THE SPACE $(\ell_\varphi/h_\varphi, weak)$ IS NOT A RADON SPACE

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Abstract. Consider the quotient Banach spaces (see [GH]) $\ell_\varphi(I)/h_\varphi(\mathcal{S})$, where I is an arbitrary set, φ is an Orlicz function, $\ell_\varphi(I)$ is the corresponding Orlicz space on I and $h_\varphi(\mathcal{S}) = \{x \in \ell_\varphi(I) : \forall\lambda > 0,\ \exists s \in \mathcal{S} \text{ such that } I_\varphi(\frac{x-s}{\lambda}) < \infty\}$, $\mathcal{S}$ being the ideal of elements of $\ell_\varphi(I)$ with finite support. These spaces are the natural generalization of the classical Banach space $\ell_\infty(I)/c_0(I)$. In [MR] it is proved that $(\ell_\infty(\mathbb{N})/c_0(\mathbb{N}), weak)$ is not a Radon space. Here we extend this result to the abroad class of quotient Banach spaces $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ by showing that every $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ space contains an isometric and order isomorphic copy of $\ell_\infty^c(\mathfrak{c})$, the subspace of $\ell_\infty(\mathfrak{c})$ integrated by the elements with countable support.

A Hausdorff topological space E is said to be a *Radon space* if every finite positive Borel measure μ on E is a Radon measure, i.e., $\mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ compact}\}$ for each Borel subset A of E. In [MR] it is proved that $(\ell_\infty(\mathbb{N})/c_0(\mathbb{N}), weak)$ is not Radon. Talagrand proved [T] that $(\ell_\infty^c(\aleph_1), weak)$ is not Radon, $\ell_\infty^c(\aleph_1)$ being the subspace of $\ell_\infty(\aleph_1)$ integrated by the elements with countable support. In this short note we show that the spaces $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ contain an isometric and order isomorphic copy of $\ell_\infty^c(\mathfrak{c})$. So, equipped with the weak topology, they are not Radon.

The quotient spaces $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ are the natural generalization of the spaces $\ell_\infty(I)/c_0(I)$ (see [GH], [LW]). Let us introduce these spaces. Let $\varphi : \mathbb{R} \to [0, +\infty]$ denote an Orlicz function, i.e. a function which is even, nondecreasing, left continuous for $x \geq 0$, $\varphi(0) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. Observe that we do not need for φ to be convex. Define:

$$a(\varphi) = \sup\{t \geq 0 : \varphi(t) = 0\} \qquad , \qquad \tau(\varphi) = \sup\{t \geq 0 : \varphi(t) < \infty\}.$$

We assume that $\tau(\varphi) > 0$. Fix an arbitrary set I and, for $x \in \mathbb{R}^I$, define $I_\varphi(x) = \sum_{i\in I} \varphi(x_i)$. Let $\ell_\varphi(I)$ be the corresponding Orlicz space, i.e.:

$$\ell_\varphi(I) = \{x \in \mathbb{R}^I : \exists\lambda > 0 \text{ such that } I_\varphi(x/\lambda) < \infty\}.$$

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Consider in $\ell_\varphi(I)$ the F-norm $|\cdot|_\varphi$:

$$|x|_\varphi := \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq \lambda\},\ \forall x \in \ell_\varphi(I),$$

and the associated distance $d(x,y) = |x-y|_\varphi$. It is known that $(\ell_\varphi(I), d)$ is a complete F-space (see [Mu]).

Let $\mathcal{S} \subseteq \ell_\varphi(I)$ be the ideal of elements of finite support. Define $h_\varphi(\mathcal{S})$ as:

$$h_\varphi(\mathcal{S}) = \{x \in \ell_\varphi(I) : \forall \lambda > 0,\ \exists s \in \mathcal{S} \text{ such that } I_\varphi(\frac{x-s}{\lambda}) < \infty\},$$

and $\delta(x)$ as:

$$\delta(x) = \inf\{\lambda > 0 : \exists s \in \mathcal{S} \text{ such that } I_\varphi(\frac{x-s}{\lambda}) < \infty\}, x \in \ell_\varphi(I).$$

Clearly, $h_\varphi(\mathcal{S})$ is a closed ideal of $\ell_\varphi(I)$ such that $\overline{\mathcal{S}} = h_\varphi(\mathcal{S})$ and, if φ is finite, we have $h_\varphi(\mathcal{S}) = \{x \in \ell_\varphi(I) : \forall \lambda > 0,\ I_\varphi(\lambda x) < \infty\}$.

In order to avoid the trivial case $\ell_\varphi(I) = h_\varphi(\mathcal{S})$, we must impose that I is infinite and $\varphi \notin \Delta_2^0$, i.e. φ doesn't satisfy the Δ_2 condition at 0.

If φ is convex we can consider the Luxemburg norm $\|\cdot\|_L$ and the Luxemburg distance d_L:

$$\|x\|_L = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\}, \qquad d_L(x,y) = \|x-y\|_L, \qquad x, y \in \ell_\varphi(I),$$

as well as the Amemiya-Orlicz norm $\|\cdot\|_o$ and the Amemiya-Orlicz distance d_o:

$$\|x\|_o = \inf_{k>0}\{\tfrac{1}{k}(1 + I_\varphi(kx))\}, \qquad d_o(x-y) = \|x-y\|_o, \qquad x, y \in \ell_\varphi(I).$$

It is known that, $\forall x \in \ell_\varphi(I), \|x\|_L \leq \|x\|_o \leq 2\|x\|_L$ and that these norms define on $\ell_\varphi(I)$ the same topology as $|\cdot|_\varphi$. Denote by B_φ^L (resp. B_φ^o) and S_φ^L (resp. S_φ^o) the closed unit ball and unit sphere of $(\ell_\varphi(I), \|\cdot\|_L)$ (resp. $(\ell_\varphi(I), \|\cdot\|_o)$). Recall that a Banach M-space is a Banach lattice $(X, \|\cdot\|)$ such that $\|x \vee y\| = \|x\| \vee \|y\|$, whenever $x, y \in X^+$.

The quotient space $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ has, among others, the following interesting properties (see [GH], [Wn]):

(1) For each $x \in \ell_\varphi(I)$ we have $\delta(x) = d(x, h_\varphi(\mathcal{S}))$ and, if φ is convex, also $\delta(x) = d_L(x, h_\varphi(\mathcal{S})) = d_o(x, h_\varphi(\mathcal{S}))$.

(2) δ is a monotone seminorm on $\ell_\varphi(I)$ such that $ker(\delta) = h_\varphi(\mathcal{S})$.

(3) Let $\|\cdot\|$ be the quotient F-norm on $\ell_\varphi(I)/h_\varphi(\mathcal{S})$. Then $(\ell_\varphi(I)/h_\varphi(\mathcal{S}), \|\cdot\|)$ is a Banach M-space such that, if $Q : \ell_\varphi(I) \to \ell_\varphi(I)/h_\varphi(\mathcal{S})$ is the canonical quotient map, then $\|Q(x)\| = \delta(x)$. Moreover, if φ is convex, the space $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ equipped with the quotient norms corresponding to the Luxemburg norm as well as to the Orlicz norm is order isomorphic and isometric to $(\ell_\varphi(I)/h_\varphi(\mathcal{S}), \|\cdot\|)$.

(4) If $a(\varphi) > 0$, then $\ell_\varphi(I)/h_\varphi(\mathcal{S}) \cong (\ell_\infty(I)/c_o(I), \|\cdot\|_\infty) \cong (C(\beta I \backslash I), \|\cdot\|_\infty)$ (order isomorphism and isometry).

(5) $h_\varphi(\mathcal{S})$ is proximinal in $(\ell_\varphi(I), |\cdot|_\varphi)$ and, if φ is convex, also in $(\ell_\varphi(I), \|\cdot\|_L)$. Recall that a subset $M \subset X$ of a metric space (X, d) is said to be *proximinal* if for each $x \in X$ there exists $y \in M$ such that $d(x, y) = \inf\{d(x, m) : m \in M\}$.

(6) From the proximinality of $h_\varphi(\mathcal{S})$ in $\ell_\varphi(I)$ (see [Go]) we get that for each $z \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ there exists $x \in \ell_\varphi(I)$ such that $Q(x) = z$, $I_\varphi(x) \leq 1$ and $I_\varphi(\lambda(x - s)) = \infty,\ \forall \lambda > 1,\ \forall s \in \mathcal{S}$.

Let I be an infinite set, $\mathfrak{m}$ =card(I) and $P_\omega(I) = \{A \subseteq I :\text{card}(A) = \aleph_0\}$. Then, clearly, card$(P_\omega(I)) = \mathfrak{m}^{\aleph_0} =: \mathfrak{n}$. Note that $\mathfrak{n} \geq \mathfrak{c}$, where $\mathfrak{c}$ =card$(\mathbb{R})$. Also there exists a family $\{A_t\}_{t\in\mathfrak{n}}$ in $P_\omega(I)$ such that card$(A_t \cap A_s) < \aleph_0$, for $t \neq s$. Indeed, let $\{I_t\}_{t\in\mathfrak{m}}$ be a family of disjoint subsets of I such that card$(I_t) = \mathfrak{m}$, $\forall t \in \mathfrak{m}$. Pick $i_t \in I_t$, $t \in \mathfrak{m}$, and choose a disjoint family $\{I_{ts}\}_{s\in\mathfrak{m}}$ of subsets of $I_t \backslash \{i_t\}$ such that card$(I_{ts}) = \mathfrak{m}$, $s \in \mathfrak{m}$. Pick $i_{ts} \in I_{ts}$ and choose a disjoint family $\{I_{tsr}\}_{r\in\mathfrak{m}}$ of subsets of $I_{ts} \backslash \{i_{ts}\}$ such that card$(I_{tsr}) = \mathfrak{m}$, $r \in \mathfrak{m}$. Pick $i_{tsr} \in I_{tsr}$, $r \in \mathfrak{m}$. By reiteration we obtain families of elements $\{i_t\}_{t\in\mathfrak{m}}$, $\{i_{ts}\}_{t,s\in\mathfrak{m}}$, etc., of I. Now consider the family $\mathfrak{T}$ of sequences of the form $(i_{t_1}, i_{t_1t_2}, i_{t_1t_2t_3}, \dots)$, $t_j \in \mathfrak{m}, j \geq 1$. It is clear that card$(\mathfrak{T}) = \mathfrak{m}^{\aleph_0} = \mathfrak{n}$, card$(T) = \aleph_0$, $\forall T \in \mathfrak{T}$ and that, if $T, S \in \mathfrak{T}$, $T \neq S$, then card$(T \cap S) < \aleph_0$.

PROPOSITION 1. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then there exists an isometric and order isomorphic copy of $(\ell^c_\infty(\mathfrak{n}), \|\cdot\|_\infty)$ in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$, with $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$ and $\mathfrak{m}$ =card(I).*

PROOF. Let $\{A_t\}_{t\in\mathfrak{n}}$ be a family of subsets of I such that card$(A_t) = \aleph_0$ and card$(A_t \cap A_s) < \aleph_0$, when $t \neq s$. Pick $x \in \ell_\varphi(I)^+$ such that $I_\varphi(x) \leq 1$, $Q(x) \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$, card(supp$(x)$) $= \aleph_0$ and $I_\varphi(\lambda(x - s)) = \infty$, $\forall \lambda > 1$, $\forall s \in \mathcal{S}$. Let supp$(x) = \{j_r\}_{r\geq 1} \subset I$. If $t \in \mathfrak{n}$ and $A_t = \{i_k\}_{k\geq 1}$, define e^t:

$$e^t_i = \begin{cases} 0\ , & \text{if } i \notin A_t \\ x_{j_r}\ , & \text{if } i = i_r\ , r \geq 1 \end{cases}.$$

Then clearly, $\{Q(e^t)\}_{t\in\mathfrak{n}}$ are positive and disjoint elements of $S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$. Consider an arbitrary countable family $\{t_k\}_{k\geq 1} \subset \mathfrak{n}$ and $(a_k) \in \ell_\infty$. We claim that o-$\sum_{k\geq 1} a_k Q(e^{t_k}) \in \ell_\varphi(I)/h_\varphi(\mathcal{S})$ and that $\|$o-$\sum_{k\geq 1} a_k Q(e^{t_k})\| = \|(a_k)\|_\infty$ (here o-$\sum$ is the order sum, i.e., if X is a vector lattice and $\{x_k\}_{k\geq 1} \subset X^+$, then o-$\sum_{k\geq 1} x_k = \sup_{n\geq 1} \sum_{k=1}^n x_k$, if this sup exists). Indeed, choose a countable family of subsets B_k, $k \geq 1$, such that: (i) $B_k \subset A_{t_k}$ and card$(A_{t_k} \setminus B_k) < \aleph_0$; (ii) $\{B_k\}_{k\geq 1}$ is a disjoint family; (iii) $I_\varphi(e^{t_k} \cdot \mathbf{1}_{B_k}) \leq 2^{-k}$. Let $y =$ o-$\sum_{k\geq 1} a_k e^{t_k} \cdot \mathbf{1}_{B_k}$. Observe that $I_\varphi(\frac{y}{\|(a_k)\|_\infty}) \leq 1$, $I_\varphi(\lambda \frac{y}{\|(a_k)\|_\infty}) = \infty$, $\forall \lambda > 1$ and $Q(e^{t_k} \cdot \mathbf{1}_{B_k}) = Q(e^{t_k})$. Thus, $y \in \ell_\varphi(I)$ and $Q(y) =$ o-$\sum_{k\geq 1} a_k Q(e^{t_k})$. Since the elements

$\{Q(e^t)\}_{t\in\mathfrak{n}} \subset S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ are disjoint and $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is an M-space, clearly, $\|Q(y)\| = \|(a_k)\|_\infty$. Thus the map $T : \ell_\infty^c(\mathfrak{n}) \to \ell_\varphi(I)/h_\varphi(\mathcal{S})$, with $T((a_t)) =$ o-$\sum_{t\in\mathfrak{n}} a_t Q(e^t) :=$ o-$\sum_{t\in\mathfrak{n}} a_t^+ Q(e^t) -$ o-$\sum_{t\in\mathfrak{n}} a_t^- Q(e^t)$, is an isometric order isomorphism. □

COROLLARY 2. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then $(\ell_\varphi(I)/h_\varphi(\mathcal{S}), weak)$ is not a Radon space.*

PROOF. If I is infinite and $\mathfrak{m} = card(I)$, we have that $\mathfrak{n} = \mathfrak{m}^{\aleph_0} \geq \mathfrak{c}$. So, by PR. 1, there exist in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ an isometric and order isomorphic copy of $\ell_\infty^c(\mathfrak{c})$. Hence, $(\ell_\varphi(I)/h_\varphi(\mathcal{S}), weak)$ is not Radon by [T]. □

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