THE GROTHENDIECK PROPERTY IN THE CLASS OF ORLICZ-TYPE MODULAR SPACES

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ABSTRACT. We introduce the class of Orlicz-type modular spaces, that includes the Orlicz-Lorentz spaces, Orlicz spaces, Musielak-Orlicz spaces, etc., and we characterize the Grothendieck property for this class of Banach spaces and some quotients.

1. Introduction and preliminaries

A Banach space X is said to have *the Grothendieck property* (for short, X is Grothendieck) if, for every sequence $\{u_n : n \geq 1\}$ of the dual X^*, u_n weak converges to 0 iff u_n weak* converges to 0. (see [2, pg. 179]). For instance, $\ell_\infty(I)$ and the space $\mathcal{B}(\Omega, \Sigma)$ of bounded real Σ-measurable functions $f : \Omega \to \mathbb{R}$ on a measurable space (Ω, Σ), equipped with the supremum norm, are Grothendieck spaces. The reader is referred to [2, p. 179] and [10, p. 348] for more information about the Grothendieck property.

The aim of this paper is to study and characterize the Grothendieck property in the class of Orlicz-type modular spaces (this class includes Orlicz spaces, Musielak-Orlicz spaces, Orlicz-Lorentz spaces, etc., see below for definitions) and some quotients spaces of this class. As an antecedent we cite the paper [4] in which it is proved that the quotient space $\ell_\varphi(I)/h_\varphi(I)$ is a Grothendieck M-space, φ being an Orlicz function, $\ell_\varphi(I)$ the Orlicz space $\ell_\varphi(I) := \{f \in \mathbb{R}^I : \exists\lambda > 0, \sum_{i\in I} \varphi(\lambda f_i) < \infty\}$ and $h_\varphi(I)$ the closure in $\ell_\varphi(I)$ of the subspace integrated by the elements with finite support.

Let us fix our notation, terminology and definitions. If I is an infinite set, let βI be the Stone-Čech compactification of I and $I^* = \beta I \setminus I$. If J is a subset of I, then ${}^cJ := I \setminus J$ will be the complement of J and $J^* = \overline{J}^{\beta I} \setminus J \subset I^*$. If X is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual.

If Σ is a σ-algebra of subsets of a set Ω, let $ba(\Sigma)$ denotes the space of bounded real signed finitely additive measures on Σ and $B(\Sigma)$ the space of bounded real Σ-measurable functions $f : \Omega \to \mathbb{R}$. Recall that $B(\Sigma)$ with the supremum norm is a Grothendieck Banach space (see [2, Cor. 1.3, p. 149]) with dual $B(\Sigma)^* = ba(\Sigma)$. In the sequel we will deal with a **Γ-finite positive measure space** (Ω, Σ, μ) where: (i) Γ is an arbitrary set such that we have a finite positive measure space $(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)$ for every $\gamma \in \Gamma$; (iii) (Ω, Σ, μ) is the sum $(\Omega, \Sigma, \mu) = \bigoplus_{\gamma\in\Gamma}(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)$, which means the following:

(1) $\Omega = \biguplus_{\gamma\in\Gamma} \Omega_\gamma$ ($\biguplus$ means disjoint union).
(2) $A \subset \Omega$ satisfies $A \in \Sigma$ if and only if $A \cap \Omega_\gamma \in \Sigma_\gamma, \forall\gamma \in \Gamma$.
(3) If $A \in \Sigma$, then $\mu(A) = \sum_{\gamma\in\Gamma} \mu_\gamma(A \cap \Omega_\gamma), \forall A \in \Sigma$. Note that $\mu(A) = \infty$ is allowed and $\mu(A) = 0$ iff $\mu_\gamma(A \cap \Omega_\gamma)0$.

We deal with Γ-finite measures, instead of σ-finite measures, in order to work with spaces like the Orlicz sequence spaces $\ell_\varphi(I)$, when I is uncountable. Of course, this strategy has some difficulties because the usual measure theory refers to σ-finite measures and so we must verify for Γ-finite measures the validity of some results, that we know hold for σ-finite measures. Let us adopt the following terminology:

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(a) Ω_a will be the union of **all the atoms** of μ and $\Omega_d := \Omega \setminus \Omega_a$. μ_a and μ_d indicate **the atomic part and the purely non atomic part of** μ, respectively. $M(\mu)$ denotes **the family of all (equivalence classes of)** μ**-measurable functions** $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$. We know that $M(\mu)$ is a σ-complete lattice with the order $x \leq y$ if and only $x(t) \leq y(t)$ μ-a.e. in Ω. $L_0(\mu)$ will be **the space of all (equivalence classes of)** μ**-measurable real functions** $f : \Omega \to \mathbb{R}$. We know that $L_0(\mu)$ is a σ-complete vector lattice, which is a sublattice of $M(\mu)$. A function $f \in L_0(\mu)$ is said to be a **real simple function** if $f = \sum_{i=1}^{n} x_i \cdot \mathbb{1}_{A_i}$, where $x_i \in \mathbb{R}$ and $A_i \in \Sigma$ with $\mu(A_i) < \infty$. Let $\mathcal{S}_0$ denote **the ideal of** $L_0(\mu)$ **generated by the subspace of real simple** Σ**-measurable functions.**

(b) A normed space $(E, \|\cdot\|)$ is called **a normed function space** over (Ω, Σ, μ) if the following requirements are fulfilled: (i) E is a subspace of $L_0(\mu)$; (ii) if $x \in E, y \in L_0(\mu)$ and $|y| \leq |x|$, μ-a.e., then $y \in E$ and $\|y\| \leq \|x\|$. A **Banach function space** is a normed function space which is complete in the norm.

(c) Let $(E, \|\cdot\|)$ be a Banach function space. A vector $x \in E$ is said to be **order-continuous** (for short, **o-continuous)** if for every downward directed set $\{x_i\}_{i \in I}$ in E such that $x_i \downarrow 0$ and $0 \leq x_i \leq |x|$, μ-a.e., for some $x \in E$, we have $\|x_i\| \downarrow 0$. Denote by E^a the closed ideal of o-continuous elements of E. If $E = E^a$, then E is called o-continuous. We say that E has the **Fatou property** if $x_n \in E, 0 \leq x_n \uparrow x$ in order for some $x \in L_0(\mu)$ and $\sup_n \|x_n\| < \infty$ imply $x \in E$ and $\|x\| = \lim_{n\to\infty} \|x_n\|$.

(d) Let Z be a real Banach lattice. Then: (i) Z is said to be an M-space if $\|x \vee y\| = \|x\| \vee \|y\|$ for every $x, y \in Z^+$; (ii) Z is said to be an L_1-space if $\|x + y\| = \|x\| + \|y\|$ for every $x, y \in Z$ with $|x| \wedge |y| = 0$; (iii) recall that Z is an M-space if and only if Z^* is an L_1-space (see [12, p. 25]); (iv) $(B(\Sigma), \|\cdot\|_\infty)$ is an M-space and so $ba(\Sigma) = B(\Sigma)^*$ is an L_1-space.

Now we introduce several notions in order to define the concept of Orlicz-type modular space, that will be the context in which we will work.

Definition 1.1. Let (Ω, Σ, μ) be a positive Γ-finite measure space. Then

(1) A mapping $\rho : M(\mu)^+ \to [0, \infty]$ is said to be

(1a) *monotone* if $\rho(x) \leq \rho(y)$ whenever $x, y \in M(\mu)^+$ and $x \leq y$.

(1b) *left-continuous* if $\rho(x_n) \uparrow \rho(x)$, whenever $x, x_n \in M(\mu)^+, n \geq 1$, and $x_n \uparrow x$.

(1c) *1-convex* if $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $x, y \in M(\mu)^+$ and $\alpha, \beta \geq 0,\ \alpha + \beta = 1$.

(2) A *Köthe semimodular* on (Ω, Σ, μ) is a mapping $\rho : L_0(\mu) \cup M(\mu)^+ \to [0, +\infty]$ such that

(2a) The restriction of ρ to $M(\mu)^+$ is a monotone left-continuous and 1-convex mapping.

(2b) $\rho(f) = \rho(|f|)$ for every $f \in L_0(\mu)$ and, if $A \in \Sigma$ with $\mu(A) < \infty$, then $\rho(\lambda \mathbb{1}_A) < \infty$ for some $\lambda > 0$.

(2c) $\rho(0) = 0$ and $f = 0$ μ-a.e. whenever $f \in M(\mu)^+$ and $\rho(\lambda f) = 0$ for all $\lambda > 0$.

(3) If ρ is a Köthe semimodular on (Ω, Σ, μ), we define the *modular space* $L_\rho(\mu)$ as follows

$$L_\rho(\mu) := \{f \in L_0(\mu) : \lim_{\lambda\to 0} \rho(\lambda f) = 0\}.$$

Clearly, $L_\rho(\mu)$ satisfies $L_\rho(\mu) = \{f \in L_0(\mu) : \exists\lambda > 0, \rho(\lambda f) < +\infty\}$. For every $f \in L_\rho(\mu)$, define the *Luxemburg norm* $\|f\|_L$ as:

$$\|f\|_L = \inf\{\lambda > 0 : \rho(f/\lambda) \leq 1\}.$$

A subset $A \subset L_\rho(\mu)$ is said to be *ρ-dense in* $L_\rho(\mu)$ if for every $f \in L_\rho(\mu)$ with $\rho(f) < +\infty$ and every $\epsilon > 0$ there exists $y_\epsilon \in A$ such that $\rho(f - y_\epsilon) < \epsilon$.

(4) ρ is *an Orlicz-type semimodular* on (Ω, Σ, μ) if and only if : (i) ρ is a Köthe semimodular on (Ω, Σ, μ) such that $\rho(f \vee g) \leq \rho(f) + \rho(g)$ for every pair $f, g \in M(\mu)^+$; (ii) ρ is **finitely determined**, that is, if $\rho(f) > a \geq 0$, for some $f \in L_0(\mu)$, there exists a subset $A \in \Sigma$ with $0 < \mu(A) < \infty$ such that $\rho(f \cdot \mathbb{1}_A) > a$.

(5) A Banach space E is said to be *an Orlicz-type modular space* if and only if there exists a Γ-finite measure space (Ω, Σ, μ) and an Orlicz-type semimodular ρ on (Ω, Σ, μ) such that $E := (L_\rho(\mu), \|\cdot\|_L)$.

Remark 1.2. (1) The Orlicz spaces, Musielak-Orlizc spaces, Lorentz spaces, etc., are Orlicz-type modular spaces. So all the theory developed in this paper holds for these spaces.

(2) Let ρ be a Köthe semimodular on (Ω, Σ, μ). The following facts are easily proved

(21) $(L_\rho(\mu), \|\cdot\|_L)$ is a σ-complete normed function space, that fulfills the Fatou property. Actually it is a Banach function space. Moreover, $\mathcal{S}_0 \subset L_\rho(\mu)$ by (2b).

(22) If $f \in M(\mu)^+$ satisfies $\rho(f) < \infty$, then $f < \infty$ μ-a.e. Indeed, let $A := \{w \in \Omega : f(w) = +\infty\}$ and observe that $t\mathbb{1}_A \leq sf$ for every $0 \leq t$ and $s > 0$. So, as ρ is monotone, for every $0 \leq t$ and $s > 0$ we have $\rho(t\mathbb{1}_A) \leq \rho(sf) \downarrow 0$ when $s \downarrow 0$. Thus $\mathbb{1}_A = 0$ μ-a.e. by (2c), that is, $\mu(A) = 0$.

2. $L_\rho(\mu)/H(\mathcal{S})$ is a Grothendieck M-space

Let (Ω, Σ, μ) be a Γ-finite measure space and ρ be an Orlicz-type semimodular on (Ω, Σ, μ). If $\mathcal{S} \subset L_\rho(\mu)$ is an ideal, we will denote

$$H(\mathcal{S}) := \{f \in L_\rho(\mu) : \forall \lambda > 0,\ \exists s \in \mathcal{S} \text{ such that } \rho\big(\frac{f-s}{\lambda}\big) < +\infty\}.$$

It is easy to see that $H(\mathcal{S})$ is a closed ideal of $L_\rho(\mu)$ such that $H(H(\mathcal{S})) = H(\mathcal{S}) = \overline{\mathcal{S}}$. For each $f \in L_\rho(\mu)$ we define

$$\delta(f) := \inf\{\lambda > 0 : \exists s \in \mathcal{S} \text{ such that } \rho\big(\frac{f-s}{\lambda}\big) < +\infty\}.$$

Observe that $\delta(f) < +\infty$, $\forall f \in L_\rho(\mu)$, by definition of $L_\rho(\mu)$.

In the following we see a series of lemmas that we need in order to establish the fundamental result of Theorem 2.8.

Lemma 2.1. *Let (Ω, Σ, μ) be a Γ-finite measure space and ρ be an Orlicz-type semimodular on (Ω, Σ, μ). If $\{f_n : n \geq 1\} \subset M(\mu)^+$, then*

$$\rho(\sup_{n\geq 1} f_n) \leq \sum_{n\geq 1} \rho(f_n).$$

Proof. Since ρ satisfies $\rho(g_1 \vee g_2) \leq \rho(g_1) + \rho(g_2)$, $\forall g_1, g_2 \in M(\mu)^+$, then

$$\rho(\sup\{f_i : i = 1, 2, \ldots, k\}) \leq \sum_{i=1}^{k} \rho(f_i).$$

As $\sup\{f_i : i = 1, 2, \ldots, k\} \uparrow \sup_{n\geq 1} f_n$ when $k \to \infty$ and ρ is left-continuous, we have

$$\rho(\sup_{n\geq 1} f_n) = \lim_{k\to\infty} \rho(\sup\{f_i : i = 1, 2, \ldots, k\}) \leq \lim_{k\to\infty} \sum_{i=1}^{k} \rho(f_i) = \sum_{i\geq 1} \rho(f_i).$$

□

Lemma 2.2. *Let (Ω, Σ, μ) be a Γ-finite measure space, ρ an Orlicz-type semimodular on (Ω, Σ, μ) and $\mathcal{S} \subset L_\rho(\mu)$ an ideal such that $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$. Then*

(1) For every $f \in L_\rho(\mu)$, the distance

$$dist(f, H(\mathcal{S})) := \inf\{\|f - h\|_L : h \in H(\mathcal{S})\}$$

from f to $H(\mathcal{S})$ satisfies $dist(f, H(\mathcal{S})) = \delta(f)$.

(2) If $x^* \in H(\mathcal{S})^\perp := \{z \in L_\rho(\mu)^* : \langle z, x\rangle = 0,\ \forall x \in H(\mathcal{S})\}$, then

$$\|x^*\| = \sup\{\langle x^*, f\rangle : f \in L_\rho(\mu) \text{ with } \rho(f) < +\infty\}.$$

Proof. (1) Let $f \in L_\rho(\mu)$ and fix $\epsilon > 0$. By definition of $\delta(f)$, there exists $s \in \mathcal{S}$ such that $\rho(\frac{f-s}{\delta(f)+\epsilon}) < +\infty$. Since $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$, there exist $y \in H(\mathcal{S})$ such that $\rho\big(\frac{f-s-y}{\delta(f)+\epsilon}\big) < 1$, which implies $dist(f, H(\mathcal{S})) \leq \delta(f) + \epsilon$, because $s + y \in H(\mathcal{S})$. Hence, as $\epsilon > 0$ is arbitrary, we get $dist(f, H(\mathcal{S})) \leq \delta(f)$. Let us prove that $dist(f, H(\mathcal{S})) \geq \delta(f)$. If $\delta(f) = 0$ this is clear. So, suppose that $\delta(f) > 0$.

Claim. For every $y \in H(\mathcal{S})$ and every positive number $0 < \lambda < \delta(f)$ we have $\rho(\frac{f-y}{\lambda}) = +\infty$.

Indeed, otherwise for some $0 < \lambda < \delta(f)$ and some $y \in H(\mathcal{S})$ we would have $\rho(\frac{f-y}{\lambda}) < +\infty$. Take $t > 0$ such that $\lambda < t < \delta(f)$ and denote $r := \lambda/t$. Then $0 < r < 1$ and there exists $s \in \mathcal{S}$ such that $\rho(\frac{y-s}{(1-r)t}) < +\infty$. Since $\frac{f-s}{t} = r\frac{f-y}{rt} + (1-r)\frac{y-s}{(1-r)t}$, we have

$$\rho\big(\tfrac{f-s}{t}\big) \leq r\rho\big(\tfrac{f-y}{\lambda}\big) + (1-r)\rho\big(\tfrac{y-s}{(1-r)t}\big) < +\infty.$$

Since $t < \delta(f)$, taking into account the definition of $\delta(f)$, we get a contradiction. So, the Claim holds.

From the Claim we deduce that $\|f - y\| \geq \delta(f),\ \forall y \in H(\mathcal{S})$, that is, $dist(f, H(\mathcal{S})) \geq \delta(f)$, and this completes the proof of (1).

(2) First, since $B(L_\rho(\mu)) \subset \{f \in L_\rho(\mu) \text{ with } \rho(f) < +\infty\}$, it is clear that

$$\|x^*\| \leq \sup\{\langle x^*, f\rangle : f \in L_\rho(\mu) \text{ with } \rho(f) < +\infty\}.$$

On the other hand, since $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$, for each $f \in L_\rho(\mu)$ with $\rho(f) < +\infty$ there exists $y_f \in H(\mathcal{S})$ such that $\rho(f - y_f) \leq 1$, which implies $f - y_f \in B(L_\rho(\mu))$. So, as $x^* \in (H(\mathcal{S}))^\perp$ we have

$$\sup\{\langle x^*, f\rangle : f \in L_\rho(\mu) \text{ with } \rho(f) < +\infty\} =$$
$$= \sup\{\langle x^*, f - y_f\rangle : f \in L_\rho(\mu) \text{ with } \rho(f) < +\infty\} \leq \|x^*\|$$

and this completes the proof. □

Throughout all this section ρ will be an Orlicz-type semimodular on (Ω, Σ, μ) and $\mathcal{S}$ an ideal of $L_\rho(\mu)$ such that $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$. Let $X = \frac{L_\rho(\mu)}{H(\mathcal{S})}$ and let $Q : L_\rho(\mu) \to \frac{L_\rho(\mu)}{H(\mathcal{S})}$ be the canonical quotient mapping. Observe that the dual space $X^* = \big(\frac{L_\rho(\mu)}{H(\mathcal{S})}\big)^*$ is isometrically isomorphic with the subspace $H(\mathcal{S})^\perp$ of $(L_\rho(\mu))^*$. So, we identify both spaces and by simplicity we write $\langle x^*, f\rangle$ instead of $\langle x^*, Q(f)\rangle$ for every $x^* \in X^*$ and $f \in L_\rho(\mu)$.

Lemma 2.3. *Let (Ω, Σ, μ) be a Γ-finite measure space and ρ an Orlicz-type semimodular on (Ω, Σ, μ), $\mathcal{S} \subset L_\rho(\mu)$ an ideal such that $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$. Then $X := \frac{L_\rho(\mu)}{H(\mathcal{S})}$ is an M-space.*

Proof. Let $x, y \in X^+$ and prove that $\|x \vee y\| = \|x\| \vee \|y\|$. Since $x \vee y \geq x,\ x \vee y \geq y$, then $\|x \vee y\| \geq \|x\|, \|y\|$, whence we get $\|x \vee y\| \geq \|x\| \vee \|y\|$. Let us prove that $\|x \vee y\| \leq \|x\| \vee \|y\|$. Pick $+\infty > \lambda > \|x\| \vee \|y\|$ and choose $f, g \in L_\rho(\mu)^+$ such that $Qf = x,\ Qg = y$ and $\|f\| < \lambda,\ \|g\| < \lambda$. So, by the definition of the Luxemburg norm, we have $\rho(f/\lambda) \leq 1$ and $\rho(g/\lambda) \leq 1$. Since ρ is an Orlicz-type semimodular we get

$$\rho\big(\frac{f \vee g}{\lambda}\big) \leq \rho\big(\frac{f}{\lambda}\big) + \rho\big(\frac{g}{\lambda}\big) \leq 2.$$

By Lemma 2.2 we have $dist(f \vee g, H(\mathcal{S})) \leq \lambda$, whence we get $\|x \vee y\| = \|Q(f \vee g)\| = dist(f \vee g, H(\mathcal{S})) \leq \lambda$. So, $\|x \vee y\| \leq \|x\| \vee \|y\|$ and this completes the proof. □

If $X = \frac{L_\rho(\mu)}{H(\mathcal{S})}$ and $\xi \in X^{*+} = H(\mathcal{S})^{\perp+}$, for each $E \in \Sigma$ and $h \in L_\rho(\mu)$ define $\xi_E : L_\rho(\mu) \to \mathbb{R}$ as $\xi_E(h) = \xi(h_E)$ where $h_E = h \cdot \mathbb{1}_E$. Clearly, $\xi_E \in X^{*+}$. Define the mapping $\nu_\xi : \Sigma \to [0, +\infty)$ as $\nu_\xi(E) = \|\xi_E\|$.

Lemma 2.4. *If $x^* \in X^{*+}$, then $\nu_{x^*} \in ba(\Sigma)$ and, given $\epsilon > 0$, there exists $f \in L_\rho(\mu)^+$ (depending on x^* and ϵ) with $\rho(f) < \epsilon$ such that*
(A) $\forall E \in \Sigma,\ \nu_{x^*}(E) = \langle x^*, f_E\rangle$; (B) $\forall g \in B(\Sigma),\ \nu_{x^*}(g) = \langle x^*, fg\rangle$.

Proof. Let $E, F \in \Sigma$ be two disjoint subsets. Then
(i) $x^*_{E\cup F} = x^*_E + x^*_F$. Obvious.
(ii) $x^*_E \wedge x^*_F = 0$. Indeed, for every $f \in L_\rho(\mu)^+$ we have

$$0 \le \langle x^*_E \wedge x^*_F, f\rangle = \inf\{\langle x^*_E, g\rangle + \langle x^*_F, f-g\rangle : 0 \le g \le f, g \in L_\rho(\mu)^+\} \le$$
$$\le \langle x^*_E, f_F\rangle + \langle x^*_F, f - f_F\rangle = 0.$$

(iii) $\|x^*_{E\cup F}\| = \|x^*_E\| + \|x^*_F\|$. First, $\|x^*_{E\cup F}\| \le \|x^*_E\| + \|x^*_F\|$ because $x^*_{E\cup F} = x^*_E + x^*_F$. On the other hand, given $\epsilon > 0$, by Lemma 2.2 there exist $f, g \in L_\rho(\mu)^+$ with $\rho(f) < +\infty,\ \rho(g) < +\infty$ such that

$$\langle x^*_E, f\rangle > \|x^*_E\| - \epsilon/2 \text{ and } \langle x^*_F, g\rangle > \|x^*_F\| - \epsilon/2.$$

So, as $\rho(f \vee g) \le \rho(f) + \rho(g) < +\infty$ we have by Lemma 2.2

$$\|x^*_{E\cup F}\| \ge \langle x^*_{E\cup F}, f \vee g\rangle \ge \langle x^*_E, f\rangle + \langle x^*_F, g\rangle > \|x^*_E\| + |x^*_F\| - \epsilon,$$

and from this fact we get $\|x^*_{E\cup F}\| = \|x^*_E\| + \|x^*_F\|$.

Therefore, the mapping $\nu_{x^*} : \Sigma \to [0. +\infty)$ such that $\nu_{x^*}(E) = \|x^*_E\|, \forall E \in \Sigma$, satisfies $\nu_{x^*} \in ba(\Sigma)^+$.

(A) Now we find the function $f \in L_\rho(\mu)^+$ fulfilling the requirements of the statement. By Lemma 2.2 for each $k \in \mathbb{N}$ we can choose a function $\tilde f_k \in L_\rho(\mu)^+$ such that $\rho(\tilde f_k) < +\infty$ and $\langle x^*, \tilde f_k\rangle \ge \|x^*\| - \frac{1}{k}$. As $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$, we can find $h_k \in H(\mathcal{S})$ such that $0 \le h_k \le \tilde f_k$ and $\rho(\tilde f_k - h_k) \le \frac{\epsilon}{2^k},\ k \ge 1$. Let $f_k = \tilde f_k - h_k,\ k \ge 1$. Then $\rho(f_k) \le \frac{\epsilon}{2^k}$ and, as $x^* \in (H(\mathcal{S}))^\perp$, also

$$\langle x^*, f_k\rangle = \langle x^*, \tilde f_k\rangle \ge \|x^*\| - \tfrac{1}{k}.$$

Let $f := \sup\{f_k : k \ge 1\}$. Since ρ is an Orlicz-type semimodular, by Lemma 2.1 we have

$$\rho(f) \le \sum_{k\le 1} \rho(f_k) \le \sum_{k\ge 1} \frac{\epsilon}{2^k} = \epsilon.$$

Hence, $f \in L_\rho(\mu)^+$ and $\|x^*\| \ge \langle x^*, f\rangle$ by Lemma 2.2. So, as $f \ge f_k$ we have

$$\|x^*\| \ge \langle x^*, f\rangle \ge \langle x^*, f_k\rangle \ge \|x^*\| - \frac{1}{k},\ \forall k \ge 1.$$

Thus $\nu_{x^*}(\Omega) = \|x^*\| = \langle x^*, f\rangle$. Let $E \in \Sigma$. Then

$$\nu_{x^*}(E) = \|x^*_E\| = \sup\{\langle x^*, h_E\rangle : h \in L_\rho(\mu), \rho(h) < +\infty\} \ge \langle x^*, f_E\rangle.$$

Analogously, if ${}^cE = \Omega \setminus E$, then $\nu_{x^*}({}^cE) \ge \langle x^*, f_{{}^cE}\rangle$. As

$$\langle x^*, f_E\rangle + \langle x^*, f_{{}^cE}\rangle = \langle x^*, f\rangle = \nu_{x^*}(\Omega) = \nu_{x^*}(E) + \nu_{x^*}({}^cE),$$

we get that $\nu_{x^*}(E) = \langle x^*, f_E\rangle$.

(B) Let $\mathcal{F} \subset B(\Sigma)$ be the subspace of real Σ-measurable step-functions $g : \Omega \to \mathbb{R}$ such that $g = \sum_{i=1}^n a_i \cdot \mathbb{1}_{A_i}$, where $a_i \in \mathbb{R}$ and $A_1, ..., A_n$ are disjoint elements of Σ. Observe that, if $g \in \mathcal{F}$, it is trivial that $\nu_{x^*}(g) = \langle x^*, gf\rangle$. So, let $g \in B(\Sigma)$. As $\mathcal{F}$ is dense in $(B(\Sigma), \|\cdot\|_\infty)$, there exists a sequence $\{g_n : n \ge 1\} \subset \mathcal{F}$ such that $\|g - g_n\|_\infty \to 0$. So, $\nu_{x^*}(g) = \lim_{n\to\infty} \nu_{x^*}(g_n)$ because $\nu_{x^*} \in ba(\Sigma) = B(\Sigma)^*$.

Claim. $g_n f \to gf$ in $(L_\rho(\mu), \|\cdot\|_L)$.

Indeed, let $\lambda > 0$ and choose $n_0 \in \mathbb{N}$ such that $\|g - g_n\|_\infty/\lambda \le (\rho(f)+1)^{-1}$ for every $n \ge n_0$. Since $|f(g-g_n)/\lambda| \le |f/(\rho(f)+1)|$, for every $n \ge n_0$ we have

$$\rho(\frac{f(g-g_n)}{\lambda}) \le \rho(\frac{f}{\rho(f)+1}) \le \frac{\rho(f)}{\rho(f)+1} \le 1.$$

So, $\|f(g-g_n)\|_L \leq \lambda$ and this proves that $g_n f \to gf$ in $(L_\rho(\mu), \|\cdot\|_L)$. Thus, we have

$$\nu_{x^*}(g) = \lim_{n\to\infty} \nu_{x^*}(g_n) = \lim_{n\to\infty} \langle x^*, fg_n\rangle = \langle x^*, fg\rangle.$$

□

Lemma 2.5. *Given $\{x_n^* : n \geq 1\} \subset X^{*+}$ and $\epsilon > 0$, there exists $f \in L_\rho(\mu)^+$ with $\rho(f) < \epsilon$ such that for every $n \geq 1$ we have*
(A) $\forall E \in \Sigma,\ \nu_{x_n^*}(E) = \langle x_n^*, f_E\rangle$; (B) $\forall g \in B(\Sigma),\ \nu_{x_n^*}(g) = \langle x_n^*, fg\rangle$.

Proof. By Lemma 2.4 for every $n \in \mathbb{N}$ there exists $f_n \in L_\rho(\mu)^+$ such that $\rho(f_n) < \frac{\epsilon}{2^n}$ and
(a) $\forall E \in \Sigma,\ \nu_{x_n^*}(E) = \langle x^*, f_{nE}\rangle$; (b) $\forall g \in B(\Sigma),\ \nu_{x_n^*}(g) = \langle x^*, f_n g\rangle$.

Let $f := \sup_{n\geq 1} f_n$. As ρ is an Orlicz-type semimodular, by Lemma 2.1 we get

$$\rho(f) \leq \sum_{n\geq 1} \rho(f_n) < \sum_{n\geq 1} \tfrac{\epsilon}{2^n} = \epsilon.$$

So, for every $n \geq 1$ and every $E \in \Sigma$ we have

$$\nu_{x_n^*}(E) = \langle x_n^*, f_{nE}\rangle \leq \langle x_n^*, f_E\rangle \leq \|x_{nE}^*\| = \nu_{x_n^*}(E),$$

whence we get $\nu_{x_n^*}(E) = \langle x_n^*, f_E\rangle$. Finally, if $g \in B(\Sigma)$ and $n \geq 1$, the argument of the part (B) of the proof of Lemma 2.4 yields that $\nu_{x_n^*}(g) = \langle x_n^*, fg\rangle$. □

If $x^* \in X^*$ and $x^* = x^{*+} - x^{*-}$ with $x^{*+}, x^{*-} \in X^{*+}$, define the mapping $\nu_{x^*} : \Sigma \to \mathbb{R}$ as $\nu_{x^*}(E) = \nu_{x^{*+}}(E) - \nu_{x^{*-}}(E)$ for every $E \in \Sigma$. Since $\nu_{x^{*+}}, \nu_{x^{*-}} \in ba(\Sigma)$, it is clear that $\nu_{x^*} \in ba(\Sigma) = (B(\Sigma))^*$ and so $\nu_{x^*}(g) = \nu_{x^{*+}}(g) - \nu_{x^{*-}}(g), \forall g \in B(\Sigma)$.

Lemma 2.6. *Given $\{x_n^* : n \geq 1\} \subset X^*$ and $\epsilon > 0$, there exists $f \in L_\rho(\mu)^+$ with $\rho(f) < \epsilon$ such that for every $n \geq 1$ we have*
(A) $\forall E \in \Sigma,\ \nu_{x_n^*}(E) = \langle x_n^*, f_E\rangle$; (B) $\forall g \in B(\Sigma),\ \nu_{x_n^*}(g) = \langle x_n^*, fg\rangle$.

Proof. By Lemma 2.5 there exists $f \in L_\rho(\mu)^+$ such that $\rho(f) < \epsilon$ and

$$\nu_{x_n^{*+}}(E) = \langle x_n^{*+}, f_E\rangle,\ \nu_{x_n^{*-}}(E) = \langle x_n^{*-}, f_E\rangle,\ \nu_{x_n^{*+}}(g) = \langle x_n^{*+}, fg\rangle,\ \nu_{x_n^{*-}}(g) = \langle x_n^{*-}, fg\rangle,$$

for every $n \in \mathbb{N}$, $E \in \Sigma$ and $g \in B(\Sigma)$. So f satisfies the statement. □

Lemma 2.7. *There exists an order-isomorphic and isometric embedding ν of the space X^* into $ba(\Sigma)$.*

Proof. For every $x^* \in X^*$ we define $\nu(x^*) = \nu_{x^*}$, which is in $ba(\Sigma)$ by Lemma 2.5.

Claim 1. ν is linear.

Indeed, let $x^*, y^* \in X^*$ and $\alpha, \beta \in \mathbb{R}$. By Lemma 2.6 there exists $f \in L_\rho(\mu)^+$ with $\rho(f) < \infty$ such that for every $E \in \Sigma$

$$\nu_{\alpha x^* + \beta y^*}(E) = \langle \alpha x^* + \beta y^*, f_E\rangle, \nu_{x^*}(E) = \langle x^*, f_E\rangle \text{ and } \nu_{y^*}(E) = \langle y^*, f_E\rangle.$$

So, for every $E \in \Sigma$ we have

$$\nu_{\alpha x^* + \beta y^*}(E) = \langle \alpha x^* + \beta y^*, f_E\rangle = \alpha\langle x^*, f_E\rangle + \beta\langle y^*, f_E\rangle = (\alpha\nu_{x^*} + \beta\nu_{y^*})(E),$$

and this proves that ν is linear.

Claim 2. $\nu_{x^*\vee y^*} = \nu_{x^*} \vee \nu_{y^*}$ for every $x^*, y^* \in X^*$.

Indeed, by Lemma 2.6 there exists $f \in L_\rho(\mu)^+$ with $\rho(f) < \infty$ such that for every $g \in B(\Sigma)$

$$\nu_{x^*\vee y^*}(g) = \langle x^* \vee y^*, gf\rangle, \nu_{x^*}(g) = \langle x^*, gf\rangle \text{ and } \nu_{y^*}(g) = \langle y^*, gf\rangle.$$

So, for every $g \in B(\Sigma)^+$ we have

$$\begin{aligned}\nu_{x^*\vee y^*}(g) = \langle x^* \vee y^*, gf\rangle &= \sup\{\langle x^*, fh_1\rangle + \langle y^*, fh_2\rangle : h_1, h_2 \in B(\Sigma)^+, h_1 + h_2 = g\} = \\ &= \sup\{\nu_{x^*}(h_1) + \nu_{y^*}(h_2) : h_1, h_2 \in B(\Sigma)^+, h_1 + h_2 = g\} = (\nu_{x^*} \vee \nu_{y^*})(g)\end{aligned}$$

and this proves that $\nu_{x^*\vee y^*} = \nu_{x^*} \vee \nu_{y^*}$.

Claim 3. ν is an isometry.

Indeed, pick $x^* \in X^{*+}$ and observe that $(\nu_{x^*})^+ = \nu_{x^{*+}}$ and $(\nu_{x^*})^- = \nu_{x^{*-}}$ by Claim 2. So, taking into account that $ba(\Sigma)$ and X^* are L_1-spaces we have

$$\|\nu_{x^*}\| = \|(\nu_{x^*})^+\| + \|(\nu_{x^*})^-\| = \|\nu_{x^{*+}}\| + \|\nu_{x^{*-}}\| = \|x^{*+}\| + \|x^{*-}\| = \|x^*\|.$$

Thus, ν is an order-isomorphic and isometric embedding of the space X^* into $ba(\Sigma)$. □

Theorem 2.8. *Let (Ω, Σ, μ) be a Γ-finite measure space, ρ an Orlicz-type semimodular on (Ω, Σ, μ) and $\mathcal{S} \subset L_\rho(\mu)$ an ideal such that $H(\mathcal{S})$ is ρ-dense in $L_\rho(\mu)$. Then the space $X = \frac{L_\rho(\mu)}{H(\mathcal{S})}$ is a Grothendieck M-space.*

Proof. First, $X = \frac{L_\rho(\mu)}{H(\mathcal{S})}$ is an M-space by Lemma 2.3. Let $\{x_n^* : n \geq 1\} \subset X^*$ be a sequence such that $x_n^* \overset{w^*}{\to} 0$. By Lemma 2.6 there exists $f \in L_\rho(\mu)^+$ with $\rho(f) < +\infty$ such that $\nu_{x_n^*}(g) = \langle x_n^*, fg\rangle$ for every $n \geq 1$ and every $g \in B(\Sigma)$. Since $fg \in L_\rho(\mu)$, $\forall g \in B(\Sigma)$, and $x_n^* \overset{w^*}{\to} 0$, then $\nu_{x_n^*}(g) \to 0$, $\forall g \in B(\Sigma)$, that is, $\nu_{x_n^*} \overset{w^*}{\to} 0$ in $(B(\Sigma)^*, w^*)$. Since $B(\Sigma)$ is Grothedieck, we get $\nu_{x_n^*} \overset{w}{\to} 0$. Finally observe that $\nu : X^* \to ba(\Sigma)$ is an isomorphism by Lemma 2.7. So, $x_n^* \overset{w}{\to} 0$ and this completes the proof. □

3. When $L_\rho(\mu)$ is Grothendieck?

Let (Ω, Σ, μ) be a Γ-finite measure space and X a Banach function space on (Ω, Σ, μ). Then:

(1) A functional $G \in X^*$ is said to be *an integral functional* if $|G|(f_n) \underset{n\to\infty}{\to} 0$ whenever $\{f_n : n \geq 1\}$ is a sequence in X such that $f_n \downarrow 0$. Let $X_i^* \subset X^*$ denote the subspace of integral functionals of X^*. The *Köthe dual* X' of X is the subspace of all elements $G \in X^*$ (in fact, $G \in X_i^*$) such that there exists a function $g \in L_0(\mu)$ fulfilling $fg \in L_1(\mu)$ and $G(f) = \int_\Omega fg d\mu$ for every $f \in X$. When μ is a σ-finite measure it is well known (see [14, Th. 3, p. 462]) that $X_i^* = X'$. We claim that the equality $X_i^* = X'$ also holds for Γ-finite measure spaces. Indeed, fix $G \in X_i^*$. By [14, Th. 3, p. 462] for each $\gamma \in \Gamma$ there exists a Σ-measurable function $g_\gamma : \Omega_\gamma \to \mathbb{R}$ such that $fg_\gamma \in L_1(\mu)$ and $G(f) = \int_{\Omega_\gamma} fg_\gamma d\mu$ for every $f \in X$ with $\mathrm{supp}(f) \subset \Omega_\gamma$. So, if $g \in L_0(\mu)$ is such that $g \upharpoonright \Omega_\gamma = g_\gamma$, then it is not hard to prove that $fg \in L_1(\mu)$ and $G(f) = \int_\Omega fg d\mu$ for every $f \in X$.

(2) A non-negative functional $G \in X^{*+}$ is called *a singular functional* whenever it follows from $G_1 \in X_i^*$ and $0 \leq G_1 \leq G$ that $G_1 = 0$. An arbitrary element $G \in X^*$ is called singular if the positive and negative components G^+, G^- of G are singular. Let X_s^* denote the subspace of singular functionals of X^*. It is well known that the subspaces X_i^* and X_s^* are mutually disjoint closed ideals of X^* such that $X^* = X_i^* \oplus X_s^*$ (see [14, Th. 2, p. 467]). It is easy to see that the subspace X^a of o-continuous elements of X satisfies $X^a = (X_s^*)^\perp$. Moreover, if $\mathrm{supp}(X^a) = \Omega$, then $(X^a)^\perp = X_s^*$ (see [14, p. 481]). So, if $\mathrm{supp}(X^a) = \Omega$, then $X^* = X_i^*$ if and only if $X = X^a$.

If a Banach lattice X is Grothendieck, then its dual X^* is o-continuous (see [10, Theorem 5.3.13, p. 355]). The converse is not true because $\ell_1(I)$ is o-continuous but $c_0(I)$ is not Grothendieck. However, as we show in the following result, when X is an Orlicz-type modular space (with some minor requirements), X is Grothendieck if and only if the subspace of integral functionals (or Köthe dual) X_i^* is o-continuous,

Theorem 3.1. *Let (Ω, Σ, μ) be a Γ-finite measure space and ρ an Orlicz-type semimodular on (Ω, Σ, μ). Assume that $\mathcal{S}_1 \subset L_\rho(\mu)$ is an ideal such that: (i) $H(\mathcal{S}_1)$ is ρ-dense in $L_\rho(\mu)$; (ii) $L_\rho(\mu)^a = H(\mathcal{S}_1)$, where $L_\rho(\mu)^a$ is the subspace of o-continuous elements of $L_\rho(\mu)$; (iii) supp$(\mathcal{S}_1) = \Omega$. Then*

(A) $L_\rho(\mu)/H(\mathcal{S}_1)$ is a Grothendieck M-space.

(B) The following statements are equivalent:
(1) $L_\rho(\mu)$ is Grothendieck; (2) $L_\rho(\mu)_i^*$ is o-continuous.

Proof. (A) follows from Proposition 2.8.

(B) First, recall that the dual space $L_\rho(\mu)^*$ has the expression $L_\rho(\mu)^* = L_\rho(\mu)^*_i \overset{d}{\oplus} L_\rho(\mu)^*_s$, where $\overset{d}{\oplus}$ means the disjoint direct sum, $L_\rho(\mu)^*_i$ is the subspace of integral functionals and $L_\rho(\mu)^*_s$ the subspace of singular functionals. Since the support or carrier $\mathrm{supp}(L_\rho(\mu)^a)$ of $L_\rho(\mu)^a$ is $\mathrm{supp}(L_\rho(\mu)^a) = \mathrm{supp}(H(\mathcal{S}_1)) = \Omega$, then $(L_\rho(\mu)^a)^\perp = L_\rho(\mu)^*_s$ (see [14, pg. 481]). On the other hand, under the conditions of the statement, $L_\rho(\mu)/H(\mathcal{S}_1)$ is Grothendieck by (A). Thus, as $L_\rho(\mu)^a = H(\mathcal{S}_1)$, we have

$$(L_\rho(\mu)/H(\mathcal{S}_1))^* = (H(\mathcal{S}_1))^\perp = (L_\rho(\mu)^a)^\perp = L_\rho(\mu)^*_s.$$

So, the sequential weak* and weak convergences coincide in $L_\rho(\mu)^*_s$. Moreover, $L_\rho(\mu)^*_s$ is o-continuous, because it is the dual of a Grothendieck Banach lattice (see [10, Theorem 5.3.13, p. 355]).

(1) ⇒ (2). Assume that $L_\rho(\mu)$ is Grothendieck. Then $L_\rho(\mu)^*$ is o-continuous (see [10, Theorem 5.3.13, p. 355]). So $L_\rho(\mu)^*_i$ is o-continuous.

(2) ⇒ (1). In order to prove that $L_\rho(\mu)$ is Grothendieck we use [10, Theorem 5.3.13, p. 355]. So, we must check the following three conditions:

(i) First condition: $L_\rho(\mu)$ has the interpolation property (I). Recall ([10, Def. 1.1.7 , p. 7]) that a vector lattice E has the interpolation property (I) if for all sequences $(x_n)_{n\geq 1},\ (y_m)_{m\geq 1} \subset E$ such that $x_n \leq y_m,\ \forall n, m \in \mathbb{N}$, there exists $u \in E$ such that $x_n \leq u \leq y_m,\ \forall n, m \in \mathbb{N}$. In our case $L_\rho(\mu)$ has the interpolation property (I) because $L_\rho(\mu)$ is σ-complete.

(ii) Second condition: $L_\rho(\mu)^*$ is o-continuous. This is true because we have the decomposition $L_\rho(\mu)^* = L_\rho(\mu)^*_i \overset{d}{\oplus} L_\rho(\mu)^*_s$ and: (i) $L_\rho(\mu)^*_i$ is o-continuous by hypothesis; (ii) $L_\rho(\mu)^*_s$ is o-continuous because it is the dual of the Grothendieck Banach lattice $L_\rho(\mu)/H(\mathcal{S}_1)$. So, $L_\rho(\mu)^*$ is o-continuous.

(iii) Third condition: if $\{z_n : n \geq 1\} \subset B(L_\rho(\mu)^*)^+$ is a pairwise disjoint sequence satisfying $z_n \overset{w^*}{\to} 0$, then $z_n \overset{w}{\to} 0$. Let us prove this condition. Consider the decomposition $z_n = z_{1n} + z_{2n}$ with $z_{1n} \in B(L_\rho(\mu)^*_i)^+$ and $z_{2n} \in B(L_\rho(\mu)^*_s)^+$ so that $\{z_{1n}, z_{2n} : n \geq 1\}$ are pairwise disjoint as elements of $L_\rho(\mu)^*$. Moreover, since $L_\rho(\mu)^*_i \subset L^0(\mu)$, each z_{1n} can be considered as a function of $L^0(\mu)^+$ such that $\mathrm{supp}(z_{1n}) \cap \mathrm{supp}(z_{1m}) = \emptyset$ if $n \neq m$.

Claim 1. $z_{1n} \overset{w^*}{\to} 0$.

Indeed, suppose that z_{1n} does not converge to 0 in the weak*-topology. Then there exists a vector $u \in B(L_\rho(\mu))$ and a positive number $0 < \epsilon \leq 1$ such that, by passing to a subsequence if necessary, we have $\langle z_{1n}, u\rangle > \epsilon,\ \forall n \geq 1$. Let $u_n = u \cdot \mathbb{1}_{\mathrm{supp}(z_{1n})},\ n \geq 1$. Notice that $u_n \in L_\rho(\mu)$, $1 \geq \|u\| \geq \|u_n\| \geq \langle z_{1n}, u_n\rangle = \langle z_{1n}, u\rangle > \epsilon$ for all $n \geq 1$ and $\langle z_{1n}, u_k\rangle = 0$, if $n \neq k$. Since maybe $\langle z_{2n}, u_n\rangle \neq 0$, we need to pass to another vector $v_n \in H(\mathcal{S}_1)$ such that $|v_n| \leq |u_n|$ and $\langle z_{2n}, v_n\rangle = 0$. Let us choose v_n. As $z_{1n}u_n \in L_1(\mu)$ and $+\infty > \int_\Omega |z_{1n}u_n| d\mu \geq \langle z_{1n}, u_n\rangle = \langle z_{1n}, u\rangle > \epsilon$, by the dominated convergence theorem there exist $0 \leq M_n < \infty$ and a finite subset $\Gamma_n \subset \Gamma$ such that, if

$$v_n := ((u_n \wedge M_n) \vee (-M_n)) \cdot \mathbb{1}_{\cup_{\gamma\in\Gamma_n}\Omega_\gamma},$$

then $v_n \in \mathcal{S}_0,\ |v_n| \leq |u_n|$, $\langle z_{2n}, v_n\rangle = 0$, $\|v_n\| \geq \langle z_n, v_n\rangle = \langle z_{1n}, v_n\rangle > \epsilon$ and $\langle z_k, v_n\rangle = \langle z_{1k}, v_n\rangle = 0$ if $k \neq n$. Define the operator $S : \ell_\infty \to L_\rho(\mu)$ as $S((t_n)_{n\geq 1}) = \sum_{n\geq 1} t_n v_n$ for every $(t_n)_{n\geq 1} \in \ell_\infty$. Observe that S is well defined because for every $(t_n)_{n\geq 1} \in \ell_\infty$ we have, on the one hand

$$\|\sum_{n\geq 1} t_n v_n\| \leq \sup\{|t_n| : n \geq 1\} \cdot \|\sum_{n\geq 1} v_n\| \leq$$
$$\leq \sup\{|t_n| : n \geq 1\}\|u\| \leq \sup\{|t_n| : n \geq 1\} = \|(t_n)_{n\geq 1}\|_{\ell_\infty},$$

and, on the other hand

$$\|\sum_{n\geq 1} t_n v_n\| \geq \sup\{|t_n| \cdot \|v_n\| : n \geq 1\} > \epsilon \sup\{|t_n| : n \geq 1\} = \epsilon\|(t_n)_{n\geq 1}\|_{\ell_\infty}.$$

So, S is an isomorphism between ℓ_∞ and $S(\ell_\infty)$ with $\|S\| \leq 1$. Define the operator $T : L_\rho(\mu) \to c_0$ as follows: for every $x \in L_\rho(\mu)$, we put $T(x) = ((\langle z_n, x\rangle)_{n\geq 1})$. As $z_n \overset{w^*}{\to} 0$, it is clear that T is a linear operator such that $\|T\| \leq 1$. Now we consider the operator $T \circ S : \ell_\infty \to c_0$ and prove the following fact.

Fact. The restriction $T \circ S\restriction c_0$ of $T \circ S$ to the canonical subspace c_0 of ℓ_∞ is a (natural) isomorphism between c_0 and the final space c_0. Moreover, $T \circ S(B(c_0)) \supset \epsilon B(c_0)$, and so, $T \circ S(B(\ell_\infty)) \supset \epsilon B(c_0)$.

Indeed, let $y := (y_1, y_2, \dots) \in \epsilon B(c_0)$. Since $\langle z_n, v_n\rangle = \langle z_{1n}, v_n\rangle > \epsilon$ and $|y_n| \leq \epsilon$, we can find $t_n \in [-1, 1]$, $n \geq 1$, such that $t_n \to 0$ and $t_n\langle z_n, v_n\rangle = y_n$. So, $t := ((t_n)_{n\geq 1}) \in B(c_0)$ and $T \circ S(t) = y$.

Thus, c_0 is a quotient of ℓ_∞, a contradiction which proves the Claim 1.

Claim 2. $(L_\rho(\mu)^*_i)^* = (L_\rho(\mu)^*_i)^*_i = L_\rho(\mu)$.

Indeed, since by hypothesis $L_\rho(\mu)^*_i$ is o-continuous we have $(L_\rho(\mu)^*_i)^* = (L_\rho(\mu)^*_i)^*_i$, that is, if $G \in (L_\rho(\mu)^*_i)^*$, there exists $g \in L_0(\mu)$ such that for every $H_f \in L_\rho(\mu)^*_i$, with $f \in L_0(\mu)$ representing the functional H_f, then $fg \in L_1(\mu)$ and $\langle G, H_f\rangle = \int_\Omega fg d\mu$. Let us prove that $(L_\rho(\mu)^*_i)^*_i = L_\rho(\mu)$. When μ is σ-finite, this fact holds true because $L_\rho(\mu)$ has the Fatou property and by [14, Th. 1, p. 470]. Let us consider the general case, that is, μ Γ-finite. First, clearly $L_\rho(\mu) \subset (L_\rho(\mu)^*_i)^* = (L_\rho(\mu)^*_i)^*_i$. Prove that $(L_\rho(\mu)^*_i)^*_i \subset L_\rho(\mu)$. It is enough to show that, if $G \in (L_\rho(\mu)^*_i)^*_i$ with $\|G\|_{(L_\rho(\mu)^*_i)^*_i} \leq 1$, $G \geq 0$, and $g \in L_0(\mu)^+$ represents G (that is, $fg \in L_1(\mu)$ and $\langle G, H_f\rangle = \int_\Omega fg d\mu$, $\forall H_f \in L_\rho(\mu)^*_i$), then $\rho(g) \leq 1$. Suppose that $\rho(g) > 1$. By the definition of Orlicz-type semimodular, there exists $A \in \Sigma$ with $\mu(A) < \infty$ such that $\rho(g\mathbb{1}_A) > 1$. Let μ_A be the restriction of μ to A. Then $L_\rho(\mu_A) := \{f\mathbb{1}_A : f \in L_\rho(\mu)\}$. Since μ_A is σ-finite and $L_\rho(\mu_A)$ has the Fatou property, then $(L_\rho(\mu_A)^*_i)^*_i = L_\rho(\mu_A)$ and the norms $\|\cdot\|_{(L_\rho(\mu_A)^*_i)^*_i}$ and $\|\cdot\|_{L_\rho(\mu_A)}$ coincide (see [14, Th. 1, p. 470]). Since $g\mathbb{1}_A \in (L_\rho(\mu_A)^*_i)^*_i = L_\rho(\mu_A)$ and $L_\rho(\mu_A) \subset L_\rho(\mu)$, then $g\mathbb{1}_A \in L_\rho(\mu)$. Moreover

$$\|g\mathbb{1}_A\|_{L_\rho(\mu)} = \|g\mathbb{1}_A\|_{L_\rho(\mu_A)} = \|g\mathbb{1}_A\|_{(L_\rho(\mu_A)^*_i)^*_i} \leq \|G\|_{(L_\rho(\mu)^*_i)^*_i} \leq 1.$$

Hence we get $\rho(g\mathbb{1}_A) \leq 1$, a contradiction, which proves that $\rho(g) \leq 1$ and so $g \in L_\rho(\mu)$.

Claim 3. $z_{1n} \xrightarrow{w} 0$.

Indeed, by Claim 2 we have $(L_\rho(\mu)^*_i)^* = (L_\rho(\mu)^*_i)^*_i = L_\rho(\mu)$. So, on $L_\rho(\mu)^*_i$ coincide the w^*-topology $\sigma(L_\rho(\mu)^*_i, L_\rho(\mu))$ and the w-topology $\sigma(L_\rho(\mu)^*_i, (L_\rho(\mu)^*_i)^*)$. Hence, we get $z_{1n} \xrightarrow{w} 0$ because $z_{1n} \xrightarrow{w^*} 0$ by Claim 1.

Claim 4. $z_{2n} \xrightarrow{w} 0$.

Indeed, as $z_{2n} = z_n - z_{1n}$, then $z_{2n} \xrightarrow{w^*} 0$, because $z_{1n} \xrightarrow{w^*} 0$ and $z_n \xrightarrow{w^*} 0$. Now we apply that $L_\rho(\mu)/H(\mathcal{S}_1)$ is Grothendieck and the fact that $L_\rho(\mu)^*_s = (L_\rho(\mu)/H(\mathcal{S}_1))^*$. So, we obtain $z_{2n} \xrightarrow{w} 0$.

Finally, from Claim 3 and Claim 4 we obtain $z_n \xrightarrow{w} 0$ and this completes the proof. □

4. The Grothendieck property for Orlicz-Lorentz spaces

Let us introduce the notion of Orlicz-Lorentz spaces. If (Ω, Σ, μ) is a complete Γ-finite measure space, for every $h \in M(\mu)$, the *distribution function* $\mu_h : [0, \infty) \to [0, \infty]$ *associated to* h is defined by

$$\mu_h(t) = \mu(\{w \in \Omega : |h(w)| > t\}),\ t \in [0, \infty),$$

and the *nonincreasing rearrangement function* $h^* : (0, \infty) \to [0, \infty]$ of h is defined by

$$h^*(t) = \inf\{\lambda > 0 : \mu_h(\lambda) \leq t\},\ \inf \emptyset = \infty.$$

Let $\varphi : \mathbb{R} \cup \{\pm\infty\} \to [0, +\infty]$ denote an Orlicz function, i.e. a convex function which is even, nondecreasing and left continuous for $x \geq 0$, $\varphi(0) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$ (see [1],[9]). Define $a(\varphi) = \sup\{t \geq 0 : \varphi(t) = 0\}$ and $\tau(\varphi) := \sup\{t \geq 0 : \varphi(t) < \infty\}$. The complementary function of φ is a new Orlicz function ψ defined for $u \geq 0$ as $\psi(u) = \sup\{tu - \varphi(t) : 0 \leq t < \infty\}$. The Orlicz function φ satisfies: (i) the Δ_2-condition at 0 (for short, $\varphi \in \Delta_2^0$) if $\varphi(t) > 0$ for $t > 0$

and $\limsup_{t\to 0}\frac{\varphi(2t)}{\varphi(t)} < \infty$; (ii) the Δ_2-condition at ∞ (for short, $\varphi \in \Delta_2^\infty$) if $\varphi(t) < \infty$ for every $0 \le t < \infty$ and $\limsup_{t\to\infty}\frac{\varphi(2t)}{\varphi(t)} < \infty$; (iii) the Δ_2-condition (for short, $\varphi \in \Delta_2$) if $\varphi \in \Delta_2^0$ and $\varphi \in \Delta_2^\infty$. A function $w : (0,\infty) \to (0,\infty)$ is said to be a *weight function* if it is nonincreasing, $\int_0^1 w(t)dt < \infty$ and $\int_0^\infty w(t)dt = \infty$.

Let φ be an Orlicz function and $w : (0,\infty) \to (0,\infty)$ a weight function. We define the mapping $I_{(\varphi,w)} : L_0(\mu) \cup M(\mu)^+ \to [0,\infty]$ associated to φ and w as

$$I_{(\varphi,w)}(f) = \int_0^\infty \varphi(f^*)wdt,\ f \in L_0(\mu).$$

Let us check that $I_{(\varphi,w)}$ is an Orlicz-type semimodular.

(0) Let $f \in M(\mu)^+$, $A := \{w \in \Omega : f(w) > a(\varphi)\}$ and $g := f\mathbb{1}_A$. Then it is easy to see that $\varphi(f^*) = \varphi(g^*)$ and so $I_{(\varphi,w)}(f) = I_{(\varphi,w)}(g)$.

(1) Clearly, $I_{(\varphi,w)}(0) = 0$, $I_{(\varphi,w)}(f) = I_{(\varphi,w)}(-f)$ and, if $I_{(\varphi,w)}(\lambda f) = 0$, $\forall\lambda > 0$, then $f = 0$. Moreover, $I_{(\varphi,w)}(f) = \infty$ as soon as $\infty = \mu(\{|f| > t\}) =: \mu_f(t)$ for some $t > a(\varphi)$.

(2) $I_{(\varphi,w)}$ is monotone because $0 \le g^* \le f^*$ and $0 \le \varphi(g^*) \le \varphi(f^*)$ whenever $|g| \le |f|$.

(3) $I_{(\varphi,w)}$ is left-continuous. Indeed, if $\{f_n : n \ge 1\} \subset M(\mu)^+$ is a sequence such that $0 \le f_n \uparrow f_0$, then $f_n^* \uparrow f_0^*$. Since φ is left-continuous, also $\varphi(f_n^*) \uparrow \varphi(f_0^*)$ and so by the monotone convergence theorem we have $I_{(\varphi,w)}(f_n) \uparrow I_{(\varphi,w)}(f_0)$.

(4) If $A \in \Sigma$ with $\mu(A) < \infty$ and $\lambda > 0$ satisfies $\varphi(\lambda) < \infty$, then $I_{(\varphi,w)}(\lambda\mathbb{1}_A) = \int_0^{\mu(A)} \varphi(\lambda)wdt < \infty$.

(5) $I_{(\varphi,w)}$ is 1-convex and satisfies $I_{(\varphi,w)}(f \vee g) \le I_{(\varphi,w)}(f) + I_{(\varphi,w)}(g)$, $\forall f, g \in L_0(\mu) \cup M(\mu)^+$. To prove these properties we proceed in several steps, namely:

Step 1. We suppose that f, g are real Σ-measurable step-functions, that is, $f := \sum_{i=1}^n a_i\mathbf{1}_{A_i}$ and $g := \sum_{i=1}^m b_i\mathbf{1}_{B_i}$ with $0 \le a_i, b_i$ and $A_i, B_p \in \Sigma$ with $A_i \cap A_j = \emptyset = B_p \cap B_q$ whenever $i \ne j, p \ne q$. In this case it is easy to see that $I_{(\varphi,w)}(\alpha f + \beta g) \le \alpha I_{(\varphi,w)}(f) + \beta I_{(\varphi,w)}(g)$, when $0 \le \alpha, \beta$, $\alpha + \beta = 1$, and $I_{(\varphi,w)}(f \vee g) \le I_{(\varphi,w)}(f) + I_{(\varphi,w)}(g)$.

Step 2. Let $f, g \in M(\mu)^+$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Find two sequences of Σ-measurable step-functions $0 \le f_n \le f_{n+1} \uparrow f$, $0 \le g_n \le g_{n+1} \uparrow g$. Then $0 \le \alpha f_n + \beta g_n \uparrow \alpha f + \beta g$ and $0 \le f_n \vee g_n \uparrow f \vee g$, whence $0 \le (\alpha f_n + \beta g_n)^* \uparrow (\alpha f + \beta g)^*$ and $0 \le (f_n \vee g_n)^* \uparrow (f \vee g)^*$. Thus, $I_{(\varphi,w)}(\alpha f_n + \beta g_n) \uparrow I_{(\varphi,w)}(\alpha f + \beta g)$ and $I_{(\varphi,w)}(f_n \vee g_n) \uparrow I_{(\varphi,w)}(f \vee g)$. By Step 1

$$I_{(\varphi,w)}(\alpha f_n + \beta g_n) \le \alpha I_{(\varphi,w)}(f_n) + \beta I_{(\varphi,w)}(g_n) \le \alpha I_{(\varphi,w)}(f) + \beta I_{(\varphi,w)}(g),$$

and

$$I_{(\varphi,w)}(f_n \vee g_n) \le I_{(\varphi,w)}(f_n) + I_{(\varphi,w)}(g_n) \le I_{(\varphi,w)}(f) + I_{(\varphi,w)}(g).$$

Therefore, we finally get

$$I_{(\varphi,w)}(\alpha f + \beta g) \le \alpha I_{(\varphi,w)}(f) + \beta I_{(\varphi,w)}(g) \text{ and } I_{(\varphi,w)}(f \vee g) \le I_{(\varphi,w)}(f) + I_{(\varphi,w)}(g).$$

(6) $I_{(\varphi,w)}$ is finitely determined. Indeed, let $f \in M(\mu)^+$ be such that $I_{(\varphi,w)}(f) > a \ge 0$. Put $A_0 := \{w \in \Omega : f(w) > a(\varphi)\}$ and $A_n := \{w \in \Omega : f(w) > a(\varphi) + \frac{1}{n}\}$, $\forall n \ge 1$. Then $f\mathbb{1}_{A_n} \uparrow f\mathbb{1}_{A_0}$ and so $I_{(\varphi,w)}(f\mathbb{1}_{A_n}) \uparrow I_{(\varphi,w)}(f\mathbb{1}_{A_0})$, because $I_{(\varphi,w)}$ is left-continuous. Since $I_{(\varphi,w)}(f\mathbb{1}_{A_0}) = I_{(\varphi,w)}(f)$ (by (0)), there exists $p \in \mathbb{N}$ such that $I_{(\varphi,w)}(f\mathbb{1}_{A_p}) > a$. If $\mu(A_p) < \infty$ we are done. Otherwise, $\mu(A_p) = \infty$ and we can choose a sequence $\{A_{pm} : m \ge 1\} \subset \Sigma$ such that $\mu(A_{pm}) < \infty$, $A_{pm} \subset A_{p,m+1} \subset A_p$, $m \ge 1$, and $\mu(A_{pm}) \uparrow \infty$. Then

$$I_{(\varphi,w)}(f\mathbb{1}_{A_{pm}}) \ge \int_0^{\mu(A_{pm})} \varphi(a(\varphi) + \tfrac{1}{p})wdt \to \infty \text{ for } m \to \infty.$$

So, there exists $q \in \mathbb{N}$ such that $I_{(\varphi,w)}(f\mathbb{1}_{A_{pq}}) > a$.

Thus, $I_{(\varphi,w)}$ is an Orlicz-type semimodular on (Ω, Σ, μ) and so we can apply the results of Sections 2 and 3 to the associated modular Banach space $(L_0(\mu))_{I_{(\varphi,w)}}$, which is called the *Orlicz-Lorentz space* and denoted by $\Lambda_{(\varphi,w)}(\mu)$. When $\varphi(t) = |t|$, then $\Lambda_{(\varphi,w)}(\mu)$ is the Lorentz space

$\Lambda_{(w)}(\mu)$. If $w(0^+) := \lim_{t\downarrow 0} w(t) < \infty$ and $\lim_{t\to\mu(\Omega)} w(t) > 0$, it is clear that $\Lambda_{(\varphi,w)}(\mu)$ is order-isomorphic to the Orlicz space $L_\varphi(\mu)$. Let $H\Lambda_{(\varphi,w)}(\mu)$ be the ideal $H\Lambda_{(\varphi,w)}(\mu) := \{f \in \Lambda_{(\varphi,w)}(\mu) : I_{(\varphi,w)}(\lambda f) < \infty,\ \forall\lambda > 0\}$.

Proposition 4.1. *Let (Ω,Σ,μ) be a complete Γ-finite measure space, φ be an Orlicz function and $w : (0,\infty) \to (0,\infty)$ a weight function. Then $\mathcal{S}_0$ is $I_{(\varphi,w)}$-dense in $\Lambda_{(\varphi,w)}(\mu)$. Moreover, if $\mu = \mu_a$ (that is, μ is purely atomic) and $\mathcal{S}_a$ is the subspace spanned by the functions of the form $\mathbb{1}_A$, A being an atom, then $\mathcal{S}_a$ is $I_{(\varphi,w)}$-dense in $\Lambda_{(\varphi,w)}(\mu)$.*

Proof. (1) Let us prove that $\mathcal{S}_0$ is $I_{(\varphi,w)}$-dense in $\Lambda_{(\varphi,w)}(\mu)$. Pick $f \in \Lambda_{(\varphi,w)}(\mu)$ such that $I_{(\varphi,w)}(f) < \infty$ and let $0 < \epsilon < \infty$. We have to find a function $g \in H(\mathcal{S}_0)$ such that $I_{(\varphi,w)}(f-g) \le \epsilon$. If $0 < a(\varphi) = \tau(\varphi)$, then $f^* \le a(\varphi)$ and so $\varphi(f^*) = 0$ and actually $I_{(\varphi,w)}(f) = 0$. Thus taking $g := 0 \in \mathcal{S}_0$ we have $I_{(\varphi,w)}(f-g) = 0 \le \epsilon$.

Assume now that $a(\varphi) < \tau(\varphi)$. Observe that the measurable set $A := \{w \in \Omega : |f(w)| > a(\varphi)\}$ is σ-finite, we say, $A := \cup_{n\ge 1}\{w \in \Omega : |f(w)| > a(\varphi) + \frac{1}{n}\}$, with $\mu(\{w \in \Omega : |f(w)| > a(\varphi)+\frac{1}{n}\}) < \infty$. Define

$$f_n := f \cdot \mathbb{1}_{\{|f|>a(\varphi)+\frac{1}{n}\}} \cdot \mathbb{1}_{\{|f|\le n\}},\ n \ge 1.$$

Observe that $f_n \in \mathcal{S}_0,\ n \ge 1$. If $g_n := f - f_n,\ n \ge 1$, it is clear that $\{|g_n| > a(\varphi) + \frac{1}{k}\} \subset \{|f| > a(\varphi) + \frac{1}{k}\},\ \forall n,k \ge 1$, and $\{|g_n| > a(\varphi) + \frac{1}{k}\} \downarrow \emptyset$ when $n \to \infty$, whence we get $,\forall k \ge 1,\ \mu_{g_n}(a(\varphi) + \frac{1}{k})\downarrow 0$ when $n \to \infty$. Thus $g_n^* \downarrow g_0$ when $n \to \infty$ for some measurable function $g_0 : [0,\infty) \to [0,\infty]$ such that $0 \le g_0 \le a(\varphi)$ and this implies $\varphi(g_n^*) \underset{n\to\infty}{\downarrow} 0$, because $a(\varphi) < \tau(\varphi)$. Therefore, as $0 \le g_n^* \le f^*$ and $I_{(\varphi,w)}(f) < \infty$, by the dominated convergence theorem we get $I_{(\varphi,w)}(f - f_n) = I_{(\varphi,w)}(g_n) \underset{n\to\infty}{\downarrow} \int_0^\infty \varphi(g_0)w dt = 0$. So, there exists $n_0 \in \mathbb{N}$ such that $I_{(\varphi,w)}(f - f_n) \le \epsilon,\ \forall n \ge n_0$, and this proves that $\mathcal{S}_0$ is $I_{(\varphi,w)}$-dense in $\Lambda_{(\varphi,w)}(\mu)$ when $a(\varphi) < \tau(\varphi)$.

(2) Assume that $\mu = \mu_a$ and pick $f \in \Lambda_{(\varphi,w)}(\mu)$ such that $I_{(\varphi,w)}(f) < \infty$ and let $0 < \epsilon < \infty$. We have to find a function $g \in \mathcal{S}_a$ such that $I_{(\varphi,w)}(f - g) \le \epsilon$. If $0 < a(\varphi) = \tau(\varphi)$, we can pick $g = 0 \in \mathcal{S}_a$ as in the part (1).

Assume that $a(\varphi) < \tau(\varphi)$. As the measurable set $A := \{w \in \Omega : |f(w)| > a(\varphi)\}$ is σ-finite, then $A = \uplus_{n\ge 1} A_n$ where $\{A_n : n \ge 1\}$ is a disjoint sequence of atoms with $\mu(A_n) < \infty$. Define

$$f_n := f \cdot \mathbb{1}_{\{|f|>a(\varphi)+\frac{1}{n}\}} \cdot \mathbb{1}_{\cup_{i=1}^n A_i},\ n \ge 1.$$

Observe that $f_n \in \mathcal{S}_a,\ \forall n \ge 1$. As in the part (1), there exists $n_0 \in \mathbb{N}$ such that $I_{(\varphi,w)}(f - f_n) \le \epsilon,\ \forall n \ge n_0$, and this proves that $\mathcal{S}_a$ is $I_{(\varphi,w)}$-dense in $\Lambda_{(\varphi,w)}(\mu)$ when $a(\varphi) < \tau(\varphi)$. ☐

Proposition 4.2. *Let (Ω,Σ,μ) be a complete Γ-finite measure space, φ be an Orlicz function and $w : (0,\infty) \to (0,\infty)$ a weight function. Then*

(A) If φ is finite we have $H\Lambda_{(\varphi,w)}(\mu) = H(\mathcal{S}_0) = (\Lambda_{(\varphi,w)}(\mu))^a$.

(B) If $\tau(\varphi) < \infty$ and $\Omega = \Omega_d \uplus \Omega_a,\ \mu = \mu_d + \mu_a$, then $H(\mathcal{S}_a) = \overline{\mathcal{S}_a} = (\Lambda_{(\varphi,w)}(\mu))^a$, where $\mathcal{S}_a$ is the subspace spanned by the functions of the form $\mathbb{1}_A$, A being an atom.

Proof. (A) Let us see that $H\Lambda_{(\varphi,w)}(\mu) = H(\mathcal{S}_0) = (\Lambda_{(\varphi,w)}(\mu))^a$ when φ is finite.

(i) First, $H\Lambda_{(\varphi,w)}(\mu) \subset H(\mathcal{S}_0)$ by the definitions of these subspaces.

(ii) As $(\Lambda_{(\varphi,w)}(\mu))^a$ is a closed ideal and $H(\mathcal{S}_0) = \overline{\mathcal{S}_0}$, in order to prove that $H(\mathcal{S}_0) \subset (\Lambda_{(\varphi,w)}(\mu))^a$, it is enough to check that $\mathbb{1}_A \in (\Lambda_{(\varphi,w)}(\mu))^a$ whenever $A \in \Sigma$ and $\mu(A) < \infty$. Since $\Lambda_{(\varphi,w)}(\mu)$ is σ-o-complete, it is enough to show that, if $0 \le f_n \le f_{n-1} \le \mathbb{1}_A$ is a sequence of elements of $\Lambda_{(\varphi,w)}(\mu)$ such that $f_n \downarrow 0$ in order, then $\|f_n\| \downarrow 0$ when $n \to \infty$. As $\mu(A) < \infty$, then $f_n^* \downarrow 0$. So, let $\lambda > 0$ and observe that $I_{(\varphi,w)}(\mathbb{1}_A/\lambda) < \infty$ (because φ is finite and $\mu(A) < \infty$) and $\varphi((\mathbb{1}_A/\lambda)^*) \ge \varphi((f_n/\lambda)^*) \downarrow 0$. Thus, we get $I_{(\varphi,w)}(f_n/\lambda) \downarrow 0$ by the dominated convergence theorem and so $\|f_n\| \downarrow 0$ when $n \to \infty$.

(iii) Finally let us see that $(\Lambda_{(\varphi,w)}(\mu))^a \subset H\Lambda_{(\varphi,w)}(\mu)$. So, pick $f \in \Lambda_{(\varphi,w)}(\mu) \setminus H\Lambda_{(\varphi,w)}(\mu)$ and prove that $f \notin (\Lambda_{(\varphi,w)}(\mu))^a$. Without loss of generality, assume that $0 \le f$ and $I_{(\varphi,w)}(f) = \infty$.

For each $(F,n) \in \mathcal{F} \times \mathbb{N}$ ($\mathcal{F}$ = finite subsets of Γ) define

$$f_{(F,n)} := (f \cdot \mathbb{1}_{\cup_{\gamma \in F} \Omega_\gamma}) \wedge n.$$

Observe that $f_{(F,n)} \in H\Lambda_{(\varphi,w)}(\mu)$ and $\{f - f_{(F,n)} : (F,n) \in \mathcal{F} \times \mathbb{N}\}$ is a downward directed set such that $f \geq f - f_{(F,n)} \downarrow 0$. Moreover

$$+\infty = I_{(\varphi,w)}(f) = I_{(\varphi,w)}((f - f_{(F,n)}) + f_{(F,n)}) =$$
$$= I_{(\varphi,w)}(\tfrac{1}{2}2(f - f_{(F,n)}) + \tfrac{1}{2}2f_{(F,n)}) \leq \tfrac{1}{2}I_{(\varphi,w)}(2(f - f_{(F,n)})) + \tfrac{1}{2}I_{(\varphi,w)}(2f_{(F,n)}),$$

whence we get $I_{(\varphi,w)}(2(f - f_{(F,n)})) = +\infty$ and so $\|f - f_{(F,n)}\| \geq 1/2$ for every $(F,n) \in \mathcal{F} \times \mathbb{N}$. Thus f is not o-continuous.

(B) First, we know that $H(\mathcal{S}_a) = \overline{\mathcal{S}_a}$. Since clearly $\mathcal{S}_a \subset (\Lambda_{(\varphi,w)}(\mu))^a$ we get $\overline{\mathcal{S}_a} \subset (\Lambda_{(\varphi,w)}(\mu))^a$ because $(\Lambda_{(\varphi,w)}(\mu))^a$ is a closed ideal.

Claim. If $f \in (\Lambda_{(\varphi,w)}(\mu))^a$, then $\text{supp}(f) \subset \Omega_a$.

Indeed, it is enough to prove that if $A \in \Sigma$ with $A \subset \Omega_d$ and $\mu(A) > 0$, then $\mathbb{1}_A \notin (\Lambda_{(\varphi,w)}(\mu))^a$. Since μ_d is diffuse and $\mu_d(A) > 0$, without loss of generality we can suppose that there exists a sequence $\{A_n : n \geq 1\} \subset \Sigma$ such that $A_n \uparrow A$ and $\mu(A \setminus A_n) > 0$, $n \geq 1$. Then $\mathbb{1}_{A \setminus A_n} \downarrow 0$ and also $I_{(\varphi,w)}((\tau(\varphi) + \epsilon) \cdot \mathbb{1}_{A \setminus A_n}) = +\infty$, $\forall \epsilon > 0$. Thus $\|\mathbb{1}_{A \setminus A_n}\| \geq \frac{1}{\tau(\varphi)} > 0$ and this yields $\mathbb{1}_A \notin \Lambda_{(\varphi,w)}(\mu)^a$.

Now let $f \in \Lambda_{(\varphi,w)}(\mu)^a$ and prove that $f \in \overline{\mathcal{S}_a}$. By the Claim we have $\text{supp}(f) \subset \Omega_a$. Let $\{A_i : i \in I\}$ be the family of atoms of μ and $f_i := f \restriction A_i$. Then $|f - \sum_{i \in J} f_i \cdot \mathbb{1}_{A_i}| \downarrow 0$ when $J \subset I$ is a finite subset. Thus $\|f - \sum_{i \in J} f_i \cdot \mathbb{1}_{A_i}\| \downarrow 0$ when $J \subset I$ is a finite subset, because f is o-continuous. Since $\sum_{i \in J} f_i \cdot \mathbb{1}_{A_i} \in \mathcal{S}_a$ for $J \subset I$ a finite subset, we conclude that $f \in \overline{\mathcal{S}_a}$. □

Proposition 4.3. *Let (Ω, Σ, μ) be a complete Γ-finite measure space, φ an Orlicz function and $w : (0,\infty) \to (0,\infty)$ a weight function. Then*

(A) If $\mathcal{S} \subset \Lambda_{(\varphi,w)}(\mu)$ is an ideal such that $H(\mathcal{S})$ is $I_{(\varphi,w)}$-dense in $\Lambda_{(\varphi,w)}(\mu)$, then $\Lambda_{(\varphi,w)}(\mu)/H(\mathcal{S})$ is a Grothendieck M-space. In particular, $\Lambda_{(\varphi,w)}(\mu)/H(\mathcal{S}_0)$ is a Grothendieck M-space

(B) If either φ is a finite Orlicz function or $\mu = \mu_a$, the following statements are equivalent:

(1) $\Lambda_{(\varphi,w)}(\mu)$ is Grothendieck ; (2) $(\Lambda_{(\varphi,w)}(\mu))^*_i$ is o-continuous.

Proof. (A) This follows from Proposition 2.8 and Proposition 4.1.

(B) This follows from Proposition 3.1, Proposition 4.1 and Proposition 4.2. □

Looking at Proposition 4.3, it is clear that, in order to see if $\Lambda_{(\varphi,w)}(\mu)$ is Grothendieck, the key is to determine who is $(\Lambda_{(\varphi,w)}(\mu))^*_i$ and when this space is o-continuous. If $\varphi(t) = |t|$ (or more generally, φ is equivalent to $g(t) = t$) we have the following result.

Proposition 4.4. *Let (Ω, Σ, μ) be a Γ-finite measure space, φ an Orlicz function equivalent to $g(t) = t$ and $w : (0,\infty) \to (0,\infty)$ a weight function. Then the space $\Lambda_{(\varphi,w)}(\mu)$, which is isomorphic to the Lorentz space $\Lambda_w(\mu)$, is not Grothendieck, provided it is infinite-dimensional.*

Proof. This holds true because it is well known (see [13, pg. 177], [3, Th. 5.1]) that every infinite dimensional Lorentz space $\Lambda_w(\mu)$ contains a complemented copy of ℓ_1. As a quotient of a Grothendieck spaces is also Grothendieck and ℓ_1 is not, we conclude the statement. □

It is well known that the Köthe-dual $(\Lambda_w(\mu))^*_i = M_S$ where $(M_S, \|\cdot\|_S)$ is the Marcinkiewicz space, which is defined as follows:

$$M_S := \{f \in L_0(\mu) : \|f\|_S := \sup_{t \in (0,\infty)} \frac{\int_0^t f^*(s)ds}{S(t)} < \infty\}, \tag{4.1}$$

$S(t)$ being $S(t) = \int_0^t w(s) \cdot ds$.

Corollary 4.5. *Let (Ω, Σ, μ) be a Γ-finite measure space and $w : (0, \infty) \to (0, \infty)$ a weight function.*

(1) If φ is an Orlicz function equivalent to $g(t) = t$, then the Köthe-dual space $(\Lambda_{(\varphi,w)}(\mu))^*_i$ is not o-continuous, provided $\Lambda_{(\varphi,w)}(\mu)$ is infinite-dimensional.

(2) The Marcinkiewicz space $(M_S, \|\cdot\|_S)$ (defined in (4.1)) is not o-continuous, provided it is infinite-dimensional.

Proof. □

Let (Ω, Σ, μ) be a complete Γ-finite measure space, φ be an Orlicz function and $w : (0, \infty) \to (0, \infty)$ a weight function. Define the functional $J_{(\varphi,w)} : L_0(\mu) \to [0, \infty]$ so that

$$J_{(\varphi,w)}(f) = \int_0^\infty \varphi(f^*/w) w dt, \ \forall f \in L_0(\mu),$$

and let

$$M_{(\varphi,w)}(\mu) := \{f \in L_0(\mu) : J_{(\varphi,w)}(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

and

$$HM_{(\varphi,w)}(\mu) := \{f \in L_0(\mu) : J_{(\varphi,w)}(\lambda f) < \infty \text{ for every } \lambda > 0\}.$$

Let us remark the following observations

(O1) The weight w is said to be *regular* when $S(2t) \geq (1 + a)S(t), \ t > 0$, for some $0 < a \leq 1$, where $S(t) = \int_0^t w(s)ds$. Thus $S(2t) - S(t) \geq aS(t) \to \infty$ when $t \to \infty$. It is very easy to see the equivalence of the following statements: (i) w is regular; (ii) $S(t) \leq btw(t), \ t \geq 0$, with $1 \leq b < \infty$; (iii) $w(t/2) \leq Cw(t), t \geq 0$, with $2 \leq C < \infty$.

(O2) If w is a regular weight, we say, $w(t/2) \leq Cw(t), t \geq 0$, with $2 \leq C < \infty$, then $J_{(\varphi,w)}(f + g) \leq J_{(\varphi,w)}(2Cf) + J_{(\varphi,w)}(2Cg)$ for every $f, g \in L_0(\mu)$. Indeed, since $(f+g)^*(t) \leq f^*(t/2)+g^*(t/2)$ and $\varphi(\frac{a}{w(t)})w(t) \leq \varphi(\frac{Ca}{w(t/2)})w(t/2)$ for every $a \geq 0$ and $t > 0$, we have

$$\begin{aligned} J_{(\varphi,w)}(f+g) &= \int_0^\infty \varphi(\frac{(f+g)^*(t)}{w(t)})w(t)dt \leq \int_0^\infty \varphi(\frac{\frac{1}{2}2f^*(t/2) + \frac{1}{2}2g^*(g/2)}{w(t)})w(t)dt \leq \\ &\leq \frac{1}{2}\int_0^\infty \varphi(\frac{2f^*(t/2)}{w(t)})w(t)dt + \frac{1}{2}\int_0^\infty \varphi(\frac{2g^*(t/2)}{w(t)})w(t)dt \leq \\ &\leq \int_0^\infty \varphi(\frac{2Cf^*(t/2)}{w(t/2)})w(t/2)d(t/2) + \int_0^\infty \varphi(\frac{2Cg^*(t/2)}{w(t/2)})w(t/2)d(t/2) = \\ &= J_{(\varphi,w)}(2Cf) + J_{(\varphi,w)}(2Cg). \end{aligned}$$

Observe that this fact implies that $M_{(\varphi,w)}(\mu)$ is a linear subspace of $L_0(\mu)$. If the weight w is not regular, the subset $M_{(\varphi,w)}(\mu)$ can be not linear.

(O3) Suppose that either $\varphi(t) = t$ or φ is an Orlicz N-function (see [9]) and w is a regular weight. Under these conditions Hudzik, Kaminska and Mastylo (see [7, Th. 2], see [9]) proved that the Köthe dual $(\Lambda_{(\varphi,w)}(\mu))^*_i$ of $\Lambda_{(\varphi,w)}(\mu)$ coincides, as a set, with the subspace $M_{(\psi,w)}(\mu)$ of $L_0(\mu)$, ψ being the Orlicz function complementary of φ. Moreover, if we define the homogeneous functional

$$|||f||| := \inf\{\lambda > 0 : J_{(\psi,w)}(\tfrac{f}{\lambda}) \leq 1\}, \ f \in M_{(\psi,w)}(\mu),$$

then $|||\cdot|||$ is a quasinorm equivalent to the dual norm of $(\Lambda_{(\varphi,w)}(\mu))^*_i$ such that by a calculus like in (O2) we have

$$\forall f, g \in M_{(\psi,w)}(\mu), \ |||f + g||| \leq 2C(|||f||| + |||g|||).$$

So, in what follows we restrict ourself to the case such that φ is an Orlicz N-function and w is a regular weight.

Lemma 4.6. *Let (Ω, Σ, μ) be a complete Γ-finite measure space, φ an Orlicz function and w a weight. Then*

(1) If $f \in L_0(\mu)$ satisfies $J_{(\varphi,w)}(f) < \infty$, then $\mu(\{|f| > t + a(\varphi)\}) < \infty, \ \forall t > 0$.

(2) If w is a regular weight, there exists $1 \le K < \infty$ such that $\frac{w(n)}{w(n+1)} \le K,\ \forall n \ge 1$.

(3) If φ is an Orlicz N-function and w is a regular weight, then $HM_{(\psi,w)}(\mu) = (M_{(\psi,w)}(\mu))^a$, ψ being the Orlicz function complementary of φ.

Proof. (1) If $\mu(\{|f| > t_0 + a(\varphi)\}) = \infty$ for some $t_0 > 0$, then $f^*(u) \ge t_0 + a(\varphi),\ \forall u \ge 0$, whence $J_{(\varphi,w)}(f) = \infty$, a contradiction.

(2) As w is regular, then $S(2t) \ge (1+a)S(t)$ for some $0 < a \le 1$, where $S(t) = \int_0^t w(s)ds$. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{S(n)} \int_n^{n+1} w(s)ds < \frac{a}{2},\ \forall n \ge n_0$. Suppose that the statement fails. Then we can pick $m \ge n_0$ such that $\frac{w(m+1)}{w(m)} < \frac{a}{2}$ and so for every $s \in (m+1, 2m]$, we have

$$\frac{w(s)}{w(s-m-1)} \le \frac{w(s)}{w(m)} \le \frac{w(m+1)}{w(m)} < \frac{a}{2}.$$

Thus

$$S(2m) - S(m) = \int_m^{2m} w(s)ds = \int_m^{m+1} w(s)ds + \int_{m+1}^{2m} w(s)ds < \tfrac{a}{2}S(m) + \int_{m+1}^{2m} \tfrac{a}{2} w(s-m-1)ds =$$

$$= \tfrac{a}{2}S(m) + \int_0^{m-1} \tfrac{a}{2} w(s)ds \le \tfrac{a}{2}S(m) + \tfrac{a}{2}S(m) = aS(m).$$

Therefore $S(2m) < (1+a)S(m)$, a contradiction.

(3) First, we show that $HM_{(\psi,w)}(\mu) \subset (M_{(\psi,w)}(\mu))^a$. Let $0 \le f \in HM_{(\psi,w)}(\mu)$ and choose a sequence $0 \le f_{n+1} \le f_n \le f$ of measurable functions such that $f_n \downarrow 0$. Then $\{f_n > t\} \subset \{f > t\}$ and $\{f_n > t\} \downarrow \emptyset$ for every $t > 0$. Since ψ is an N-function, then $a(\psi) = 0$. Thus $\mu(\{f > t\}) < \infty,\ \forall t > 0$, by (1) and we get $\mu(\{f_n > t\}) \downarrow 0,\ \forall t > 0$, whence $f_n^* \downarrow 0$. Let $\lambda > 0$ and observe that $\int_0^\infty \psi(\frac{f^*}{\lambda w})wdt < \infty$ because $f \in HM_{(\psi,w)}(\mu)$. Thus, since $\psi(\frac{f^*}{\lambda w})w \ge \psi(\frac{f_n^*}{\lambda w})w \downarrow 0$, by the dominated convergence theorem we get $J_{(\psi,w)}(f_n/\lambda) \downarrow 0$ and so $|||f_n||| \downarrow 0$. As f is σ-complete, this proves that f is o-continuous.

Let us prove the converse inclusion, that is, $(M_{(\psi,w)}(\mu))^a \subset HM_{(\psi,w)}(\mu)$. Pick $f \in M_{(\psi,w)}(\mu) \setminus HM_{(\psi,w)}(\mu)$ and prove that $f \notin (M_{(\psi,w)}(\mu))^a$. Without loss of generality, assume that $0 \le f$ and $J_{(\psi,w)}(f) = \infty$. For each $(F,n) \in \mathcal{F} \times \mathbb{N}$ ($\mathcal{F}$ = finite subsets of Γ) define

$$f_{(F,n)} := (f \cdot \mathbb{1}_{\cup_{\gamma \in F}\Omega_\gamma}) \wedge n.$$

Observe that $f_{(F,n)} \in HM_{(\psi,w)}(\mu)$ and $\{f - f_{(F,n)} : (F,n) \in \mathcal{F} \times \mathbb{N}\}$ is a downward directed set such that $f \ge f - f_{(F,n)} \downarrow 0$. Moreover, if the regular weight w satisfies $w(t/2) \le Cw(t)$ with $2 \le C < \infty$, then by (O2)

$$+\infty = J_{(\psi,w)}(f) = J_{(\psi,w)}((f - f_{(F,n)}) + f_{(F,n)}) \le J_{(\psi,w)}(2C(f - f_{(F,n)})) + J_{(\psi,w)}(2Cf_{(F,n)}).$$

Since $J_{(\psi,w)}(2Cf_{(F,n)}) < \infty$, we get $J_{(\psi,w)}(2C(f - f_{(F,n)})) = +\infty$ and so $|||f - f_{(F,n)}||| \ge 1/(2C)$, for every $(F,n) \in \mathcal{F} \times \mathbb{N}$. Thus f is not o-continuous. □

Lemma 4.7. *Let (Ω, Σ, μ) be a complete Γ-finite measure space, ψ an Orlicz N-function and w a regular weight. Then*

(1) If μ is atomless and $0 < \mu(\Omega) < \infty$, then $M_{(\psi,w)}(\mu)$ is o-continuous if and only if $\psi \in \Delta_2^\infty$.

(2) If μ is atomless and $\mu(\Omega) = \infty$, then $M_{(\psi,w)}(\mu)$ is o-continuous if and only if $\psi \in \Delta_2$.

(3) If (Ω, Σ, μ) is the counting measure space on an infinite set I, then $M_{(\psi,w)}(\mu)$ is o-continuous if and only if $\psi \in \Delta_2^0$.

Proof. First, observe that we ask for ψ and w to be an Orlicz N-function and a regular weight, respectively, because only under these conditions we know that $M_{(\psi,w)}(\mu)$ is, as a set, the Köthe dual $(\Lambda_{(\varphi,w)}(\mu))_i^*$ of $\Lambda_{(\varphi,w)}(\mu)$ and the quasinorm $|||\cdot|||$ is equivalent to the norm of $(\Lambda_{(\varphi,w)}(\mu))_i^*$ (see [7]), φ being the Orlicz N-function complementary of ψ.

(1) Suppose that $\psi \in \Delta_2^\infty$ and pick some $0 \le f \in M_{(\psi,w)}(\mu)$. We have to prove that f is o-continuous. Assume that $J_{(\psi,w)}(f) < \infty$ and let $\lambda > 1$. Since $\psi \in \Delta_2^\infty$, there exists $0 < K_\lambda < \infty$

such that $\psi(\lambda t) \leq K_\lambda \psi(t),\ \forall t > 1$. Thus, as $\mu(\Omega) < \infty$, we have

$$J_{(\psi,w)}(\lambda f) = \int_0^{\mu(\Omega)} \psi(\frac{\lambda f^*}{w}) w dt = \int_{\{\frac{f^*}{w} \leq 1\}} \psi(\frac{\lambda f^*}{w}) w dt + \int_{\{\frac{f^*}{w} > 1\}} \psi(\frac{\lambda f^*}{w}) w dt \leq$$

$$\leq \psi(\lambda) \int_0^{\mu(\Omega)} w dt + K_\lambda \int_0^{\mu(\Omega)} \psi(\frac{f^*}{w}) w dt < \infty.$$

Since $\lambda > 1$ is arbitrary we get $f \in HM_{(\psi,w)}(\mu)$. This proves that f is o-continuous because $HM_{(\psi,w)}(\mu) = (M_{(\psi,w)}(\mu))^a$ by Lemma 4.6.

Now we prove the converse statement. Without loss of generality, we assume that $\Omega = [0,1]$ and that μ is the Lebesgue measure on $[0,1]$. We suppose that $\psi \notin \Delta_2^\infty$ and we construct in $M_{(\psi,w)}(\mu)$ an order-isomorphic copy of ℓ_∞. Let $2 \leq C < \infty$ be such that $w(t) \leq Cw(2t)$ for every $t > 0$. Since $\psi \notin \Delta_2^\infty$, there exists a sequence $\{u_k : k \geq 1\} \subset (0,\infty)$ such that $0 < u_k \uparrow \infty$, $\psi((\frac{C+1}{C})u_k) > 2^{k+1}\psi(u_k)$, $\frac{2}{3\psi(u_1)} \leq \int_0^1 w dt$ and $\frac{1}{2^{k+1}\psi(u_{k+1})} \leq \frac{1}{4}\frac{1}{2^k\psi(u_k)},\ k \geq 1$. Then

$$\sum_{k\geq 1} \tfrac{1}{2^k\psi(u_k)} \leq \tfrac{1}{2\psi(u_1)}(1 + 4^{-1} + 4^{-2} + \cdots) = \tfrac{2}{3\psi(u_1)} \leq \int_0^1 w dt.$$

Let $\{r_k : k \geq 1\} \subset (0,1]$, $r_k \downarrow 0$, be such that $\sum_{k\geq 1} \frac{1}{2^k\psi(u_k)} = \int_0^{r_1} w dt$ and $\frac{1}{2^k\psi(u_k)} = \int_{r_{k+1}}^{r_k} w dt,\ k \geq 1$. Observe that for $n \geq 1$

$$w(r_{n+1})r_{n+1} \leq \int_0^{r_{n+1}} w dt = \sum_{k>n} \tfrac{1}{2^k\psi(u_k)} \leq \tfrac{1}{2^n\psi(u_n)}(4^{-1} + 4^{-2} + \cdots) =$$

$$= \tfrac{1}{3}\frac{1}{2^n\psi(u_n)} = \tfrac{1}{3}\int_{r_{n+1}}^{r_n} w dt \leq \tfrac{1}{3}(r_n - r_{n+1})w(r_{n+1}),$$

whence $4r_{n+1} \leq r_n$. Denote $p_k := r_k - r_{k+1}$ and define $f_k := u_k w \mathbb{1}_{(r_{k+1},r_k]},\ k \geq 1$, and $f := \sum_{k\geq 1} f_k$. Then

(a) Clearly the functions f_k have disjoint supports, $f^* = f$ and

$$J_{(\psi,w)}(f) = \int_0^{r_1} \psi(\frac{f^*}{w}) w dt = \sum_{k\geq 1} \int_{r_{k+1}}^{r_k} \psi(u_k) w dt = \sum_{k\geq 1} 2^{-k} = 1.$$

Thus $|||f_k||| \leq |||f||| \leq 1$.

(b) Let us compute $J_{(\psi,w)}((C+1)f_k)$. Observe that: (I) $f_k^*(t) = u_k w(t + r_{k+1})$, if $0 < t \leq p_k$, and $f_k^*(t) = 0$ if $t > p_k$; (II) if $t \geq r_{k+1}$, then $2t \geq t + r_{k+1}$ and so $w(t) \leq Cw(2t) \leq Cw(t + r_{k+1})$. Thus

$$J_{(\psi,w)}((C+1)f_k) = \int_0^{p_k} \psi\big((C+1)\frac{f_k^*}{w}\big) w dt \geq \int_{r_{k+1}}^{p_k} \psi\big((C+1)\frac{u_k w(t + r_{k+1})}{Cw(t + r_{k+1})}\big) w dt =$$

$$= \int_{r_{k+1}}^{p_k} \psi(\frac{C+1}{C}u_k) w dt > 2^{(k+1)} \int_{r_{k+1}}^{p_k} \psi(u_k) w dt.$$

Since w is decreasing and $p_k \geq r_{k+1} + \frac{p_k}{2}$ (that is, p_k is equal or greater than the middle point $r_{k+1} + \frac{p_k}{2}$ of the interval $[r_{k+1}, r_k]$) then

$$\int_{r_{k+1}}^{p_k} \psi(u_k) w dt \geq \tfrac{1}{2}\int_{r_{k+1}}^{r_k} \psi(u_k) w dt = 2^{-k-1}.$$

Thus, finally we have $J_{(\psi,w)}((C+1)f_k) > 1$ and so $|||f_k||| \geq (C+1)^{-1},\ k \geq 1$. Taking into account the equivalence between the quasinorm $|||\cdot|||$ and the dual norm of $(\Lambda_{(\varphi,w)}(\mu))_i^*$, it is clear that the mapping $T : \ell_\infty \to M_{(\psi,w)}(\mu)$ such that $T((a_k)_{k\geq 1}) = \sum_{k\geq 1} a_k f_k,\ (a_k)_{k\geq 1} \in \ell_\infty$, is an order-isomorphism between ℓ_∞ and $T(\ell_\infty)$. Therefore $M_{(\psi,w)}(\mu)$ contains an order-isomorphic copy of ℓ_∞, a contradiction, because $M_{(\psi,w)}(\mu)$ is o-continuous by hypothesis.

(2) Suppose that $\psi \in \Delta_2$ and prove that $M_{(\psi,w)}(\mu)$ is o-continuous. Let $f \in M_{(\psi,w)}(\mu)$ and assume that $J_{(\psi,w)}(f) < \infty$. Then for every $\lambda > 0$ we have $J_{(\psi,w)}(\lambda f) < \infty$ because $\psi \in \Delta_2$, that is, $f \in HM_{(\psi,w)}(\mu)$. Now apply that $HM_{(\psi,w)}(\mu) = (M_{(\psi,w)}(\mu))^a$ by Lemma 4.6.

Suppose that $M_{(\psi,w)}(\mu)$ is o-continuous and prove that $\psi \in \Delta_2$. First, $\psi \in \Delta_2^\infty$ by the part (1). Assume that $\psi \notin \Delta_2^0$. We have to construct in $M_{(\psi,w)}(\mu)$ an order-isomorphic copy of ℓ_∞ by using an argument similar to the one of the part (1). Without loss of generality, we suppose that $\Omega = [0,\infty)$ and that μ is the Lebesgue measure. Let $2 \leq C < \infty$ be such that $w(t) \leq Cw(2t)$ for every $t > 0$. Since $\psi \notin \Delta_2^0$, there exists a sequence $\{u_k : k \geq 1\} \subset (0,\infty)$ and a two sequences of integers $\{r_k : k \geq 1\}$ and $\{p_k : k \geq 1\}$ such that:

(i) $0 < u_{k+1} < u_k$ and $\psi((\frac{C+1}{C})u_k) > 2^{k+2}\psi(u_k),\ k \geq 1$.

(ii) $0 = r_1 < r_2 < r_3 < \cdots$ with $p_k = r_{k+1} - r_k$, $p_k \geq 2r_k,\ k \geq 2$, and $2^{-k-1} \leq \int_{r_k}^{r_{k+1}} \psi(u_k)w(t)dt \leq 2^{-k},\ k \geq 1$.

To do this construction we proceed step by step:

Step 1. Choose $u_1 > 0$ such that $\psi((\frac{C+1}{C})u_1) > 2^{1+2}\psi(u_1)$ and $\int_0^1 \psi(u_1)wdt < 2^{-1-1}$. So, there exists $r_2 \in \mathbb{N},\ r_2 \geq 2$, such that $2^{-1-1} \leq \int_0^{r_2} \psi(u_1)w(t)dt \leq 2^{-1}$.

Step 2. Choose $0 < u_2 < u_1$ such that $\psi((\frac{C+1}{C})u_2) > 2^{2+2}\psi(u_2)$ and $\int_{r_2}^{3r_2} \psi(u_2)wdt < 2^{-2-1}$. So, there exists $r_3 \in \mathbb{N},\ r_3 \geq 3r_2$, such that $2^{-2-1} \leq \int_{r_2}^{r_3} \psi(u_2)w(t)dt \leq 2^{-2}$. Observe that, if $p_2 := r_3 - r_2$, then $p_2 \geq 2r_2$.

Further we proceed by iteration. For $k \geq 1$ define $f_k \in L_0(\mu)$ as $f_k := u_k \cdot w \cdot \mathbb{1}_{(r_k, r_{k+1}]}$. Let $f := \sum_{k\geq 1} f_k$. Then

(a) Clearly the functions f_k have disjoint supports, $f^* = f$ and

$$J_{(\psi,w)}(f) = \int_0^\infty \psi(\frac{f}{w})wdt = \sum_{k\geq 1}\int_{r_k}^{r_{k+1}} \psi(u_k)wdt \leq \sum_{k\geq 1} 2^{-k} = 1.$$

Thus $|||f_k||| \leq |||f||| \leq 1$.

(b) Let us compute $J_{(\psi,w)}((C+1)f_k)$. Observe that: (I) $f_k^*(t) = u_k w(t+r_k)$, if $0 < t \leq p_k$, and $f_k^*(t) = 0$ if $t > p_k$; (II) if $t \geq r_k$, then $2t \geq t + r_k$ and so $w(t) \leq Cw(2t) \leq Cw(t+r_k)$. Thus

$$J_{(\psi,w)}((C+1)f_k) = \int_0^{p_k} \psi\big((C+1)\frac{f_k^*}{w}\big)wdt \geq$$

$$\geq \int_{r_k}^{p_k} \psi\big((C+1)\frac{u_k w(t+r_k)}{Cw(t+r_k)}\big)wdt = \int_{r_k}^{p_k} \psi\big(\frac{C+1}{C}u_k\big)wdt > 2^{(k+2)}\int_{r_k}^{p_k}\psi(u_k)wdt.$$

Since w is decreasing and $p_k \geq r_k + \frac{p_k}{2}$ (that is, p_k is equal or greater than the middle point $r_k + \frac{p_k}{2}$ of the interval $[r_k, r_k + p_k]$) then

$$\int_{r_k}^{p_k} \psi(u_k)wdt \geq \tfrac{1}{2}\int_{r_k}^{r_k+p_k} \psi(u_k)wdt \geq 2^{-k-2}.$$

Thus, finally we have $J_{(\psi,w)}((C+1)f_k) > 1$ and so $|||f_k||| \geq (C+1)^{-1},\ k \geq 1$. Taking into account the equivalence between the quasinorm $|||\cdot|||$ and the dual norm of $(\Lambda_{(\varphi,w)}(\mu))_i^*$, it is clear that the mapping $T : \ell_\infty \to M_{(\psi,w)}(\mu)$ such that $T((a_k)_{k\geq 1}) = \sum_{k\geq 1} a_k f_k,\ (a_k)_{k\geq 1} \in \ell_\infty$, is an order-isomorphism between ℓ_∞ and $T(\ell_\infty)$. Therefore $M_{(\psi,w)}(\mu)$ contains an order-isomorphic copy of ℓ_∞, a contradiction, because $M_{(\psi,w)}(\mu)$ is o-continuous by hypothesis.

(3) Suppose that $\psi \in \Delta_2^0$ and prove that $M_{(\psi,w)}(\mu)$ is o-continuous. Pick $f := (f_i)_{i\in I} \in M_{(\psi,w)}(\mu)$ and assume that $J_{(\psi,w)}(f) < \infty$.

Claim. $f,\ f^*$ and $\frac{f^*}{w}$ are bounded functions.

Indeed, first $f_i \to 0$ when $i \in I$ (that is, $(f_i)_{i\in I} \in c_0(I)$) because otherwise $f^* \geq d > 0$ on $[0,\infty)$ and this contradicts the fact that $J_{(\psi,w)}(f) < \infty$. So, f and f^* are bounded and $f^*(t) \downarrow 0$ when $t \to \infty$. Moreover, f^*/w is bounded. Indeed, assume that f^*/w is not bounded on $(0,\infty)$. As f^*/w is bounded on the interval $(0,\delta)$ for every $\delta > 0$, there exists a sequence $\{x_k : k \geq 1\} \subset \mathbb{R}$ such that $x_k \uparrow \infty$ and $f^*(x_k)/w(x_k) \geq k,\ \forall k \geq 1$. Then, taking into account that f^* and w are non-increasing functions, and the fact $w(t/2) \le Cw(t),\ \forall t > 0$, we have:

$$\forall k \ge 1,\ \forall t \in [\tfrac{x_k}{2}, x_k],\ \frac{f^*(t)}{w(t)} \ge \frac{f^*(x_k)}{w(x_k/2)} \ge \frac{f^*(x_k)}{Cw(x_k)} \ge \frac{k}{C}.$$

So, for every $k \ge 1$ we have

$$J_{(\psi,w)}(f) = \int_0^\infty \psi(\frac{f^*}{w})wdt \ge \int_{x_k/2}^{x_k} \psi(\frac{f^*}{w})wdt \ge \int_{x_k/2}^{x_k} \psi(\frac{k}{C})wdt = \psi(\frac{k}{C})(S(x_k) - S(x_k/2)).$$

On the other hand, $S(x_k) - S(x_k/2) \to +\infty$ when $k \to \infty$ by (O1) and so $\psi(k/C)(S(x_k) - S(x_k/2)) \to \infty$, a contradiction because $J_{(\psi,w)}(f) < \infty$.

Therefore, as $\psi \in \Delta_2^0$, we have $J_{(\psi,w)}(2^n f) < \infty$, $\forall n \ge 1$, and so $f \in HM_{(\psi,w)}(\mu)$. Hence $M_{(\psi,w)}(\mu)$ is o-continuous by Lemma 4.6.

Now we prove the converse statement. We suppose that $\psi \notin \Delta_2^0$ and we construct in $M_{(\psi,w)}(\mu)$ an order-isomorphic copy of ℓ_∞. Without loss of generality, we assume that $I = \mathbb{N}$. Moreover, if λ is the Lebesgue measure on $[0,\infty)$, we shall work in the subspace $M^0_{(\psi,w)}([0,\infty),\lambda)$ of $M_{(\psi,w)}([0,\infty),\lambda)$ consisting of those elements which are constant in each interval $(n-1,n]$, $n \ge 1$, because $(M_{(\psi,w)}(I,\mu), |||\cdot|||)$ is order-isometric to $(M^0_{(\psi,w)}([0,\infty),\lambda), |||\cdot|||)$. Since $\psi \notin \Delta_2^0$, by the above part (2) there exist a sequence $\{u_k : k \ge 1\} \subset \mathbb{R}$, $0 < u_{k+1} < u_k < ..$, a sequence of integers $0 = r_1 < r_2 < \cdots$, and a sequence $\{f_k : k \ge 1\} \subset M_{(\psi,w)}([0,\infty),\lambda)$ with disjoint supports $\mathrm{supp}(f_k) \subset (r_k, r_{k+1}]$ such that the mapping $T((a_k)_{k\ge1}) = \sum_{k\ge1} a_k f_k$, $(a_k)_{k\ge1} \in \ell_\infty$, is an order-isomorphism. For every $k \ge 1$ define

$$x_k := \sum_{j=r_k}^{-1+r_{k+1}} u_k w(j+1) \cdot \mathbb{1}_{(j,j+1]}.$$

Clearly, $\{x_k : k \ge 1\}$ is a sequence in $M^0_{(\psi,w)}([0,\infty),\lambda)$ whose elements are pairwise disjoint. As w is a regular weight, there exists (see Lemma 4.6) a constant $1 \le K < \infty$ such that $\frac{w(n)}{w(n+1)} \le K$, $\forall n \ge 1$. It is easy to see that for $k \ge 2$ we have

$$x_k \le f_k \le \sum_{j=r_k}^{-1+r_{k+1}} u_k w(j) \cdot \mathbb{1}_{(j,j+1]} \le K x_k.$$

So, for $k \ge 2$ we have: (i) $|||x_k||| \le |||\sum_{i\ge2} x_i||| \le |||\sum_{i\ge2} f_i||| \le 1$; (ii) $|||x_k||| \ge K^{-1}|||f_k||| \ge K^{-1}(C+1)^{-1}$. Thus the mapping $S((a_k)_{k\ge1}) = \sum_{k\ge1} a_k x_{k+1}$, $(a_k)_{k\ge1} \in \ell_\infty$, is an order-isomorphism, and this completes the proof of the part (3). ☐

Corollary 4.8. *Let (Ω,Σ,μ) be a complete Γ-finite measure space, φ an Orlicz N-function, ψ the complementary Orlicz function of φ and w a regular weight. Then*

(1) If μ is atomless and $0 < \mu(\Omega) < \infty$, then $\Lambda_{(\varphi,w)}(\mu)$ is Grothendieck if and only if $\psi \in \Delta_2^\infty$.

(2) If μ is atomless and $\mu(\Omega) = \infty$, then $\Lambda_{(\varphi,w)}(\mu)$ is Grothendieck if and only if $\psi \in \Delta_2$.

(3) If (Ω,Σ,μ) is the counting measure space on an infinite set I, then $\Lambda_{(\varphi,w)}(\mu)$ is Grothendieck if and only if $\psi \in \Delta_2^0$.

Proof. The proof follows from Proposition 4.3, Lemma 4.7 and the result of [7] that states that, under the given conditions, the Köthe dual $(\Lambda_{(\varphi,w)}(\mu))_i^*$ of $\Lambda_{(\varphi,w)}(\mu)$ coincides, as a set, with the space $M_{(\psi,w)}(\mu)$. ☐

5. The Grothendieck property for Orlicz spaces

Let (Ω,Σ,μ) be a Γ-finite measure space, $\mathcal{S}_0$ the ideal of $L_0(\mu)$ generated by the simple Σ-measurable real functions, φ and Orlicz function and $I_\varphi : L_0(\mu) \cup M(\mu)^+ \to [0,+\infty]$ be the Orlicz functional such that $I_\varphi(f) = \int_\Omega \varphi(f) \cdot d\mu$. It is easy to see that I_φ is an Orlicz-type semimodular. Let $L_\varphi(\mu) := \{f \in L_0(\mu) : \exists\lambda > 0$ such that $I_\varphi(f/\lambda) < +\infty\}$ be he corresponding Orlicz space with the Luxemburg norm $\|f\| = \inf\{\lambda > 0 : I_\varphi(f/\lambda) \le 1\}$. When we work with the counting measure μ on a set I, we put $\ell_\varphi(I)$ instead of $L_\varphi(\mu)$. Observe that the functional I_φ and the Orlicz space $L_\varphi(\mu)$ coincide with the functional $I_{(\varphi,w)}$ and the Orlicz-Lorentz space $\Lambda_{(\varphi,w)}(\mu)$, respectively, when w is the regular weight $w(t) = 1$, $\forall t \in (0,\infty)$. So, we can apply all the results of the previous section. Let

$$H_\varphi(\mu) := \{f \in L_\varphi(\mu) : \forall \lambda > 0, I_\varphi(\lambda f) < +\infty\}.$$

Clearly, $H_\varphi(\mu)$ is a closed ideal of $L_\varphi(\mu)$ such that, if $\tau(\varphi) < \infty$, then $H_\varphi(\mu) = \{0\}$.

Proposition 5.1. *Let (Ω, Σ, μ) be a complete Γ-finite measure space, φ an Orlicz function and ψ the Orlicz function complementary of φ. Let $L_\varphi(\mu)^* = L_\varphi(\mu)^*_i \overset{d}{\oplus} L_\varphi(\mu)^*_s$ be the disjoint decomposition of $L_\varphi(\mu)^*$ into the subspace of integral functionals $L_\varphi(\mu)^*_i$ and singular functionals $L_\varphi(\mu)^*_s$. Then*

(A) $L_\varphi(\mu)^*_i = L_\psi(\mu)$.

(B) $H(\mathcal{S}_0)$ is I_φ-dense in $L_\varphi(\mu)$ and, if $\mu = \mu_a$, then $H(\mathcal{S}_a)$ is I_φ-dense in $L_\varphi(\mu)$, $\mathcal{S}_a$ being the ideal generated by the functions of the form $\mathbb{1}_A$, A being an atom of μ.

(C) If φ is finite then $H_\varphi(\mu) = H(\mathcal{S}_0) = L_\varphi(\mu)^a$.

(D) If $\tau(\varphi) < \infty$, then $L_\varphi(\mu)^a = H(\mathcal{S}_a) = \overline{\mathcal{S}_a}$.

Proof. (A) It is well known that the subspace of integral functionals $L_\varphi(\mu)^*_i$ on an Orlicz space $L_\varphi(\mu)$ with the Luxemburg norm is the Orlicz space $L_\psi(\mu)$ with the Amemiya-Orlicz norm, ψ being the Orlicz function complementary of φ.

(B), (C) and (D) follow from Proposition 4.1 and Proposition 4.2. □

Proposition 5.2. *Let (Ω, Σ, μ) be a complete Γ-finite measure space, φ an Orlicz function and ψ the Orlicz function complementary of φ. Then*

(A) $L_\varphi(\mu)/H(\mathcal{S}_0)$ is a Grothendieck M-space.

(B) If either φ is a finite Orlicz function or $\mu = \mu_a$, the following statements are equivalent:

(1) $L_\varphi(\mu)$ is Grothendieck ; (2) $L_\psi(\mu)$ is o-continuous.

Proof. This follows from Proposition 5.1 and Proposition 4.3. □

Lemma 5.3. *Let (Ω, Σ, μ) be a complete Γ-finite measure space and ψ an Orlicz function. Then*

(1) If μ is atomless and $0 < \mu(\Omega) < \infty$, then $L_\psi(\mu)$ is o-continuous if and only if $\psi \in \Delta_2^\infty$.

(2) If μ is atomless and $\mu(\Omega) = \infty$, then $L_\psi(\mu)$ is o-continuous if and only if $\psi \in \Delta_2$.

(3) If (Ω, Σ, μ) is the counting measure space on a infinite set I, then $\ell_\psi(I)$ is o-continuous if and only if $\psi \in \Delta_2^0$.

Proof. The proof is analogous to the one of Lemma 4.7, using the regular weight $w(t) = t$ and taking into account that now we do not ask for ψ to be an Orlicz N-function and so it can be $a(\psi) > 0$ and $\tau(\psi) < \infty$. □

Proposition 5.4. *Let (Ω, Σ, μ) be a complete Γ-finite measure space, μ_a and μ_d the atomic and purely non-atomic parts of μ, respectively, and ψ an Orlicz function. Then the following statements are equivalent:*

(1) $L_\psi(\mu)$ is o-continuous.

(2) The following conditions $C(\psi, \mu_d)$ and $C(\psi, \mu_a)$ are fulfilled:

(A) Condition $C(\psi, \mu_d)$. If $\mu_d = 0$, then $C(\psi, \mu_d)$ is nothing. Suppose that $\mu_d > 0$. Then $C(\psi, \mu_d)$ is the following condition: (A1) if $0 < \mu(\Omega_d) < \infty$, $\psi \in \Delta_2^\infty$, and (A2) if $\mu(\Omega_d) = \infty$, $\psi \in \Delta_2$.

(B) Condition $C(\psi,\mu_a)$. If the family of atoms $\{A_i : i \in I\}$ is finite, then $C(\psi,\mu_a)$ is nothing. Suppose that the family of atoms $\{A_i :\in I\}$ is infinite. Then $C(\psi,\mu_a)$ is the fact that $(\mu(A_i)\psi)_{i\in I}$ satisfies the condition δ_2^0 (see the next section).

Proof. First, as $L_\psi(\mu) = L_\psi(\mu_d) \overset{d}{\oplus} L_\psi(\mu_a)$, it is clear that $L_\psi(\mu)$ is o-continuous if and only if both $L_\psi(\mu_d)$ and $L_\psi(\mu_a)$ are o-continuous.

(A) Suppose that $0 < \mu_d(\Omega_d)$. Then by Lemma 5.3 $L_\psi(\mu_d)$ is o-continuous if and only the condition $C(\psi,\mu_d)$ is fulfilled.

(B) If the family of atoms $\{A_i : i \in I\}$ is finite, then $L_\psi(\mu_a)$ is finite-dimensional and so it is o-continuous. Suppose that the family of atoms $\{A_i : i \in I\}$ is infinite. Observe that the space $L_\psi(\mu_a)$ is order-isometric to the Musielak-Orlicz space $\ell_\varphi(I)$ (see the next section), where $\varphi := (\mu(A_i)\psi)_{i\in I}$. Thus $L_\psi(\mu_a)$ is o-continuous if and only the condition $C(\psi,\mu_a)$ is fulfilled, because it is well known that a Musielak-Orlicz sequence space $\ell_\varphi(I)$ (where $\varphi := (\varphi_i)_{i\in I}$ is a family of Orlicz functions) is o-continuous if and only if φ satisfies the condition δ_2^0. □

Corollary 5.5. *Let I be an infinite set, φ an Orlicz function and ψ the Orlicz function complementary of φ. Then $\ell_\varphi(I)$ is Grothendieck if and only if $\psi \in \Delta_2^0$.*

Proof. By Proposition 5.2, $\ell_\varphi(I)$ is Grothendieck if and only if $\ell_\psi(I)$ is o-continuous if and only if $\psi \in \Delta_2^0$ by Lemma 5.3. □

Corollary 5.6. *Let φ be an Orlicz function and ψ the Orlicz function complementary of φ. Then:*

(A) $L_\varphi([0,+\infty))$ is Grothendieck if and only if $\psi \in \Delta_2$.

(B) $L_\varphi([0,1])$ is Grothendieck if and only if $\psi \in \Delta_2^\infty$.

Proof. This follows from Proposition 5.2 and Proposition 5.4. □

6. The Grothendieck property for Musielak-Orlicz spaces

If (Ω,Σ,μ) is a complete Γ-finite measure space, a function $\varphi : \Omega \times (\mathbb{R}\cup\{\pm\infty\}) \to [0,\infty]$ is said to be a Musielak-Orlicz function (see [11, p. 33]) if: (i) $\varphi(w,\cdot) : \mathbb{R}\cup\{\pm\infty\} \to [0,\infty]$ is an Orlicz function for each $w \in \Omega$; (ii) $\varphi(\cdot,t) : \Omega \to [0,\infty]$ is a Σ-measurable function for every $t \in \mathbb{R}\cup\{\pm\infty\}$. Define the functional $I_\varphi : L_0(\mu) \cup M(\mu) \to [0,\infty]$ as follows

$$\forall f \in L_0(\mu) \cup M(\mu),\ I_\varphi(f) = \int_\Omega \varphi(w,f(w))d\mu.$$

It is easy to see that I_φ is an Orlicz-type semimodular. The Musielak-Orlicz space $L_\varphi(\mu)$ is the modular space $(L_0(\mu))_{I_\varphi}$ associated to the semimodular I_φ, that is

$$L_\varphi(\mu) := \{f \in L_0(\mu) : \exists\lambda > 0 \text{ such that } I_\varphi(\lambda f) < \infty\}.$$

When μ is the counting measure on a set I, we put $\ell_\varphi(I)$ instead of $L_\varphi(\mu)$, φ being in this case a family of Orlicz functions $\varphi := (\varphi_i)_{i\in I}$. We consider in $L_\varphi(\mu)$ the Luxemburg norm

$$\|f\| := \inf\{\lambda > 0 : I_\varphi(f/\lambda) \le 1\}.$$

Proposition 6.1. *Let (Ω,Σ,μ) be a complete Γ-finite measure space and φ a Musielak-Orlicz function. Then*

(A) $H(\mathcal{S}_0)$ is I_φ-dense in $L_\varphi(\mu)$ and, if $\mu = \mu_a$, then $H(\mathcal{S}_a)$ is I_φ-dense in $L_\varphi(\mu)$.

(B) If either φ is locally integrable (that is, $\int_A \varphi(w,t)d\mu < \infty$ for every $t \in \mathbb{R}$ and every $A \in \Sigma$ with $\mu(A) < \infty$) or $\mu = \mu_a$, then the following statements are equivalent

(a) $L_\varphi(\mu)$ is Grothendieck; (b) $(L_\varphi(\mu))_i^*$ is o-continuous.

Proof. (A) Let us prove that $H(\mathcal{S}_0)$ is I_φ-dense in $L_\varphi(\mu)$. Let $f \in L_\varphi(\mu)$ be such that $I_\varphi(f) < \infty$ and $\epsilon > 0$. Since $\int_\Omega \varphi(w,f(w))d\mu < \infty$, then $\Gamma_0 := \{\gamma \in \Gamma : \int_{\Omega_\gamma} \varphi(w,f(w))d\mu > 0\}$ is countable, we say, $\Gamma_0 := \{\gamma_n : n \ge 1\}$. Moreover, $\int_\Omega \varphi(w,f(w))d\mu = \int_{\cup_{n\ge1}\Omega_{\gamma_n}} \varphi(w,f(w))d\mu$. Let $f_n := ((f \upharpoonright$

$\cup_{i=1}^{n}\Omega_{\gamma_i}) \wedge n) \vee (-n)$. Clearly $f_n \in \mathcal{S}_0$, $|f| \geq |f - f_n|$ and so $\varphi(w, f(w)) \geq \varphi(w, (f - f_n)(w))$ a.e. and $\varphi(w, (f - f_n)(w)) \downarrow 0$ a.e. Thus by the dominated convergence theorem we have $I_\varphi(f - f_n) \downarrow 0$ and there exists $p \in \mathbb{N}$ such that $I_\varphi(f - f_p) < \epsilon$.

Analogously it is proved that $H(\mathcal{S}_a)$ is I_φ-dense in $L_\varphi(\mu)$, if $\mu = \mu_a$.

(B) We consider two cases, namely: (I) φ is locally integrable; (II) $\mu = \mu_a$.

(I) Suppose that φ is locally integrable. Then $I_\varphi(\lambda f) < \infty$ for every $\lambda \in \mathbb{R}$ and every $f \in \mathcal{S}_0$ and also for every $f \in H(\mathcal{S}_0)$. Actually $H(\mathcal{S}_0) = \{f \in L_\varphi(\mu) : I_\varphi(\lambda f) < \infty, \ \forall \lambda > 0\}$. By Proposition 3.1 it is enough to prove that $H(\mathcal{S}_0) = L_\varphi(\mu)^a$.

(i) As $(L_\varphi(\mu))^a$ is a closed ideal and $H(\mathcal{S}_0) = \overline{\mathcal{S}_0}$, in order to see that $H(\mathcal{S}_0) \subset (L_\varphi(\mu))^a$, it is enough to verify that $\mathbb{1}_A \in (L_\varphi(\mu))^a$ whenever $A \in \Sigma$ and $\mu(A) < \infty$. Since $L_\varphi(\mu)$ is σ-o-complete, it is enough to show that, if $0 \leq f_n \leq f_{n-1} \leq \mathbb{1}_A$ is a sequence of elements of $L_\varphi(\mu)$ such that $f_n \downarrow 0$ in order, then $\|f_n\| \downarrow 0$ when $n \to \infty$. So, let $\lambda > 0$ and observe that $I_\varphi(\mathbb{1}_A/\lambda) < \infty$ (because $\mu(A) < \infty$ and φ is locally integrable) and $\mathbb{1}_A/\lambda \geq f_n/\lambda \downarrow 0$. Thus, we get $I_\varphi(f_n/\lambda) \downarrow 0$ by the dominated convergence theorem and so $\|f_n\| \downarrow 0$ when $n \to \infty$.

(ii) Let us see that $(L_\varphi(\mu))^a \subset H(\mathcal{S}_0)$. So, pick $f \in L_\varphi(\mu) \setminus H(\mathcal{S}_0)$ and prove that $f \notin (L_\varphi(\mu))^a$. Without loss of generality, assume that $0 \leq f$ and $I_\varphi(f) = \infty$. For each $(F, n) \in \mathcal{F} \times \mathbb{N}$ ($\mathcal{F}$ = finite subsets of Γ) define

$$f_{(F,n)} := (f \cdot \mathbb{1}_{\cup_{\gamma \in F}\Omega_\gamma}) \wedge n.$$

Observe that $f_{(F,n)} \in \mathcal{S}_0$ and $\{f - f_{(F,n)} : (F, n) \in \mathcal{F} \times \mathbb{N}\}$ is a downward directed set such that $f \geq f - f_{(F,n)} \downarrow 0$. Moreover

$$+\infty = I_\varphi(f) = I_\varphi((f - f_{(F,n)}) + f_{(F,n)}) =$$
$$= I_\varphi(\tfrac{1}{2}2(f - f_{(F,n)}) + \tfrac{1}{2}2f_{(F,n)}) \leq \tfrac{1}{2}I_\varphi(2(f - f_{(F,n)})) + \tfrac{1}{2}I_\varphi(2f_{(F,n)}),$$

whence we get $I_\varphi(2(f - f_{(F,n)})) = +\infty$ and so $\|f - f_{(F,n)}\| \geq 1/2$ for every $(F, n) \in \mathcal{F} \times \mathbb{N}$. Thus f is not o-continuous.

(II) By (A) and Proposition 3.1 it is enough to prove that $H(\mathcal{S}_a) = L_\varphi(\mu)^a$. Since clearly $\mathbb{1}_A \in L_\varphi(\mu)^a$ when $A \in \Sigma$ is an atom, then $\mathcal{S}_a \subset L_\varphi(\mu)^a$ and so $H(\mathcal{S}_a) = \overline{\mathcal{S}_a} \subset L_\varphi(\mu)^a$.

Now let $f \in L_\varphi(\mu)^a$ and prove that $f \in \overline{\mathcal{S}_a}$. Let $\{A_i : i \in I\}$ be the family of atoms of μ. Then $|f - \sum_{i \in J} f_i \cdot \mathbb{1}_{A_i}| \downarrow 0$ when $J \subset I$ is a finite subset. Thus $\|f - \sum_{i \in J} f_i \cdot \mathbb{1}_{A_i}\| \downarrow 0$ when $J \subset I$ is a finite subset, because f is o-continuous. Since $\sum_{i \in J} f_i \cdot \mathbb{1}_{A_i} \in \mathcal{S}_a$ when $J \subset I$ is a finite subset, we conclude that $f \in \overline{\mathcal{S}_a}$. □

We are interested in the Musielak-Orlicz sequence space $\ell_\varphi(I)$, I being a set and $\varphi := (\varphi_i)_{i \in I}$ a family of Orlicz functions. For this space clearly $\mathcal{S}_0 = \mathcal{S}_a$.

Definition 6.2. *A family of Orlicz functions $\varphi := (\varphi_i)_{i \in I}$ satisfies the δ_2^0 condition if there are two positive constants a and K, a finite subset $I_0 \subset I$ and a family $\{c_i : i \in I\} \subset [0, \infty]$ such that $\sum_{i \in I \setminus I_0} c_i < \infty$ and for every $i \in I$ and $u \in \mathbb{R}$ satisfying $\varphi_i(u) \leq a$ there holds $\varphi_i(2u) \leq K\varphi_i(u) + c_i$.*

Corollary 6.3. *Let I be a set, $\varphi := (\varphi_i)_{i \in I}$ a family of Orlicz functions and $\psi := (\psi_i)_{i \in I}$ the complementary function of φ. Then*

(1) $\frac{\ell_\varphi(I)}{H(\mathcal{S}_0)}$ is a Grothendieck M-space.

(2) The following statement are equivalent

(a) $\ell_\varphi(I)$ is Grothendieck; (b) $\psi \in \delta_2^0$.

Proof. (1) This follows from Proposition 3.1 and Proposition 6.1.

(2) Observe that $(\ell_\varphi(I))_i^* = \ell_\psi(I)$, where $\psi := (\psi_i)_{i \in I}$ and ψ_i is the Orlicz function complementary of φ_i for all $i \in I$. On the other hand, it is well known that a Musielak-Orlicz sequence space $\ell_\psi(I)$ is o-continuous if and only if $\psi \in \delta_2^0$. Now it is enough to apply Proposition 6.1. □

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