{m}$, h and $h(x)$ be as in Exercise 7.1. Assume that $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$–module of rank r.

a) Prove that if $h_1, \dots, h_l \in M$ are such that $h_1(x), \dots, h_l(x)$ are a basis for $M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m}$, then $h_1, \dots, h_l$ are a basis for $M_\mathfrak{m}$ and hence $l = r$.

b) Conclude that any basis of $M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m}$ "lifts" to a basis of $M_\mathfrak{m}$.

c) Prove that if $h_1, \dots, h_r$ are as in a), then there exists $f \in \mathbf{k}[X_1, \dots, X_n]$, not vanishing at x, such that $h_1, \dots, h_r$ are a basis for M_f as an A_f–module.

d) Conclude that if $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$–module of rank r, then there exists $f \in \mathbf{k}[X_1, \dots, X_n]$, not vanishing at x, such that M_f is a free A_f–module of rank r.

e) Conclude that $M_\mathfrak{n}$ is also a free $A_\mathfrak{n}$–module of rank r if $\mathfrak{n}$ is the ideal of any $y \in U = V \smallsetminus V(f)$.

f) Conclude also that, if $h_1, \dots, h_r$ are as in a) and $\mathfrak{n}$ is as in e), then $h_1(y), \dots, h_l(y)$ are a basis for $M_\mathfrak{n}/\mathfrak{n}M_\mathfrak{n}$, for any $y \in U$.

3) Let M be a finite A–module. We say that M is *locally free of rank* r if for any $\mathfrak{p} \in \mathrm{Spec}A$, $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$–module of rank r.

a) Prove that if $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$–module of rank r, then there exists $f \in A$, $f \notin \mathfrak{p}$ such that M_f is also a free A_f–module of rank r.

b) Conclude that, under the hypothesis of a), $M_{\mathfrak{q}}$ is also a free $A_{\mathfrak{q}}$–module of rank r for all $\mathfrak{q} \notin \mathscr{V}(f)$ (note that this is the same as saying that M_f is a locally free module over A_f or rank r). Conclude also that $M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}}$ is a vector space of dimension r over the residue field, $\mathbf{k}(\mathfrak{q})$, of $\mathfrak{q}$.

c) Prove that if M is such that $M_{\mathfrak{m}}$ is free of rank r for any $\mathfrak{m}$ maximal ideal of A, then M is a locally free module of rank r.

d) Prove that if M is a locally free module over A of rank r, then there exists a family Λ of elements of A such that

i) the complements of the $\mathscr{V}(f)$, $f \in \Lambda$ form an open covering of $\mathrm{Spec}A$;

ii) M_f is a free A_f–module for all $f \in \Lambda$.

4) Let now V, A, M, x, $\mathfrak{m}$, h and $h(x)$ be as in Exercise 7.1. For any $y \in V$, let $\mathfrak{n}$ be the ideal of y in A.

a) Prove that, if $\mathbf{k}$ is algebraically closed, M is locally free if and only if $M_{\mathfrak{n}}$ is a free $A_{\mathfrak{n}}$–module for all $y \in V$.

Assume now that M is a locally free module over A of rank r.

b) Prove that $M_{\mathfrak{n}}/\mathfrak{n}M_{\mathfrak{n}}$ is a $\mathbf{k}$–vector space of dimension r.

c) Prove that there exists a family Λ of elements of $\mathbf{k}[X_1, \ldots, X_n]$ such that

i) the sets $V \setminus V(f)$, $f \in \Lambda$ form a Zariski open covering of V;

ii) M_f is a free A_f–module for all $f \in \Lambda$;

- **Geometric interpretation of Exercises 7.2, 7.3 and 7.4**: A module M which after localizing at $\mathfrak{p} \in \mathrm{Spec}A$, becomes free, "remains" free when "restricted" to a suitable open neighborhood of $\mathfrak{p}$. In this situation, a basis of $M_{\mathfrak{p}}$ "extends" to a basis for the "restriction" of M to U. If M is a locally free module there is an open "basic" (see Exercise 5.7, g)) covering of $\{U_f | f \in \Lambda\}$ of $\mathrm{Spec}A$ (or of the variety V) such that the "restriction" of M to each U_f is free. If a module M over the coordinate ring A of an algebraic variety V of $\mathbf{A}^n_{\mathbf{k}}$ is locally free of rank r, then the "restriction" of M to every point of V is a k–vector space of the same dimension r.

5) Let A be an integral domain. Give examples of ideals of A which are free A–modules and therefore, locally free A–modules.

6) Consider the ideal $I = (X, Y)$ of $A = \mathbf{k}[X, Y]$.

a) Find the support of I as an A–module.

b) Prove that I is not a locally free A–module.

c) Find the largest open set U of $\mathbf{A}^2_{\mathbf{k}}$ such that for any $y \in U$, $I_{\mathfrak{n}}$ is free, where $\mathfrak{n}$ is the ideal of y; of what rank?

7) Consider the ideal $I = (X, Y)$ of $A = \mathbf{k}[X, Y]$. Consider an exact sequence

$$0 \longrightarrow K \longrightarrow A^2 \xrightarrow{\alpha} A \xrightarrow{\pi} A/I \longrightarrow 0,$$

where π is the canonical projection from A to A/I and K is the kernel of α. Find the supports of A/I and K.

8) Prove, directly from the definition, that (Y, X^2) is an indecomposable ideal of $\mathbf{k}[X, Y]$.

9) Let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be two $\mathfrak{p}$-primary ideals. Prove that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is also $\mathfrak{p}$-primary.

10) Let $A = \mathbf{k}[X, Y]$ and let $I = (X^2, XY)$.

a) Prove that I is not primary.

b) Prove that $I = (X) \cap (X, Y)^2$ is a shortest primary decomposition of I. Give a geometric interpretation of it.

c) Prove that for any $\gamma \in \mathbf{k}$, $I = (X) \cap (X^2, Y - \gamma X)$ is a shortest primary decomposition of I. Conclude that I does not have a unique shortest primary decomposition. Give a geometric interpretation of them.

d) Find, using directly the definition, $\mathrm{Supp} A/I$ and $\mathrm{Ass} A/I$.

11) Find a shortest primary decomposition of the ideal $(XY + XZ + YZ, XYZ)$ of $\mathbf{k}[X, Y, Z]$. Is it unique?

12) Prove that if $\varphi : A \longrightarrow B$ is a ring homomorphism and $\mathfrak{q} \subset B$ is a $\mathfrak{p}$–primary ideal, then $\varphi^{-1}(\mathfrak{q})$ is a $\varphi^{-1}(\mathfrak{p})$–primary ideal.

13) Consider the ideal $I = (Y^2 - X^3, X^4 - YZ)$ of $\mathbf{k}[X, Y, Z]$. Find a shortest primary decomposition of I. Is it unique? Hint: do primary decomposition in $\mathbf{k}[X, Y]/(Y^2 - X^3)$ and use Exercise 7.13.

14) For any $t \in \mathbf{k}$, find a primary decomposition in $\mathbf{k}[X, Y]/(Y - X^2)$ of the ideal generated by the clase of $Y - t$, where $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$.

15) Find a primary decomposition in $\mathbf{Z}[i]$, $\mathbf{Z}[\sqrt{3}]$ y $\mathbf{Z}[\sqrt{7}]$ of the ideals generated by 2, 3, 5, 7 and 11.

16) Find a primary decomposition of the ideal $(X^2 + Y^2 + 1, X^2 + 2XY - Y^2 + 1)$ in $\mathbf{R}[X, Y]$ and $\mathbf{Q}[X, Y]$.

17) Let S be a multiplicative set of A.

a) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A, prove that $S \cap \mathfrak{p} = \emptyset$ if and only if $S \cap \mathfrak{q} = \emptyset$.

b) Under the same conditions, prove that $f/s \in S^{-1}\mathfrak{q}$ if and only if $f \in \mathfrak{q}$.

c) Again under the same conditions, prove that $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary.

d) Again under the same conditions, prove that if $\varphi : A \longrightarrow S^{-1}A$ is the natural homomorphism from A to $S^{-1}A$, then $\varphi^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$.

e) Conclude the existence of a one–to–one correspondence between the primary ideals of A that do not intersect S and the primary ideals of $S^{-1}A$.

18) Let A be a Noetherian ring and let M be a finite A–module. Prove that $\mathrm{Ass} M$ is a finite set in the following way:

a) Prove the existence of a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A.

b) Conclude that $\mathrm{Ass} M \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Remark: if I is an ideal of A and $M = A/I$, we already knew that M has finitely many associated primes, since this follows from the existence of primary decompositions and from the first uniqueness theorem.

19) Let $A = \mathbf{k}[X, Y, Z]$ and $I = (XY, X - YZ)$.

a) Find a shortest primary decomposition of I and the associated primes of A/I. Hint: use the fact that $\mathbf{k}[X, Y, Z]/(X - YZ)$ is isomorphic to $\mathbf{k}[Y, Z]$ and Exercise 7.12.

b) For any associated prime $\mathfrak{p}$ of A/I, find $\overline{a} \in A/I$ such that $\mathrm{ann}\ \overline{a} = \mathfrak{p}$.

20) Let $A = \mathbf{k}[X, Y, Z]/(XZ - Y^2)$, let $I = (\overline{X}, \overline{Y}) \subset A$ and let $M = A/I^2$.

a) Prove that $I^2 = (\overline{X}) \cap (\overline{X}, \overline{Y}, \overline{Z})^2$ and that this is a shortest primary decomposition of I^2 (I provides an example of a prime ideal whose square is not primary)

b) Find Ass$M$.

c) For each associated prime $\mathfrak{p}$ of M find an element $m \in M$ such that ann $m = \mathfrak{p}$.

d) Find a chain of submodules of M like the one in Exercise 7.18, a).

21) a) Give an example of an ideal I of $\mathbf{R}[X, Y]$, neither radical nor primary, having a unique shortest primary decomposition and such that $V(Y - X + 1) \subset V(I) \subset \mathbf{A}^2_{\mathbf{R}}$.

b) Give an example of an ideal J of $\mathbf{R}[X, Y]$ not having a unique shortest primary decomposition and such that $V(Y - X + 1) = V(I) \subset \mathbf{A}^2_{\mathbf{R}}$.

22) Consider the ideal $J = (X^2, XY, XZ, YZ)$ of $B = \mathbf{C}[X, Y, Z]$ and the B–module $M = \mathbf{C}[X, Y, Z]/J$.

a) Find $\sqrt{J}$ and the irreducible components of $V(J)$.

b) Prove that $J = (X, Y) \cap (X, Z) \cap (X^2, Y, Z)$.

c) Find Ass$M$.

c) Decide if J has a unique shortest primary decomposition. If it does not, find two different shortest primary decompositions of J.