On James boundaries in dual Banach spaces

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Abstract

Let X be a Banach space, $K \subset X^*$ a w^*-compact subset and B a boundary of K. We study when the fact $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$ allows to "localize" inside K, even inside B, a copy of the basis of $\ell_1(\mathfrak{c})$ and a structure that we call a w^*-$\mathbb{N}$-family. Among other things, we prove that: (i) if either K is w^*-metrizable or B is a w^*-countable determined boundary of K, the fact $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$ implies that K contains a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$; (ii) if either $B = Ext(K)$ or B is a w^*-$\mathcal{K}$ analytic boundary of K, then K contains a copy of the basis of $\ell_1(\mathfrak{c})$ (resp., a w^*-$\mathbb{N}$-family) if and only if B does.

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1. Introduction

If K is a w^*-compact subset of a dual Banach space X^*, a subset B of K is said to be a *(James) boundary* of K if every $x \in X$ attains on B its maximum on K. For instance, K itself and the set of extreme points $Ext(K)$ of K are boundaries of K. If B is a boundary of K, then $\overline{\mathrm{co}}^{w^*}(B) = \overline{\mathrm{co}}^{w^*}(K)$ and also $\overline{\mathrm{co}}(B) = \overline{\mathrm{co}}^{w^*}(K)$ in some cases. But, in general, $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$. The aim of this paper is to study "local" consequences of the fact $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$. In particular, we investigate:

(1) "Localization results" (localization of copies of the basis of $\ell_1(\mathfrak{c})$ and localization of w^*-$\mathbb{N}$-families (see below for definitions)), which are consequences of the inequality $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$. If $\overline{\mathrm{co}}(K) \neq \overline{\mathrm{co}}^{w^*}(K)$, K contains a copy of the basis of $\ell_1(\mathfrak{c})$ ($\mathfrak{c}$ is the cardinality of $\mathbb{R}$) by [10, Proposition 3.5]. So, it is natural to ask if the same holds when $\overline{\mathrm{co}}(B) \neq \overline{\mathrm{co}}^{w^*}(K)$, B being a boundary of K. The answer to this question is, in general, negative (see the Counterexamples

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of Section 3), but in many cases a copy of the basis of $\ell_1(\mathfrak{c})$ can be "localized" inside K, even inside B.

(2) Estimations of distances to some spaces of 1-Baire functions. Actually, given $d > 0$ and a vector $\psi \in X^{**}$ such that $\sup \langle \psi, \overline{\text{co}}^{w^*}(K)\rangle > \sup \langle \psi, B\rangle + d$, we relate d with the distance from ψ to different subspaces: the subspace of $\ell_\infty(K)$ of 1-Baire bounded functions on K, the subspace of X^{**} of 1-Baire functions on $(B(X^*), w^*)$, etc. We use these estimations as auxiliary results for the technique of localizations.

(3) Finally, we apply the above results to give extensions of the Theorem of Talagrand [19] for the w^*-topology of X^* and the boundaries B of w^*-compact subsets K of X^*. Recall that Talagrand Theorem asserts that, given an arbitrary subset A of a Banach space X, A contains a copy of the basis of $\ell_1(\mathfrak{c})$ iff $\overline{\text{co}}(A)$ does iff $\overline{[A]}$ does. So, it is natural to ask whether B contains a copy of the basis of $\ell_1(\mathfrak{c})$ when $\overline{\text{co}}^{w^*}(K)$ does, B being a boundary of $K \subset X^*$. Of course, if $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$, the answer is affirmative by Talagrand Theorem, but when $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$ the ideas of Talagrand Theorem do not work. However, using "localization" techniques we get some extension of the Talagrand Theorem and this shows the importance of the "localization" point of view.

Concerning the inequality $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$ and connected with the subject of this paper, many and interesting results have been obtained. In particular, this paper is indebted and closely related to the papers [5] and [4].

The paper is organized as follows. In Section 2 we estimate distances to some spaces of 1-Baire functions. In Section 3 we apply these results to characterize when the fact $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$ -B being a boundary of the w^*-compact subset K- implies the existence inside K of a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$. In Section 4 we study the w^*-countably determined boundaries. Finally, in Section 5 we consider w^*-$\mathcal{K}$ analytic boundaries and the special boundary $B = Ext(K)$ and give extensions of Talagrand Theorem for these kind of boundaries.

Our notation is standard. If $(X, \|\cdot\|)$ is a real Banach space (we shall consider only Banach spaces over $\mathbb{R}$), let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. The weak*-topology of X^* is denoted by w^* and the weak topology of X by w. If A is a subset of X, then $[A]$ and $\overline{[A]}$ denote the linear hull and the closed linear hull of A, respectively. $\text{co}(A)$ denotes the convex hull of the set A, $\overline{\text{co}}(A)$ is the $\|\cdot\|$-closure of $\text{co}(A)$ and, if $A \subset X^*$, we put $\overline{\text{co}}^{w^*}(A)$ for the w^*-closure of $\text{co}(A)$. If C is a convex subset of X^*, for $x^* \in X^*$ and $A \subset X^*$, let $dist(x^*, C) = \inf\{\|x^* - c\| : c \in C\}$ be the distance from x^* to C and $dist(A, C) = \sup\{dist(a, C) : a \in A\}$ the distance from A to C. Observe that $dist(A, C) = dist(\text{co}(A), C) = dist(\overline{\text{co}}(A), C)$.

If A is a subset of X^*, let $\mathcal{S}eq(X^{**}, A)$ be the family of those elements $z \in X^{**}$ such that there exists a sequence $(x_n)_{n\geq 1} \subset X$ with $\langle a, x_n\rangle \to \langle z, a\rangle$, $\forall a \in A$. Clearly, $\mathcal{S}eq(X^{**}, A)$ is a subspace of X^{**}. We put $\mathcal{S}eq(X^{**})$ instead of $\mathcal{S}eq(X^{**}, X^*)$. $\mathcal{S}eq(X^{**})$ is a norm-closed subspace of X^{**} ([15]).

Let $X_c := \bigcup\{\overline{Y}^{w^*} \subset X^{**} : Y$ a separable subspace of $X\}$. It is easy to see that X_c is a norm-closed subspace of X^{**}. Observe that $B_0 := X_c \cap B(X^{**})$ is always a boundary of $B(X^{**})$.

If K is a Hausdorf compact space, $M(K)$ denotes the space of Radon Borel measures on K and $\mathcal{P}(K)$ the family of Radon Borel probabilities on K. If $k \in K$, δ_k will be the Dirac measure with mass 1 on k. If K is a w^*-compact subset of a dual Banach space X^* and μ a Radon Borel probability on K, $r(\mu)$ will denote the barycenter of μ.

2. Distances to the space of 1-Baire functions

If (T, τ) is a Hausdorf topological space, define $\mathcal{B}_1(T)$ and $\mathcal{B}_1^\epsilon(T)$ as follows:

(a) $\mathcal{B}_1(T)$ (resp., $\mathcal{B}_{1b}(T)$) will denote the family of 1-Baire real functions (resp., real bounded functions) on T. Recall that a function $f : T \to \mathbb{R}$ is said to be an 1-Baire function if there exists a sequence $\{f_n : n \geq 1\}$ in the space of real continuous functions $C(T)$ such that $f_n \to f$ pointwise on T. Observe that $\mathcal{B}_{1b}(T)$ is a closed subspace of $\ell_\infty(T)$.

(b) If $\epsilon \geq 0$ let $\mathcal{B}_1^\epsilon(K)$ (resp., $\mathcal{B}_{1b}^\epsilon(T)$) denote the family of functions (resp., bounded functions) $f : T \to \mathbb{R}$ such that for every $\eta > \epsilon$ and every non-empty subset $F \subset T$ there exists an open subset $V \subset T$ such that $V \cap F \neq \emptyset$ and $\mathrm{diam}(f(V \cap F)) \leq \eta$. $\mathcal{B}_{1b}^0(T)$ is a closed subspace of $\ell_\infty(T)$. In general, $\mathcal{B}_{1b}(T) \neq \mathcal{B}_{1b}^0(T)$ but, if (T, τ) is a complete metrizable space, then $\mathcal{B}_{1b}(T) = \mathcal{B}_{1b}^0(T)$ ([2, 1E, 1C]).

If K is a Hausdorf compact space and $\varphi \in \mathcal{B}_{1b}(K)$, then $\tilde{\varphi} : C(K)^* \to \mathbb{R}$ will be:

$$\forall \mu \in M(K),\ \tilde{\varphi}(\mu) := \int_K \varphi d\mu$$

Observe that: (i) $\tilde{\varphi} \in \mathcal{S}eq(C(K)^{**})$; (ii) the mapping $\mathcal{B}_{1b}(K) \ni \varphi \to \tilde{\varphi} \in C(K)^{**}$ is an isometric isomorphism between $\mathcal{B}_{1b}(K)$ -endowed with the supremum norm of $\ell_\infty(K)$- and $\mathcal{S}eq(C(K)^{**})$.

Lemma 2.1. *Let X be a Banach space, H a w^*-compact convex subset of X^*, $T : X \to C(H)$ be the continuous operator such that $Tx := x \restriction H$, $\forall x \in X$, and $\psi \in X^{**}$. We have:*

(A) If $\varphi \in \mathcal{B}_{1b}(H) \subset \ell_\infty(H)$, then $\|\psi \restriction H - \varphi\| \leq \|T^{**}\psi - \tilde{\varphi}\| \leq 3\|\psi \restriction H - \varphi\|$.

(B) $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) \leq dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) \leq 3dist(\psi \restriction H, \mathcal{B}_{1b}(H))$.

Proof. (A) First, observe that

$$\|\psi \restriction H - \varphi\| = \sup\{|(\psi - \varphi)(h)| : h \in H\} = \sup\{|\langle T^{**}\psi - \tilde{\varphi}, \delta_h\rangle| : h \in H\} \leq$$
$$\leq \sup\{\langle T^{**}\psi - \tilde{\varphi}, \mu\rangle : \mu \in B(C(H)^*)\} = \|T^{**}\psi - \tilde{\varphi}\|.$$

Let us see that $\|T^{**}\psi - \tilde{\varphi}\| \leq 3\|\psi \restriction H - \varphi\|$. For this we assume that $\|\psi \restriction H - \varphi\| < \frac{1}{2\eta}$ in $\ell_\infty(H)$ for some $\eta > 0$ and we are going to prove that $\|T^{**}\psi - \tilde{\varphi}\| < \frac{3}{2\eta}$. Choose $0 < \eta' < \eta$ such that $\|\psi \restriction H - \varphi\| < \frac{1}{2}\eta'$ in the supremum norm of $\ell_\infty(H)$. Thus $\psi \restriction H \in \mathcal{B}_{1b}^{\eta'}(H)$ because $\varphi \in \mathcal{B}_{1b}^0(H)$ and $\|\psi \restriction H - \varphi\| < \frac{1}{2}\eta'$.

Claim 1. $T^{**}\psi$ and $T^{**}\psi - \tilde{\varphi}$ belong to $\mathcal{B}_{1b}^{\eta'}((\mathcal{P}(H), w^*))$.

Indeed, observe that $T^*(\mathcal{P}(H)) = H$ because H is a convex w^*-compact subset. Let $A \subset \mathcal{P}(H)$ be a subset and $\epsilon > \eta'$. Since $\psi \upharpoonright H \in \mathcal{B}_{1b}^{\eta'}(H)$, there exists an open set V of X^* such that $V \cap T^*(A) \neq \emptyset$ and $diam(\psi(V \cap T^*(A))) \leq \epsilon$. So, if $W := T^{*-1}(V) \cap \mathcal{P}(H)$, then W is an open subset of $(\mathcal{P}(H), w^*)$ such that

$$W \cap A \neq \emptyset \quad \text{and} \quad diam\langle T^{**}\psi, W \cap A\rangle = diam\langle \psi, V \cap T^*(A)\rangle \leq \epsilon.$$

Thus $T^{**}\psi \in \mathcal{B}_{1b}^{\eta'}((\mathcal{P}(H), w^*))$ and so $T^{**}\psi - \tilde{\varphi} \in \mathcal{B}_{1b}^{\eta'}((\mathcal{P}(H), w^*))$ because $-\tilde{\varphi} \in \mathcal{B}_{1b}^{0}((\mathcal{P}(H), w^*))$ and $\mathcal{B}_{1b}^{\eta'}((\mathcal{P}(H), w^*)) + \mathcal{B}_{1b}^{0}((\mathcal{P}(H), w^*)) = \mathcal{B}_{1b}^{\eta'}((\mathcal{P}(H), w^*))$.

Claim 2. Let $\mathcal{P}_a(H)$ denote the family of purely atomic elements of $\mathcal{P}(H)$. Then for every $\mu \in \mathcal{P}_a(H)$ we have $|\langle T^{**}\psi - \tilde{\varphi}, \mu\rangle| < \frac{1}{2}\eta'$.

Indeed, if $\mu \in \mathcal{P}_a(H)$, then $\mu = \sum_{n\geq 1} \lambda_n \delta_{p_n}$, where $p_n \in H$, $\lambda_n \geq 0$, $\sum_{n\geq 1} \lambda_n = 1$, and δ_{p_n} is the Dirac probability with mass 1 on p_n. Since $|\psi(p_n) - \varphi(p_n)| < \frac{1}{2}\eta'$, $\forall n \geq 1$, we have:

$$|\langle T^{**}\psi - \tilde{\varphi}, \mu\rangle| = |\sum_{n\geq 1} \lambda_n(\psi(p_n) - \varphi(p_n))| \leq \sum_{n\geq 1} \lambda_n |\psi(p_n) - \varphi(p_n)| <$$
$$< \sum_{n\geq 1} \lambda_n \tfrac{1}{2}\eta' = \tfrac{1}{2}\eta'.$$

Assume that $\|T^{**}\psi - \tilde{\varphi}\| \geq \frac{3}{2}\eta$. Since ,$\forall z \in C(H)^{**}$, $\|z\| = \sup\{|\langle z, \mu\rangle| : \mu \in \mathcal{P}(H)\}$ (this is an easy exercise), there exists $\nu \in \mathcal{P}(H)$ and $d > 0$ such that $|\langle T^{**}\psi - \tilde{\varphi}, \nu\rangle| > \frac{3}{2}\eta' + d$. Without loss of generality, suppose that $\langle T^{**}\psi - \tilde{\varphi}, \nu\rangle > \frac{3}{2}\eta' + d$. In the sequel we use an argument due to Odell and Rosenthal (see [16, p. 380]). By the Radon-Nikodým theorem we can identify $L_1(\nu)$ with the subspace $\{\rho \in M(H) : \rho << \nu\}$ of $M(H)$ ($\rho << \nu$ means that ρ is absolutely continuous with respect to ν). Thus $T^{**}\psi - \tilde{\varphi} \in L_1(\nu)^* = L_\infty(\nu)$ and so there exists a Borel bounded function $\phi : H \to \mathbb{R}$ such that for every Radon measure $\rho << \nu$ we have

$$\langle T^{**}\psi - \tilde{\varphi}, \rho\rangle = \int_H \phi \frac{d\rho}{d\nu} d\nu = \int_H \phi d\rho, \tag{2.1}$$

whence

$$\frac{3}{2}\eta' + d < \langle T^{**}\psi - \tilde{\varphi}, \nu\rangle = \int_H \phi d\nu \leq \int_H \phi^+ d\nu.$$

Let $E := \{k \in H : \phi^+(k) \geq \frac{3}{2}\eta' + d\}$. Then $\nu(E) > 0$. Define $\mu \in \mathcal{P}(H)$ such that $\mu(B) := \frac{\nu(B\cap E)}{\nu(E)}$ for every Borel subset B of H. Clearly $\mu << \nu$. Let S be the support of μ, which is a compact subset of H such that $\mathcal{P}(S)$ is a convex compact subset of $\mathcal{P}(H)$. We have the following facts:

(i) Let $\mathcal{P}_\mu := \{\tau \in \mathcal{P}(H) : \tau << \mu\}$. Then $\mathcal{P}_\mu \subset \mathcal{P}(S)$ and, moreover, $\mathcal{P}_\mu$ is w^*-dense in $\mathcal{P}(S)$ (this is an easy exercise).

(ii) If $\rho \in \mathcal{P}_\mu$, then $\rho << \mu << \nu$ and $\rho({}^cE) = 0$. Thus by (2.1)

$$\langle T^{**}\psi - \tilde{\varphi}, \rho\rangle = \int_H \phi d\rho = \int_E \phi d\rho \geq \frac{3}{2}\eta' + d.$$

(iii) $\mathcal{P}_a(S)$ is clearly w^*-dense in $\mathcal{P}(S)$ and $\langle T^{**}\psi - \tilde{\varphi}, \rho\rangle < \frac{1}{2}\eta'$, $\forall \rho \in \mathcal{P}_a(S)$, by Claim 2.

Thus for every open subset V of $\mathcal{P}(H)$ with $V \cap \mathcal{P}(S) \neq \emptyset$ we have $V \cap \mathcal{P}_\mu \neq \emptyset \neq V \cap \mathcal{P}_a(S)$ and this implies

$$diam(\langle T^{**}\psi - \tilde{\varphi}, V \cap \mathcal{P}(S)\rangle) > \frac{3}{2}\eta' + d - \frac{1}{2}\eta' = \eta' + d.$$

Therefore $T^{**}\psi - \tilde{\varphi}$ does not belong to $\mathcal{B}_{1b}^{\eta'}((\mathcal{P}(H), w^*))$, a contradiction to Claim 1. Thus $\|T^{**}\psi - \tilde{\varphi}\| < \frac{3}{2}\eta$. Finally, we get $dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) < \frac{3}{2}\eta$, because $\tilde{\varphi} \in \mathcal{S}eq(C(H)^{**})$.

(B) follows immediately from (A).

□

Lemma 2.2. *Let X be a Banach space, $H \subset X^*$ a w^*-compact convex subset, $T : X \to C(H)$ such that $Tx := x \restriction H$, $\forall x \in X$, B a boundary of H, $w_0 \in H$, $\psi \in X^{**}$ and $d > 0$ be such that*

$$\langle \psi, w_0\rangle > \sup\langle \psi, \overline{\text{co}}(B)\rangle + d. \tag{2.2}$$

Then $dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) \geq \frac{1}{2}d$.

Proof. (A) First, we suppose that $\|\psi\| = 1$. Assume that

$$dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) < \tfrac{1}{2}d.$$

Then, there exist $\varphi \in \mathcal{B}_{1b}(H)$, a number d' with $0 < d' < d$ and a vector $e \in B(C(H)^{**})$ such that $T^{**}\psi = \tilde{\varphi} + \frac{d'}{2}e$ in $C(H)^{**}$. Let $r_0 := \sup\langle \psi, \overline{\text{co}}(B)\rangle$ and define

$$U := \{z \in B(X^{**}) : \langle z, w_0\rangle \geq r_0 + d\} \text{ and } V := \{x \in B(X) : \langle w_0, x\rangle \geq r_0 + d\}.$$

Clearly, $U = \overline{V}^{w^*}$ and $\psi \in U$. Let $T : X \to C(H)$ be the restriction operator such that $Tx := x \restriction H$, $\forall x \in X$. Then $T^{**}\psi \in T^{**}U = \overline{TV}^{w^*}$ and

$$\tilde{\varphi} = T^{**}\psi - \tfrac{d'}{2}e \in \overline{TV}^{w^*} + \tfrac{d'}{2}B(C(H)^{**}) = \overline{TV + \tfrac{d'}{2}B(C(H))}^{w^*}.$$

Since $\tilde{\varphi} \in \mathcal{S}eq(C(H)^{**})$, by [16, REMARK, p. 379] there exist sequences $\{x_n : n \geq 1\} \subset V$ and $\{f_n : n \geq 1\} \subset B(C(H))$ such that

$$T^{**}x_n + \tfrac{d'}{2}f_n \to \tilde{\varphi} \text{ in } (C(H)^{**}, w^*). \tag{2.3}$$

By the Simons equality [18, SUP-LIMSUP THEOREM] we have

$$\sup_{p\in B}\limsup_{n\to\infty}\langle p,x_n\rangle=\sup_{h\in H}\limsup_{n\to\infty}\langle h,x_n\rangle.$$

On the one hand, if $p\in B$ and δ_p is the Dirac probability with mass 1 on p, we have

$$\langle T^{**}x_n+\tfrac{d'}{2}f_n,\delta_p\rangle \underset{n\to\infty}{\to} \langle\tilde{\varphi},\delta_p\rangle=\varphi(p) \text{ by (2.3)},$$

whence we get

$$\limsup_{n\to\infty}\langle p,x_n\rangle=\limsup_{n\to\infty}\langle T^{**}x_n,\delta_p\rangle=\limsup_{n\to\infty}\big[\langle T^{**}x_n+\tfrac{d'}{2}f_n,\delta_p\rangle-\langle\tfrac{d'}{2}f_n,\delta_p\rangle\big]=$$
$$=\varphi(p)+\limsup_{n\to\infty}\big[-\langle\tfrac{d'}{2}f_n,\delta_p\rangle\big]\le\varphi(p)+\tfrac{d'}{2}=\psi(p)-\tfrac{d'}{2}\langle e,\delta_p\rangle+\tfrac{d'}{2}\le r_0+d'.$$

On the other hand, taking into account that $w_0\in H$ and that $x_n\in V$, we have:

$$\sup_{h\in H}\limsup_{n\to\infty}\langle h,x_n\rangle\ge\limsup_{n\to\infty}\langle w_0,x_n\rangle\ge r_0+d.$$

So, we conclude that $r_0+d'\ge r_0+d$, that is, $d'\ge d$, a contradiction, that completes the proof in this case (A).

(B) Let $\psi\in X^{**}$ be arbitrary (but $\psi\neq0$). From the inequality (2.2) we get $\langle\psi/\|\psi\|,w_0\rangle>\sup\langle\psi/\|\psi\|,\overline{\mathrm{co}}(B)\rangle+d/\|\psi\|$. Thus by (A) we obtain

$$dist(T^{**}(\psi/\|\psi\|),\mathcal{S}eq(C(H)^{**}))\ge\frac{d}{2\|\psi\|},$$

and finally $dist(T^{**}\psi,\mathcal{S}eq(C(H)^{**}))\ge\frac{1}{2}d$. □

Definition 2.3. *If X is a Banach space and K a w^*-compact subset of X^*, the B-index of K (in short, $Bindex(K)$), is defined as*

$$Bindex(K):=$$
$$=\sup\{dist(\overline{\mathrm{co}}^{w^*}(W),\overline{\mathrm{co}}(B)):W\subset K\ w^*\text{-compact and } B \text{ a boundary of } W\}.$$

Theorem 2.4. *Let X be a Banach space and $H\subset X^*$ a w^*-compact convex subset, B a boundary of H, $w_0\in H$, $\psi\in X^{**}$ and $d>0$ be such that $\langle\psi,w_0\rangle>\sup\langle\psi,\overline{\mathrm{co}}(B)\rangle+d$. We have*

(1) $dist(\psi\restriction H,\mathcal{B}_{1b}(H))\ge\frac{1}{6}d$ in $\ell_\infty(H)$ and so $dist(S(X^{**})\restriction H,\mathcal{B}_{1b}(H))\ge\frac{1}{6}Bindex(H)$ in $\ell_\infty(H)$.

(2) If $H\subset B(X^*)$ then $dist(\psi,Seq(X^{**}))\ge dist(\psi,Seq(X^{**},H))\ge\frac{d}{2}$.

Proof. (1) This follows from Lemma 2.1 and Lemma 2.2.

(2) Let $T:X\to C(H)$ be the restriction operator such that $Tx=x\restriction H$, $\forall x\in X$. Observe that $\|T\|\le1$ because $H\subset B(X^*)$.

Claim. $T^{**}(\mathcal{S}eq(X^{**},H))\subset\mathcal{S}eq(C(H)^{**})$.

Indeed, let $z \in \mathcal{S}eq(X^{**}, H)$. Then $z \upharpoonright H \in \mathcal{B}_{1b}(H)$ and from Lemma 2.1 we get

$$dist(T^{**}z, \mathcal{S}eq(C(H)^{**})) \leq 3dist(z \upharpoonright H, \mathcal{B}_{1b}(H)) = 0.$$

As $\mathcal{S}eq(C(H)^{**})$ is a closed subspace of $C(H)^{**}$ (by [15]), we conclude that $T^{**}z \in \mathcal{S}eq(C(H)^{**})$.

So, as $\|T\| \leq 1$, we get

$$dist(\psi, Seq(X^{**}, H)) \geq dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})).$$

Now an application of Lemma 2.2 gives that $dist(\psi, Seq(X^{**}, H)) \geq \frac{d}{2}$. Finally, the inequality $dist(\psi, Seq(X^{**})) \geq dist(\psi, Seq(X^{**}, H))$ is obvious because $Seq(X^{**})$ is a subspace of $Seq(X^{**}, H)$. □

Corollary 2.5. *Let X be a Banach space, $H \subset X^*$ a w^*-compact convex subset, $\psi \in X^{**}$ and $C \subset X$ a convex subset with $\psi \in \overline{C}^{w^*}$. The following are equivalent:*

(1) $\psi \upharpoonright H \in \mathcal{B}_{1b}(H)$.

(2) There exists a sequence $\{x_n : n \geq 1\} \subset C$ such that $x_n(h) \to \psi(h)$ for every $h \in H$.

Proof. As (2) $\Rightarrow$ (1) is obvious, we prove (1) $\Rightarrow$ (2). Let $T : X \to C(H)$ be the restriction operator $T(x) = x \upharpoonright H$. Since $\psi \upharpoonright H \in \mathcal{B}_{1b}(H)$, from Lemma 2.1 we get

$$dist(T^{**}\psi, \mathcal{S}eq(C(H)^{**})) \leq 3dist(\psi \upharpoonright H, \mathcal{B}_{1b}(H)) = 0.$$

As $\mathcal{S}eq(C(H)^{**})$ is a closed subspace of $C(H)^{**}$ (by [15]), we conclude that $T^{**}\psi \in \mathcal{S}eq(C(H)^{**})$. Finally, the implication (1) $\Rightarrow$ (2) follows from [16, REMARK, p. 379] because $T^{**}\psi \in \overline{T(C)}^{w^*}$. □

Corollary 2.6. *Let X be a Banach space, H a convex w^*-compact subset of X^* and $\psi \in X^{**}$. Then:*

(a) If $\psi \in \mathcal{S}eq(X^{**}, H)$, there exists a sequence $(x_n)_{n\geq 1} \subset X$ with $\|x_n\| \leq \|\psi\|$ such that $x_n \to \psi$ on H.

(b) $\psi \upharpoonright H \in \mathcal{B}_{1b}(H)$ if and only if $\psi \in \mathcal{S}eq(X^{**}, H)$.

Proof. (a) By hypothesis $\psi \upharpoonright H \in \mathcal{B}_{1b}(H)$ and $\psi \in \overline{\|\psi\|B(X)}^{w^*}$. Now it is enough to apply Corollary 2.5.

(b) This follows from Corollary 2.5 and the fact $\psi \in \overline{\|\psi\|B(X)}^{w^*}$. □

3. Localization of w^*-$\mathbb{N}$-families and copies of the basis of $\ell_1(\mathfrak{c})$

In this Section we deal with a very useful tool introduced in [10]: the w^*-$\mathbb{N}$-families. Let us define this notion, that will have a very important role in order to localize copies of the basis of $\ell_1(\mathfrak{c})$(see [10, Definition 3.3] and [11, Definition 2.1]).

Definition 3.1. *Let X be a Banach space. A bounded subset $\mathcal{F}$ of X^* is said to be a w^*-$\mathbb{N}$-family of width$(\mathcal{F}) \geq d > 0$ if $\mathcal{F}$ has the form*

$$\mathcal{F} = \{\eta_{M,N} : M, N \text{ disjoint subsets of } \mathbb{N}\},$$

and there are two sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$ such that for every pair of disjoint subsets M, N of $\mathbb{N}$ we have

$$\eta_{M,N}(x_m) \geq r_m + d, \ \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leq r_n, \ \forall n \in N.$$

The index $Width(Y)$ of a subset $Y \subset X^*$ is defined as follows ($(\sup\{\emptyset\} = 0)$):

$$Width(Y) :=$$
$$= \sup\{d > 0 : \text{ exists a } w^*\text{-}\mathbb{N}\text{-family } \mathcal{A} \subset Y \text{ such that } width(\mathcal{A}) \geq d\}.$$

Among the properties of the w^*-$\mathbb{N}$-families (see [10, Remark 3.4] and [11, Remark 2.2]), we highlight the following facts: (i) a w^*-$\mathbb{N}$-family $\mathcal{A}$ always contains a copy of the basis of $\ell_1(\mathfrak{c})$; (ii) the family $\{x_m : m \geq 1\} \subset B(X)$ associated with a w^*-$\mathbb{N}$-family $\mathcal{A}$ is equivalent to the basis of ℓ_1.

If K is a w^*-compact subset of a dual Banach space X^*, the inequality $\overline{\text{co}}(K) \neq \overline{\text{co}}^{w^*}(K)$ always implies that K contains a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$ (see [10, Lemma 3.2],[11, Proposition 2.5]). However, from the fact $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$, B being a mere boundary of K, we cannot localize, in general, inside K neither a w^*-$\mathbb{N}$-family nor a copy of the basis of $\ell_1(\mathfrak{c})$. Let us see some counterexamples.

Counterexample 1. The following counterexample shows that, if K is not w^*-metrizable, the fact $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$, B being a mere boundary of K, does not imply, in general, the localization inside K of a w^*-$\mathbb{N}$-family. Let I be an uncountable set, $X := c_0(I)$ and $B := \{e_i : i \in I\}$ be the canonical basis of $X^* = \ell_1(I)$. Clearly, B is a boundary of the w^*-compact subset $K := \overline{\{e_i : i \in I\}}^{w^*} = \{e_i : i \in I\} \cup \{0\}$. As $0 \notin \overline{\text{co}}(B)$ then $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$. However, K fails to have a w^*-$\mathbb{N}$-family because X does not have a copy of ℓ_1. Observe that B itself is a copy of the basis of $\ell_1(I)$. $\square$

Counterexample 2. In the following counterexample we show a Banach space X such that X^* has neither a w^*-$\mathbb{N}$-family nor a copy of ℓ_1, but there exist a w^*-compact subset K of X^* and a boundary B of K such that $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$. Let X be the long James space $J(\omega_1)$ and Y be its isometric predual (see [3, 7.7.4 Proposition, p. 348]). Then:

(i) Y and all its successive dual spaces are Asplund. So, $X^* = Y^{**} = J(\omega_1)^*$ has neither a copy of $\ell_1(\mathfrak{c})$ nor a w^*-$\mathbb{N}$-family.

(ii) Let $K := B(X^*)$ and $B_0 := Y_c \cap K$, where $Y_c := \cup\{\overline{[A]}^{w^*} : A \subset Y \text{ countable}\}$. It is easy to see that B_0 is a boundary of K such that $\overline{\text{co}}(B_0) \subset Y_c$.

(iii) With the notation of [3, p. 346], the vector e_{ω_1} satisfies $e_{\omega_1} \in B(X^*)$ but $e_{\omega_1} \notin Y_c$ and so $e_{\omega_1} \notin \overline{\text{co}}(B_0)$. In fact, if $A \subset Y$ is a countable family, there exists $\alpha_0 < \omega_1$ such that $A \subset \overline{[\{e_\alpha : \alpha \leq \alpha_0, \alpha \text{ a non limit ordinal}\}]}$. So, if $\alpha_0 < \beta < \omega_1$, the basic vector $h_\beta := \mathbf{1}_{(\beta,\omega_1]}$ of $Y^* = J(\omega_1)$ satisfies $\langle a, h_\beta \rangle = 0$, $\forall a \in A$, but $\langle e_{\omega_1}, h_\beta \rangle = 1$. $\square$

In spite of these counterexamples, in many cases the fact $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$ implies that K -and sometimes the boundary B itself- has a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$. Our approach to this problem consists of two steps:

Step 1. We suppose that K is w^*-metrizable. In this case K always contains a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$, if $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$.

Step 2. The general case. We obtain a characterization that actually reduces this case to the metrizable one.

The metrizables case.

In this case we suppose that K is a w^*-compact metrizable subset of a dual Banach space X^*, B a boundary of K such that $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$ and prove that K has a w^*-$\mathbb{N}$-family and a copy of the basis of $\ell_1(\mathfrak{c})$. Moreover we estimate the index $Width(K)$ in terms of the index $Bindex(K)$. Observe that K is w^*-metrizable iff $\overline{\text{co}}^{w^*}(K)$ is (an easy exercise).

Theorem 3.2. *Let X be a Banach space, $H \subset X^*$ a convex w^*-compact subset and B a boundary of H such that $dist(H, \overline{\text{co}}(B)) > d > 0$. If H is w^*-metrizable, H has a w^*-$\mathbb{N}$-family $\mathcal{A}$ of $width(\mathcal{A}) \geq \frac{d}{3}$ and a copy of the basis of $\ell_1(\mathfrak{c})$. Thus $Width(H) \geq \frac{1}{3} dist(H, \overline{\text{co}}(B))$.*

Proof. Since $dist(H, \overline{\text{co}}(B)) > d > 0$, there exist $w_0 \in H$ and $\psi \in S(X^{**})$ such that

$$\langle \psi, w_0 \rangle > \sup \langle \psi, \overline{\text{co}}(B) \rangle + d.$$

Thus, $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) > \frac{1}{6} d$ by Theorem 2.4. Since H is w^*-metrizable, $dist(\psi \restriction H, \mathcal{B}_{1b}(H)) = \frac{1}{2} Frag(\psi \restriction H, H)$ by [9, Proposition 6.4], where $Frag(\psi \restriction H, H)$ is the fragmentation index of $\psi \restriction H$ in H. Recall (see [9, p. 231]) that for a function $f : H \to \mathbb{R}$ *the fragmentation index* $Frag(f, H)$ is the infimum of the family of numbers $\epsilon \geq 0$ such that for every $\eta > \epsilon$ and every non-empty subset $F \subset H$, there exists an open subset $V \subset H$ such that $V \cap F \neq \emptyset$ and $diam(f(V \cap F)) \leq \eta$. It is clear that, $\forall \epsilon \geq 0$, $\mathcal{B}_{1b}^{\epsilon}(H) = \{f \in \ell_\infty(H) : Frag(f, H) \leq \epsilon\}$.

So, $Frag(\psi \restriction H, H) > d/3 > 0$, whence we get $\psi \restriction H \notin \mathcal{B}_{1b}^{d/3}(H)$. By [9, Proposition 6.1] there exist a non-empty w^*-compact subset $F \subset H$ and two real numbers $s < t$ with $t - s > d/3$ such that $\overline{F \cap \{\psi \leq s\}}^{w^*} = F = \overline{F \cap \{\psi \geq t\}}^{w^*}$. Thus there exist two real numbers s', t' with $s < s' < t' < t$ and $t' - s' > d/3$ such that every w^*-open subset $V \subset X^*$ with $V \cap F \neq \emptyset$ satisfies

$$\inf \langle \psi, V \cap F \rangle \leq s < s' < t' < t \leq \sup \langle \psi, V \cap F \rangle.$$

This fact implies that F (and so H) contains a w^*-$\mathbb{N}$-family $\mathcal{A}$ such that $width(\mathcal{A}) \geq d/3$ (see [10, proof of LEMMA 3.2] or [11, Proof of 2 $\Rightarrow$ 3]) and a copy of the basis of $\ell_1(\mathfrak{c})$. Finally, the inequality $Width(H) \geq \frac{1}{3} dist(H, \overline{\text{co}}(B))$ follows from the above results and the definition of $Width(H)$ (see Definition 3.1) $\square$

Let us see the quantitative connection between $Width(H)$ and $Bindex(H)$.

Corollary 3.3. *Let X be a Banach space and H a w^*-compact subset of X^*. Then*

(1) $Width(H) \leq Bindex(H)$.

(2) If H is w^*-metrizable, $Width(H) = 0$ if and only if $Bindex(H) = 0$.

(3) If H is convex and w^*-metrizable then $Width(H) \leq Bindex(H) \leq 3Width(H)$

Proof. (1) This follows from [11, Lemma 2.4].

(2) First, $Width(H) = 0$ whenever $Bindex(H) = 0$ by (1). Now we suppose that $Bindex(H) > 0$ and prove that $Width(H) > 0$. The fact $Bindex(H) > 0$ means that there exist a w^*-compact subset $W \subset H$ and a boundary B of W such that $dist(\overline{\text{co}}^{w^*}(W), \overline{\text{co}}(B)) > 0$. Thus $Width(\overline{\text{co}}^{w^*}(W)) > 0$ by Theorem 3.2. From [11, Proposition 2.5, Proposition 3.8] we get $Width(W) > 0$ and so $Width(H) > 0$, and this completes the proof of (2).

(3) follows from (1) and Theorem 3.2. □

The general case.

The general case can be reduced to the metrizable case as follows.

Definition 3.4. *If X is a Banach space and K a w^*-compact subset of X^*, we define the index $Bindex_c(K)$ of K as the supremum of $Bindex(i^*(K))$, where i^* is the adjoint operator of the canonical inclusion mapping $i : Y \to X$ and Y is a separable subspace of X.*

Remark. Let K be a w^*-compact subset of the dual Banach space X^*. If X is separable (or if K is w^*-metrizable), it is clear that $Bindex(K) \leq Bindex_c(K)$. But if X is non-separable we can have $Bindex(K) > 0$ and $Bindex_c(K) = 0$. This happens in the above Counterexamples.

Proposition 3.5. *Let X be a Banach space and H a w^*-compact subset of X^*. Then*

(A) $Width(H) \leq Bindex_c(H)$.

(B) If H is convex then $Width(H) \leq Bindex_c(H) \leq 3Width(H)$.

Proof. (A) Let $\mathcal{F} \subset H$ be a w^*-$\mathbb{N}$-family of $width(\mathcal{F}) > d > 0$ associated with the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(X)$. Let $Y := \overline{[\{x_m : m \geq 1\}]}$ and $i : Y \to X$ be the canonical inclusion mapping. Obviously, $i^*(\mathcal{F})$ is a w^*-$\mathbb{N}$-family of $i^*(H)$ such that $width(i^*(\mathcal{F})) > d > 0$ associated with the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_m : m \geq 1\} \subset B(Y)$. Thus by Corollary 3.3

$$d \leq Width(i^*(H)) \leq Bindex(i^*(H)) \leq Bindex_c(H).$$

Therefore, $Width(H) \leq Bindex_c(H)$.

(B) Suppose that H is convex. First, $Width(H) \leq Bindex_c(H)$ by (A). Now we assume that $Bindex_c(H) > d > 0$ and prove that $d/3 < Width(H)$. The fact $Bindex_c(H) > d > 0$ implies that there exists a separable closed subspace $Y \subset X$ such that $Bindex(i^*(H)) > d$, $i : Y \to X$ being the canonical inclusion mapping. Then $Width(i^*(H)) > d/3$ by Corollary 3.3 and so there exists in $i^*(H)$ a w^*-$\mathbb{N}$-family $\mathcal{A}'$ of $width(\mathcal{A}') > d/3$ associated with certain sequences $\{y_n : n \geq 1\} \subset B(Y)$ and $\{r_n : n \geq 1\} \subset \mathbb{R}$. For each $a \in \mathcal{A}'$ choose $k_a \in H$ such that $i^*(k_a) = a$. Then $\mathcal{A} := \{k_a : a \in \mathcal{A}'\}$ is a w^*-$\mathbb{N}$-family of $width(\mathcal{A}) > d/3$ associated with the sequences $\{i(y_n) : n \geq 1\} \subset B(X)$ and $\{r_n : n \geq 1\} \subset \mathbb{R}$. Thus $Width(H) > d/3$ and so $3Width(H) \geq Bindex_c(H)$. □

Corollary 3.6. *Let X be a Banach space and K a w^*-compact subset of X^*. The following are equivalent:*

(1) $Width(\overline{\mathrm{co}}^{w^*}(K)) = 0$; (1') $Bindex_c(\overline{\mathrm{co}}^{w^*}(K)) = 0$.

(2) $Width(K) = 0$; (2') $Bindex_c(K) = 0$.

Proof. (1) $\Leftrightarrow$ (1′) and (2′) $\Rightarrow$ (2) follow from Proposition 3.5. (1) $\Leftrightarrow$ (2) is proved in [11, Prop. 2.5, Prop. 3.8]. Finally, (1′) $\Rightarrow$ (2′) is obvious. □

The γ topology of a dual Banach space X^* is the topology of X^* of the convergence on countable bounded subsets of X. The topology γ has been used by Cascales, Muñoz, Orihuela, etc., in several papers (see [4],[5]). It is easy to see that $(X^*, \gamma)^* = X_c$.

Proposition 3.7. *Let K be a w^*-compact subset of the dual Banach space X^*. The following are equivalent:*

(1) $Bindex_c(K) = 0$.

(2) $\overline{\mathrm{co}}^{\gamma}(B) = \overline{\mathrm{co}}^{w^*}(H)$ for every w^*-compact subset H of K and every boundary B of H.

Proof. (1) $\Rightarrow$ (2). Suppose that $\overline{\mathrm{co}}^{\gamma}(B) \neq \overline{\mathrm{co}}^{w^*}(H)$ for some w^*-compact subset H of K and some boundary B of H. This means that there exists a point $w_0 \in \overline{\mathrm{co}}^{w^*}(H)$ that can be strictly separated from $\mathrm{co}(B)$ using elements of X_c. Precisely, there exist a separable closed subspace $Y \subset X$, a vector $\psi \in S(Y^{**}) = S(\overline{Y}^{w^*})$ and a positive number $d > 0$ such that

$$\langle \psi, w_0 \rangle > \sup \langle \psi, \mathrm{co}(B) \rangle + d.$$

So, if $i : Y \to X$ is the canonical inclusion mapping, then

$$\langle \psi, i^*(w_0) \rangle > \sup \langle \psi, \mathrm{co}(i^*(B)) \rangle + d. \tag{3.1}$$

As $i^*(w_0) \in \overline{\mathrm{co}}^{w^*}(i^*(H))$, from (3.1) we get $dist(\overline{\mathrm{co}}^{w^*}(i^*(H)), \overline{\mathrm{co}}(i^*(B))) > d > 0$. Hence $Bindex(i^*(H)) > d$ because $i^*(B)$ is a boundary of $i^*(H)$. Thus $Bindex_c(K) > d$, a contradiction which proves the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Suppose that $Bindex_c(K) > 0$. Then, by definition of $Bindex_c(K)$, there exists a closed separable subspace $Y \subset X$ such that $Bindex(i^*(K)) > 0$,

where $i : Y \to X$ is the canonical inclusion mapping. From Corollary 3.3 we get that $Width(i^*(K)) > 0$. Hence, by [11, Lemma 2.4] there exists a w^*-compact subset $L \subset i^*(K)$ such that $\overline{\text{co}}(L) \neq \overline{\text{co}}^{w^*}(L)$. So there exist $\psi \in S(Y^{**}) = S(\overline{Y}^{w^*})$, $d > 0$ and $v_0 \in \overline{\text{co}}^{w^*}(L)$ such that

$$\langle \psi, v_0 \rangle > \sup \langle \psi, \text{co}(L) \rangle + d. \tag{3.2}$$

Let $W := i^{*-1}(L) \cap K$.

Claim. $\overline{\text{co}}^{\gamma}(W) \neq \overline{\text{co}}^{w^*}(W)$.

Indeed, let $w_0 \in \overline{\text{co}}^{w^*}(W)$ be such that $i^*(w_0) = v_0$. Then, taking into account that $\text{co}(L) = i^*(\text{co}(W))$, $\psi = i^{**}\psi$ and (3.2), we get

$$\begin{aligned}\langle \psi, w_0 \rangle = \langle i^{**}\psi, w_0 \rangle = \langle \psi, i^*(w_0) \rangle = \langle \psi, v_0 \rangle > \sup \langle \psi, i^*(\text{co}(W)) \rangle + d = \\ = \sup \langle \psi, \text{co}(W) \rangle + d.\end{aligned}$$

As $\psi \in X_c$, we conclude that $w_0 \in \overline{\text{co}}^{w^*}(W) \setminus \overline{\text{co}}^{\gamma}(W)$, and this contradicts the hypothesis and completes the proof. $\square$

Remark. In [5, Theorem 5.4] it is proved that X fails to have a copy of ℓ_1 iff $\overline{\text{co}}^{\gamma}(B) = \overline{\text{co}}^{w^*}(K)$, for every w^*-compact subset K of X^* and every boundary B of K. The above Proposition 3.7 implies [5, Theorem 5.4]. Indeed,

$$\begin{aligned}&\overline{\text{co}}^{\gamma}(B) = \overline{\text{co}}^{w^*}(H)\ \forall H \subset X^* \ \ w^*\text{-compact subset and every boundary } B \text{ of } H \\ &\overset{Prop.\ 3.7}{\Leftrightarrow}\ Bindex_c(K) = 0\ \forall K \subset X^* \ \ w^*\text{-compact subset} \\ &\overset{Cor.\ 3.6}{\Leftrightarrow}\ Width(K) = 0\ \forall K \subset X^* \ \ w^*\text{-compact subset} \\ &\overset{Def.\ 3.1}{\Leftrightarrow}\ X \ \text{ does not have a copy of } \ell_1.\end{aligned}$$

4. w^*-countably determined boundaries

In [4] it is proved that a Banach space X fails to have a copy of ℓ_1 if and only if $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$ for every w^*-compact subset of X^* and every w^*-$\mathcal{K}$ analytic boundary B of K. In this Section we give a "localized " version of this result and show that it also holds with "w^*-$\mathcal{CD}$" instead of "$w^*\mathcal{KA}$ " .

Let us recall that, if (T, τ) is a topological space, $\Sigma' \subset \Sigma := \mathbb{N}^{\mathbb{N}}$ and $\Phi : \Sigma' \to 2^T$ is a set valued mapping, Φ is said to be usco if ,$\forall \sigma \in \Sigma'$, $\Phi(\sigma)$ is a compact non-empty subset of T and Φ is *upper-semicontinuous*, that is, for each $\sigma \in \Sigma'$ and for an open subset U of T such that $\Phi(\sigma) \subset U$ there exists a neighborhood G of σ with $\Phi(G) \subset U$. A subset $Y \subset T$ is said to be *countably determined* (in short, $\mathcal{CD}$) in (T, τ) if there exists a subset $\Sigma' \subset \Sigma := \mathbb{N}^{\mathbb{N}}$ and a set-valued usco mapping $\Phi : \Sigma' \to 2^T$ such that $Y = \bigcup_{\sigma \in \Sigma'} \Phi(\sigma)$ (see [17, p. 11]). When $\Sigma' = \Sigma$, Y is said to be $\mathcal{K}$-*analytic* (in short, $\mathcal{KA}$) in (T, τ).

Lemma 4.1. *Let X be a Banach space, $\emptyset \neq \Sigma' \subset \Sigma := \mathbb{N}^{\mathbb{N}}$ and $\Phi : \Sigma' \to 2^{X^*}$ a set-valued usco mapping. Define*

$$C := \bigcup \{\overline{\text{co}}^{w^*}(\Phi(F)) : F \textit{ finite subset of } \Sigma'\}.$$

Then C is a convex subset such that $\overline{C} = \overline{C}^{\gamma}$.

Proof. The proof can be done as in [4, Proposition 5.5]. Actually, Proposition 5.5 of [4] holds for any w^*-usco mapping $B : \Sigma' \to 2^{X^*}$ and any subset $\Sigma' \subset \mathbb{N}^{\mathbb{N}}$. □

Proposition 4.2. *Let X be a Banach space, H a w^*-compact subset of X^* and B a $w^*\mathcal{CD}$ boundary of H fulfilling $dist(\overline{\text{co}}^{w^*}(H), \overline{\text{co}}(B)) > d > 0$ and $\overline{\text{co}}^{\gamma}(B) = \overline{\text{co}}^{w^*}(H)$. Then there exists in B a w^*-$\mathbb{N}$-family $\mathcal{A}$ of $width(\mathcal{A}) > d > 0$ and a copy of the basis of $\ell_1(\mathfrak{c})$. Therefore, $Width(B) \geq dist(\overline{\text{co}}^{w^*}(H), \overline{\text{co}}(B))$ in this case.*

Proof. As $dist(\overline{\text{co}}^{w^*}(H), \overline{\text{co}}(B)) > d > 0$, there exists $w_0 \in \overline{\text{co}}^{w^*}(H)$ such that $dist(w_0, \overline{\text{co}}(B)) > d > 0$. Since B is $w^*\mathcal{CD}$ by hypothesis, there exist a subset $\Sigma' \subset \Sigma := \mathbb{N}^{\mathbb{N}}$ and an usco mapping $\Phi : \Sigma' \to 2^{X^*}$ such that $B = \bigcup_{\sigma \in \Sigma'} \Phi(\sigma)$. By Lemma 4.1, if

$$C := \bigcup \{\overline{\text{co}}^{w^*}(\Phi(F)) : F \text{ a finite subset of } \Sigma'\},$$

then C is convex and $\overline{C} = \overline{C}^{\gamma}$. Hence $\overline{C} = \overline{\text{co}}^{w^*}(H)$, because $\overline{\text{co}}^{\gamma}(B) = \overline{\text{co}}^{w^*}(H)$ and $B \subset C$. Thus, given $0 < \epsilon < dist(w_0, \overline{\text{co}}(B)) - d$, there exists a finite subset $F \subset \Sigma'$ such that $dist(w_0, \overline{\text{co}}^{w^*}(\Phi(F))) < \epsilon$. Let $v_0 \in \overline{\text{co}}^{w^*}(\Phi(F))$ be such that $\|w_0 - v_0\| < \epsilon$. Then, if K is the w^*-compact subset $K := \Phi(F)$, we have

$$dist(v_0, \overline{\text{co}}(K)) \geq$$
$$\geq dist(w_0, \overline{\text{co}}(K)) - \|w_0 - v_0\| \geq dist(w_0, \overline{\text{co}}(B)) - \|w_0 - v_0\| > d > 0.$$

As $v_0 \in \overline{\text{co}}^{w^*}(K)$, by [10, Lemma 3.2] there exist in K, and so in B, a w^*-$\mathbb{N}$-family $\mathcal{A}$ of width $\geq d$ and a copy of the basis of $\ell_1(\mathfrak{c})$. As d can be taken arbitrarily close to $dist(\overline{\text{co}}^{w^*}(H), \overline{\text{co}}(B))$, we finally get $Width(B) \geq dist(\overline{\text{co}}^{w^*}(H), \overline{\text{co}}(B))$. □

Let us say that a subset Y of a dual Banach space X^* is *a Pettis set* or Y *has the property* (P) (in short, $Y \in (P)$) if $\overline{\text{co}}(K) = \overline{\text{co}}^{w^*}(K)$ for every w^*-compact subset K of Y (see [11],[20, p. 79]). Recall that: (i) by [12] X^* has the property (P) iff X fails to have a copy of ℓ_1; (ii) by [11, Proposition 2.5] a w^*-compact subset $K \subset X^*$ satisfies $K \in (P)$ iff $Width(K) = 0$, that is, K does not contains a w^*-$\mathbb{N}$-family. The notion of w^*-compact Pettis set was also considered in [5], under the name of $P(B(D))$-set with $D = B(X)$, but viewed from a perspective different from ours; this paper does not deal with "localization" problems inside K.

Theorem 4.3. *Let X be a Banach space and Y a subset of X^*. The following are equivalent:*

(A) $Y \in (P)$.

(B) For every w^*-compact subset K of Y and every boundary B of K we have $\overline{\text{co}}^{\gamma}(B) = \overline{\text{co}}^{w^*}(K)$.

(C) For every w^*-compact subset K of Y and every $w^*\mathcal{CD}$ boundary B of K we have $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$.

(D) For every w^*-compact subset K of Y and every $w^*\mathcal{KA}$ boundary B of K we have $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$.

Proof. (A)$\Leftrightarrow$(B) follows from Corollary 3.6 and Proposition 3.7.

(A)+(B) $\Rightarrow$ (C) follows from Proposition 4.2.

(C) $\Rightarrow$ (D) is clear because every $w^*\mathcal{KA}$ subset is $w^*\mathcal{CD}$.

(D) $\Rightarrow$ (A). It is enough to observe that every w^*-compact subset K of X^* is a $w^*\mathcal{KA}$ subset and also a boundary of K. $\square$

Remarks. (1) The equivalence of points $(A), (B)$ and (D) of Proposition 4.3, when $Y = X^*$, is the result obtained in [5, Theorem 5.4] and [4, Theorem 5.6].

(2) The above Counterexample 1 shows that, if K is a w^*-compact subset of a dual Banach space X^* with $K \in (P)$ and B is a non-w^*-$\mathcal{CD}$ boundary of K, it can be $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$. Note that the boundary $B = \{e_i : i \in I\}$ of Counterexample 1 is a non-$w^*\mathcal{CD}$ set, because B is an uncountable discrete space (so, it is not Lindelof) and every $\mathcal{CD}$ space is Lindelof.

(3) When the w^*-compact subset $K \subset X^*$ is w^*-metrizable, every subset of K is w^*-$\mathcal{CD}$ and Theorem 4.3 asserts, in this case, that $K \in (P)$ iff $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(W)$ for every w^*-compact subset $W \subset K$ and every boundary B of W. Note that this statement coincides with that obtained in Corollary 3.3.

5. Applications to some special boundaries

When K is a w^*-compact subset of X^* and B is either a $w^*\mathcal{KA}$ boundary of K or $B := Ext(K)$, we obtain better results than in the general case. First, we see a series of lemmas.

Lemma 5.1. *(Talagrand [19]) Let τ be a cardinal with cofinality cf$(\tau) > \aleph_0$, X a Banach space and A a subset of X. The following are equivalent*

(1) A has a copy of the basis of $\ell_1(\tau)$.

(2) $\overline{\text{co}}(A)$ has a copy of the basis of $\ell_1(\tau)$.

(3) $\overline{[A]}$ has a copy of $\ell_1(\tau)$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Let $E := \overline{[A]}$ and $T : \ell_1(\tau) \to E$ be an isomorphism between $\ell_1(\tau)$ and $T(\ell_1(\tau))$. The adjoint operator $T^* : E^* \to \ell_\infty(\tau)$ is a quotient mapping w^*-w^*-continuous. Let $0 < \eta$ be such that $\eta B(\ell_\infty(\tau)) \subset T^*(B(E^*))$ and $W := T^{*-1}(B(\ell_\infty(\tau))) \cap \frac{1}{\eta}B(E^*)$. Clearly, we can suppose that W is the

unit closed ball of E^* for certain dual norm $|||\cdot|||$ equivalent to the given norm. It is obvious that $T^*(B((E^*, |||\cdot|||))) = T^*(W) = B(\ell_\infty(\tau)) = [-1,1]^\tau$. By [19, Theorem 4] we conclude that A has a copy of the basis of $\ell_1(\tau)$. $\square$

Recall that a regular Hausdorff space T is angelic if: (i) every relatively countably compact subset $W \subset T$ is relatively compact; (ii) the closure of a relatively compact subset $W \subset T$ is precisely the set of limits of its sequences.

Lemma 5.2. *Let X be a separable Banach space and E be a norm-closed $w^*\mathcal{KA}$ subspace of X^* such that $E \in (P)$. If $w_1^* = \sigma(E^*, E)$ then $(B(E^*), w_1^*)$ is angelic.*

Proof. First, observe that $(B(E), w^*)$ is analytic because it is metrizable and a $w^*\mathcal{KA}$ set (it is w^*-closed in (E, w^*)) (see [17, Theorem 5.5.1]). Since $E \in (P)$, then E and so the unit closed ball $B(E)$ fail to have a w^*-$\mathbb{N}$-family by [11, Prop. 3.8]. Let $i : E \to X^*$ be the inclusion mapping and $A := i^*(B(X)) \subset B(E^*)$. Then:

(i) Clearly, $\overline{A}^{w_1^*} = B(E^*)$, where $w_1^* := \sigma(E^*, E)$.

(ii) $(B(E^*), w_1^*)$ is a compact subset of $\mathbb{R}^{B(E)}$.

(iii) The space of 1-Baire functions $(\mathcal{B}_1(B(E), w^*), \tau_p)$, τ_p being the topology on $\mathcal{B}_1(B(E), w^*)$ of pointwise convergence on $B(E)$, is a topological subspace os $\mathbb{R}^{B(E)}$ such that $A \subset \mathcal{B}_1(B(E), w^*)$.

Since $B(E)$ fails to have a w^*-$\mathbb{N}$-family, if $\alpha < \beta$ and $(x_n)_{n\geq 1}$ is a sequence in $B(X)$, there is a subset $I \subset \mathbb{N}$ such that

$$\{t \in B(E) : \langle x_n, t\rangle \leq \alpha, \forall n \in I, \langle x_m, t\rangle \geq \beta, \forall m \in \mathbb{N} \setminus I\} = \emptyset.$$

So, by [2, 4G. Corollary] we obtain that A is relatively compact in the space of 1-Baire functions $(\mathcal{B}_1(B(E), w^*), \tau_p)$. Thus $(\overline{A}^{\tau_p}, \tau_p) = (\overline{A}^{w_1^*}, w_1^*) = (B(E^*), w_1^*)$ is a compact subset of $(\mathcal{B}_1(B(E), w^*), \tau_p)$. Since $(B(E), w^*)$ is analytic, $(\mathcal{B}_1(B(E), w^*), \tau_p)$ is angelic by [2, 3G. Corollary]. Thus $(B(E^*), w_1^*)$ is angelic. $\square$

Lemma 5.3. *Let X be a separable Banach space, K be a w^*-compact subset of X^* containing a w^*-$\mathbb{N}$-family and B a $w^*\mathcal{KA}$ boundary of K. Then B contains a w^*-$\mathbb{N}$-family.*

Proof. Suppose that B fails to contain a w^*-$\mathbb{N}$-family and let $E := \overline{[B]}$. Clearly, E is a $w^*\mathcal{KA}$ subspace of X^* such that $E \in (P)$ and so E fails to contain a w^*-$\mathbb{N}$-family (see [11, Lemma 3.7, Proposition 3.8]). Then $(B(E^*), \sigma(E^*, E))$ is angelic by Lemma 5.2. Thus $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$ by [7, Theorem I.2] and so E contains a w^*-$\mathbb{N}$-family, a contradiction that proves the statement. $\square$

Lemma 5.4. *Let K be a w^*-compact subset of a dual Banach space X^* and B a boundary of K such that, for every continuous operator $T : \ell_1 \to X$,*

$T^*(B)$ *contains a* $w^*\mathcal{KA}$ *boundary of* $T^*(K)$*. Then the following statements are equivalent:*

(1) K *contains a* w^**-*$\mathbb{N}$*-family; (2)* B *contains a* w^**-*$\mathbb{N}$*-family.*

Proof. As (2) $\Rightarrow$ (1) is obvious, we prove (1) $\Rightarrow$ (2). Let $\mathcal{F}$ be a w^*-$\mathbb{N}$-family inside K of $width(\mathcal{F}) \geq d > 0$ associated with the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{x_n : n \geq 1\} \subset B(X)$. Let $T : \ell_1 \to X$ be the continuous operator such that $T(e_n) = x_n$, $\forall n \geq 1$, where $\{e_n : n \geq 1\}$ is the canonical basis of ℓ_1. Clearly, T is an isomorphism between ℓ_1 and $T(\ell_1)$ such that $T^*(\mathcal{F})$ is a w^*-$\mathbb{N}$-family inside the w^*-compact subset $T^*(K)$ of $width(T^*(\mathcal{F})) \geq d > 0$ associated with the sequences $\{r_m : m \geq 1\} \subset \mathbb{R}$ and $\{e_n : n \geq 1\} \subset B(\ell_1)$. By hypothesis $T^*(B)$ contains a $w^*\mathcal{KA}$ boundary B_0 of $T^*(K)$. By Lemma 5.3 the boundary B_0 contains a w^*-$\mathbb{N}$-family $\mathcal{A}$ of $width(\mathcal{A}) \geq \delta > 0$ associated with certain sequences $\{s_m : m \geq 1\} \subset \mathbb{R}$ and $\{u_n : n \geq 1\} \subset B(\ell_1)$. For each $a \in \mathcal{A}$ we find $v_a \in B$ such that $T^*(v_a) = a$. Now it is easy to see that $\mathcal{H} := \{v_a : a \in \mathcal{A}\}$ is a w^*-$\mathbb{N}$-family inside B of $width(\mathcal{H}) \geq \delta > 0$, associated with the sequences $\{s_m : m \geq 1\} \subset \mathbb{R}$ and $\{T(u_n) : n \geq 1\} \subset B(X)$. $\square$

Now we can give a version of the above Talagrand Theorem, for $\tau = \mathfrak{c}$, the w^*-topology of X^* and either $B = Ext(K)$ or B a $w^*\mathcal{KA}$ boundary.

Theorem 5.5. *Let* X *be a Banach space and* K *a* w^**-compact subset of* X^**. Let* $B \subset K$ *be either a* $w^*\mathcal{KA}$ *boundary of* K *or* $B = Ext(K)$*. Then*

(A) K *contains a* w^**-*$\mathbb{N}$*-family if and only if* B *contains a* w^**-*$\mathbb{N}$*-family.*

(B) K *contains a copy of the basis of* $\ell_1(\mathfrak{c})$ *if and only if* B *does.*

Proof. (A) It is enough to see that B satisfies the requirements of Lemma 5.4. So, let $T : \ell_1 \to X$ be a continuous operator. If B is a $w^*\mathcal{KA}$ boundary of K, then it is easy to see that $T^*(B)$ is a $w^*\mathcal{KA}$ boundary of $T^*(K)$ and so we are done in this case. Suppose that $B = Ext(K)$. Then $Ext(T^*(K)) \subset T^*(Ext(K))$. As $(T^*(K), w^*)$ is a metrizable compact set (because $(B(\ell_\infty), w^*)$ is), $Ext(T^*(K))$ is a $\mathcal{G}_\delta$ subset of $T^*(K)$ ([6, 27.3 Corollary]) and so a w^*-analytic subset. Thus we can apply Lemma 5.4 in this case.

(B) We prove that B contains a copy of the basis of $\ell_1(\mathfrak{c})$ when K does. We consider two cases, namely:

Case 1. $\overline{\text{co}}(B) = \overline{\text{co}}^{w^*}(K)$. The cardinal $\mathfrak{c}$ satisfies $\text{cf}(\mathfrak{c}) > \aleph_0$ because $\text{cf}(2^\alpha) > \alpha$ for every infinite cardinal α (see [13, p. 78]) and because $\mathfrak{c} = 2^{\aleph_0}$. Thus, we can apply Lemma 5.1 and so there exists a copy of the basis of $\ell_1(\mathfrak{c})$ inside B.

Case 2. $\overline{\text{co}}(B) \neq \overline{\text{co}}^{w^*}(K)$. First, assume that B is a $w^*\mathcal{KA}$ boundary of K. By Proposition 4.3 and [11, Proposition 2.5] there exists a w^*-$\mathbb{N}$-family inside K and so inside B by part (A). Thus B contains a copy of the basis of $\ell_1(\mathfrak{c})$ because every w^*-$\mathbb{N}$-family does.

Now assume that $B = Ext(K)$. By the proof of [12, 3.1.Proposition] there exist $\varphi \in S(X^{**})$, a non-empty subset $S \subset Ext(K)$ and two real numbers r, δ with $\delta > 0$ such that, if V is a w^*-open subset of X^* with $V \cap S \neq \emptyset$, then we can find two vectors $\xi, \eta \in \overline{\text{co}}^{w^*}(V \cap S)$ such that $\varphi(\xi) > r + \delta$ and $\varphi(\eta) < r$. So, K contains a w^*-$\mathbb{N}$-family (see the proof of [10, LEMMA 3.2]). Thus, B contains a w^*-$\mathbb{N}$-family by part (A). Finally, B contains a copy of the basis of $\ell_1(\mathfrak{c})$ because every w^*-$\mathbb{N}$-family does. □

Corollary 5.6. *Let K be a w^*-compact subset of the dual Banach space X^*. The following are equivalent:*

(1) K fails to have a w^*-$\mathbb{N}$-family, that is, $K \in (P)$.

(2) $\overline{\text{co}}(Ext(W)) = \overline{\text{co}}^{w^*}(W)$ for every w^*-compact subset W of K.

(3) $Ext(K)$ fails to have a w^*-$\mathbb{N}$-family.

Proof. (1) $\Rightarrow$ (2). Let $W \subset K$ a w^*-compact subset. Then $W \in (P)$ because $k \in (P)$. Thus $\overline{\text{co}}^{\gamma}(Ext(W)) = \overline{\text{co}}^{w^*}(W)$ by Theorem 4.3. Finally $\overline{\text{co}}$ □

In [8] Godefroy and Talagrand study and characterize the *representable and universally representable* Banach spaces. A Banach space X is said to be *representable* if X is isomorphic to a $w^*\mathcal{KA}$ subspace of ℓ_∞. A Banach space X is said to be *universally representable* if X is representable and every subspace Y of ℓ_∞ isomorphic to X is a $w^*\mathcal{KA}$ subset of ℓ_∞. The Lemma 5.2 allows us to extend [8, Théorème 6] in the following way.

Theorem 5.7. *Let Y be a separable Banach space and X a $w^*\mathcal{KA}$ closed subspace of the dual Y^*. The following statements are equivalent:*

(a) X is universally representable.

(b) X fails to have a copy of $\ell_1(\mathfrak{c})$.

(c) $(B(X^*), \sigma(X^*, X))$ is an angelic space.

(d) X is universally (P), that is, if Z is a subspace of the dual Banach space V^* and Z is isomorphic to X, then Z fulfills the property (P) inside V^*.

(e) $X \in (P)$ inside Y^*.

Proof. The equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ is the Théorème 6 of [8].

$(b) \Rightarrow (d)$. Suppose that X fails to have a copy of $\ell_1(\mathfrak{c})$. Then, if Z is isomorphic to X and also a subspace of certain dual Banach space V^*, $Z \in (P)$ inside V^* because Z does not have a w^*-$\mathbb{N}$-family in V^*.

$(d) \Rightarrow (e)$ is obvious and $(e) \Rightarrow (c)$ follows from Lemma 5.2. □

For a Banach space X, let $NA(X)$ denote the subset of elements of the dual X^* which attain their norm on $B(X)$. The following Proposition 5.8 and Proposition 5.9 generalize Lemma 2.10 and Proposition 2.14 of [1], respectively.

Proposition 5.8. *Let X be a Banach space, $J : X \to X^{**}$ the canonical inclusion, M a closed subspace of X^* and $i : M \to X^*$ be the canonical inclusion mapping such that $i^* \circ J(B(X))$ contains* *(i) either a $w^*\mathcal{KA}$ boundary B of $B(M^*)$; (ii) or $Ext(B(M^*))$.*

Then:

(A) If X does not have a copy of $\ell_1(\mathfrak{c})$, $i^* \circ J : X \to M^*$ is a canonical quotient.

(B) If M is infinite dimensional, there is an infinite dimensional quotient space of X which is isomorphic to a dual space.

Proof. (A) Let $B \subset i^* \circ J(B(X))$ be either a $w^*\mathcal{KA}$ boundary of $B(M^*)$ or $B = Ext(B(M^*))$. Clearly B does not to have a copy of the basis of $\ell_1(\mathfrak{c})$ because X does not.

Claim. $\overline{\text{co}}(B) = B(M^*)$.

Indeed, suppose that $\overline{\text{co}}(B) \neq B(M^*)$. Then:

(i) Assume that B is a $w^*\mathcal{KA}$ boundary of $B(M^*)$. By Proposition 4.3 and [11, Proposition 2.5] there exists a w^*-$\mathbb{N}$-family inside $B(M^*)$ and so inside B by Proposition 5.5. Thus, B contains a basis of $\ell_1(\mathfrak{c})$, a contradiction.

(ii) Assume that $B = Ext(B(M^*))$.

Por la Proposición 4.3 existe una w^*-$\mathbb{N}$-familia dentro de $B(M^*)$. Por la Proposición 5.5 también existe una w^*-$\mathbb{N}$-familia dentro de B. En consecuencia, B posee una copia de la base de $\ell_1(\mathfrak{c})$.

(b) Sea $B = Ext(B(M^*))$. Por la Proposición 5.6 el conjunto B posee una w^*-$\mathbb{N}$-familia y, por tanto, una copia de la base de $\ell_1(\mathfrak{c})$.

Hemos llegado a una contradicción que prueba el Aserto.

Finalmente, observemos que el hecho $\overline{\text{co}}(B) = B(M^*)$ implica que $i^* \circ J : X \to M^*$ es un 1-cociente. Let $B \subset i^* \circ J(B(X))$ be a $w^*\mathcal{KA}$ boundary of $B(M^*)$. Clearly, B does not have a basis of $\ell_1(\mathfrak{c})$, because X fails to have a copy of $\ell_1(\mathfrak{c})$.

Claim. $\overline{\text{co}}(B) = B(M^*)$.

Indeed, assume that $\overline{\text{co}}(B) \neq B(M^*)$. By Proposition 4.3 and [11, Proposition 2.5] there exists a w^*-$\mathbb{N}$-family inside $B(M^*)$ and so inside B by Proposition 5.5. Thus, B contains a basis of $\ell_1(\mathfrak{c})$, a contradiction which proves the Claim.

Finally, observe that the fact $\overline{\text{co}}(B) = B(M^*)$ implies that $i^* \circ J : X \to M^*$ is a canonical quotient. □

Proposition 5.9. *Let X be a Banach space, $J : X \to X^{**}$ the canonical inclusion and M an infinite dimensional closed subspace of $NA(X)$ such that $i^* \circ J(B(X))$ contains a $w^*\mathcal{KA}$ boundary of $B(M^*)$, $i : M \to X^*$ being the canonical inclusion mapping. Then there is an infinite dimensional quotient space of X which is isomorphic to a dual space.*

Proof. If X does not have a copy of $\ell_1(\mathfrak{c})$, we apply Proposition 5.8. If X contains a copy of $\ell_1(\mathfrak{c})$, then ℓ_∞ is a quotient of X. □

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