Groups and topologies related to *D*-sequences

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p-adic topologies.

Definition

Let p be a prime number. The family

$$\{p^n\mathbb{Z} : n \in \mathbb{N}_0\}$$

is a neighborhood basis for a group topology on $\mathbb{Z}$. This topology is called the p-adic topology, $\lambda_{\mathbf{p}}$.

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is a neighborhood basis for a group topology on $\mathbb{Z}$. This topology is called the p-adic topology, $\lambda_{\mathbf{p}}$.

Properties

- $\lambda_{\mathbf{p}}$ *is metrizable.*
- $\lambda_{\mathbf{p}}$ *is precompact.*
- $\lambda_{\mathbf{p}}$ *is linear.*
- $\lambda_{\mathbf{p}}$ *is locally quasi-convex.*
- $(\mathbb{Z}, \lambda_{\mathbf{p}})^{\wedge} = \mathbb{Z}(\mathbf{p}^{\infty})$.

D-sequences.

Definition

Let $\mathbf{b} = (b_n)_{n \in \mathbb{N}_0} \subset \mathbb{N}$, satisfying that:

- $b_0 = 1$.
- $b_n \mid b_{n+1}$.
- $b_n \neq b_{n+1}$.

Then $\mathbf{b}$ is a D-sequence. Let $\mathscr{D}$ be the family of all D-sequences. For a D-sequence we define the sequence of ratios $q_n := \frac{b_n}{b_{n-1}}$.

Example

- $b_n = p^n$ is a D-sequence and $q_n = p$ for all n.
- $b_n = (n+1)!$ is a D-sequence and $q_n = n+1$ for all n.
- Let q_n be a sequence of natural numbers satisfying $q_n \neq 1$ for all n. Then $b_n := \prod_{i=1}^{n} q_i$ is a D-sequence.

Proposition

Let $\mathbf{b}$ *be a D-sequence. Suppose that* $q_{j+1} \neq 2$ *for infinitely many* j*. For each integer number* $L \in \mathbb{Z}$*, there exists a natural number* $N = N(L)$ *and* ***unique*** *integers* $k_0, \ldots, k_N$*, such that:*

(1) $L = \sum_{j=0}^{N} k_j b_j$.

(2) $k_j \in \left(-\frac{q_{j+1}}{2}, \frac{q_{j+1}}{2}\right]$, *for* $0 \leq j \leq N$.

D-sequences (3)

Proposition

Let $\mathbf{b}$ *be a* D*-sequence. Then any* $y \in \mathbb{R}$ *can be written* ***uniquely*** *in the form*

$$y = \sum_{n=0}^{\infty} \frac{\beta_n}{b_n} = \frac{\beta_0}{b_0} + \frac{\beta_1}{b_1} + \frac{\beta_2}{b_2} + \cdots + \frac{\beta_s}{b_s} + \cdots, \tag{1}$$

where $\beta_n \in \mathbb{Z}$ *and* $|\beta_{n+1}| \leq \frac{q_{n+1}}{2}$*. Further,*

$$-\frac{1}{2b_n} < y - \sum_{j=0}^{n} \frac{\beta_j}{b_j} \leq \frac{1}{2b_n}$$

holds for all $n \in \mathbb{N}_0$*.*

Definition

The sequence

$$(k_0, k_1, \cdots, k_{N(L)}, 0, \dots) \in \prod_{n=1}^{\infty} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right],$$

will be called the **b-coordinates of** L.

The sequence

$$(\beta_0, \beta_1, \dots) \in \mathbb{Z} \times \prod_{n=1}^{\infty} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$$

will be called the **b-coordinates of** y.

D-sequences (5)

- $\mathscr{D} := \{\mathbf{b} : \mathbf{b} \text{ is a } D\text{-sequence}\}$
- $\mathscr{D}_\infty := \{\mathbf{b} \in \mathscr{D} : \frac{b_{n+1}}{b_n} \to \infty\}$.
- $\mathscr{D}_\infty^\ell := \{\mathbf{b} \in \mathscr{D} : \frac{b_{n+\ell}}{b_n} \to \infty\}$.
- $\mathscr{D}_\infty(\mathbf{b}) := \{\mathbf{c} \sqsubset \mathbf{b} : \mathbf{c} \in \mathscr{D}_\infty\}$.
- $\mathscr{D}_\infty^\ell(\mathbf{b}) := \{\mathbf{c} \sqsubset \mathbf{b} : \mathbf{c} \in \mathscr{D}_\infty^\ell\}$.
- $\mathbf{b}$ has bounded ratios if there exists L such that $q_n < L$ for all n.
- $\mathbf{b}$ is basic if q_n is a prime number for all n.
- $\mathbb{Z}(\mathbf{b}^\infty) := \bigcup_{n \in \mathbb{N}_0} \left\langle \frac{1}{b_n} + \mathbb{Z} \right\rangle \leq \mathbb{T}$.
- $\mathbb{Z}_{\mathbf{b}} := \prod_{n \in \mathbb{N}} \left(\left(-\frac{q_n}{2}, \frac{q_n}{2} \right] \cap \mathbb{Z} \right)$.

b-adic topologies.

Definition

Let **b** be a *D*-sequence. The family

$$\{b_n\mathbb{Z} : n \in \mathbb{N}_0\}$$

is a neighborhood basis for a group topology on $\mathbb{Z}$. This topology is called the **b**-adic topology, $\lambda_{\mathbf{b}}$.

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Properties

- $\lambda_{\mathbf{b}}$ *is metrizable.*
- $\lambda_{\mathbf{b}}$ *is precompact.*
- $\lambda_{\mathbf{b}}$ *is linear.*
- $\lambda_{\mathbf{b}}$ *is locally quasi-convex.*
- $(\mathbb{Z}, \lambda_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

In (Chasco et. al, 1999), the authors state that any locally quasi-convex group topology is the topology of uniform convergence on certain subsets of the dual. In order to define a topology of uniform convergence on $\mathbb{Z}$ we choose a family that is formed by only one subset which is precisely the range of a sequence $\underline{b}$ in $\mathbb{T}$.

Proposition

Let $\mathbf{b}$ *be a D-sequence. Fix* $\underline{\mathbf{b}} := \{\frac{1}{b_n} + \mathbb{Z} : n \in \mathbb{N}_0\} \subset \mathbb{T}$*. Define* $V_{\mathbf{b},m} := \left\{z \in \mathbb{Z} : \frac{z}{b_n} + \mathbb{Z} \in \mathbb{T}_m \text{ for all } n \in \mathbb{N}_0\right\}$ *and* $\mathscr{V}_{\mathbf{b}} := \{V_{\mathbf{b},m} : m \in \mathbb{N}\}$*. Then* $\mathscr{V}_{\mathbf{b}}$ *is a neighborhood basis for the topology of uniform convergence on* $\underline{\mathbf{b}}$ *in the group of the integers. We call this topology* $\tau_{\mathbf{b}}$*.*

Basic properties of $\tau_{\mathbf{b}}$.

Properties

Let $\mathbf{b}$ *be a D-sequence. Then:*

- $\tau_{\mathbf{b}}$ *is a metrizable topology.*
- $V_{\mathbf{b},m} = \bigcap_{\chi \in \underline{\mathbf{b}}} \chi^{-1}(\mathbb{T}_m)$. *Hence* $\tau_{\mathbf{b}}$ *is locally quasi-convex.*
- $\mathbb{Z}(\mathbf{b}^\infty) = \langle \underline{\mathbf{b}} \rangle \leq (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$.

Theorem

Let $\mathbf{b}$ *be a D-sequence such that* $\mathbf{b} \in \mathscr{D}_\infty^\ell$. *Then* $(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

The topology $\delta_{\mathbf{b}}$.

Definition

Let $\mathbf{b}$ be a D-sequence. Define

$$\delta_{\mathbf{b}} := \sup\{\tau_{\mathbf{c}} : \mathbf{c} \in \mathscr{D}^{\ell}_{\infty}(\mathbf{b})\}.$$

Definition

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$$\delta_{\mathbf{b}} := \sup\{\tau_{\mathbf{c}} : \mathbf{c} \in \mathscr{D}^{\ell}_{\infty}(\mathbf{b})\}.$$

Properties

- $\delta_{\mathbf{b}}$ *is locally quasi-convex.*
- $(\mathbb{Z}, \delta_{\mathbf{b}})^{\wedge} = \mathbb{Z}(\mathbf{b}^{\infty})$.

$\delta_{\mathbf{b}}$ if **b** has bounded ratios.

Theorem

Let **b** *be a basic D-sequence with bounded ratios and let* $(x_n) \subset \mathbb{Z}$ *be a non-quasiconstant sequence such that* $x_n \xrightarrow{\lambda_{\mathbf{b}}} 0$*.*
Then there exists a metrizable locally quasi-convex compatible group topology $\tau (= \tau_{\mathbf{c}}$ *for some subsequence* **c** *of* **b**$)$ *on* $\mathbb{Z}$ *satisfying:*
(a) τ *is compatible with* $\lambda_{\mathbf{b}}$*.*
(b) $x_n \stackrel{\tau}{\not\to} 0$*.*
(c) $\lambda_b < \tau$*.*

Theorem

Let **b** *a basic D-sequence with bounded ratios. Then the topology* $\delta_{\mathbf{b}}$ *has* ***no*** *non-trivial convergent sequences. Hence, it cannot be non-discrete metrizable. Since* $(\mathbb{Z}, \delta_{\mathbf{b}})^\wedge \neq \mathbb{T}$*, it is non-discrete.*

$\delta_{\mathbf{b}}$ if $\mathbf{b}$ has bounded ratios. (2)

Remark

If $\mathbf{b}$ *is a basic D-sequence with bounded ratios, then* $\mathscr{D}^k_\infty(\mathbf{b}) \neq \mathscr{D}^{k+1}_\infty(\mathbf{b})$*.*

Remark

If $\mathbf{b}$ *is a basic D-sequence with bounded ratios, then* $\tau_{\mathbf{b}}$ *is discrete.*

$\delta_{\mathbf{b}}$ if $\mathbf{b} \in \mathscr{D}_\infty^\ell$.

Proposition

Let $\mathbf{b}$ *be a basic D-sequence such that* $\mathbf{b} \in \mathscr{D}_\infty^\ell$. *The following facts can be easily proved:*

- *The sequence* $\mathbf{b}$ *has unbounded ratios.*
- *We have that* $\mathscr{D}_\infty^k(\mathbf{b}) = \mathscr{D}_\infty^\ell(\mathbf{b})$, *for any natural number* $k \geq \ell$.
- *We have* $\delta_{\mathbf{b}} = \tau_{\mathbf{b}}$.
- *The topology* $\delta_{\mathbf{b}}$ *is metrizable. Hence, it has non-trivial convergent sequences.*

Proposition

Let $\mathbf{b}, \mathbf{c}$ *be the D-sequences defined as follows:*

$$\bullet b_{3n+1} = 2 \cdot b_{3n} \quad \bullet b_{3n+2} = p_{n+1} b_{3n+1} \quad \bullet b_{3n+3} = 3 \cdot b_{3n+2}$$

and

$$\bullet c_{3n+1} = 3 \cdot c_{3n} \quad \bullet c_{3n+2} = p_{n+1} c_{3n+1} \quad \bullet c_{3n+3} = 2 \cdot c_{3n+2},$$

where p_n *is the n-th prime number.*

Then $\mathbb{Z}(\mathbf{b}^\infty) = \mathbb{Z}(\mathbf{c}^\infty)$ *and* $\delta_{\mathbf{b}} \neq \delta_{\mathbf{c}}$.

The usual topology on $\mathbb{Z}(\mathbf{b}^\infty)$.

Notation

We shall denote by τ_U the topology in $\mathbb{Z}(\mathbf{b}^\infty)$ inherited from the one of the complex plane.

Proposition (Aussenhofer, Chasco)

Let H be a dense and metrizable subgroup of a topological group G. Then $G^\wedge = H^\wedge$.

Corollary

$(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge$ *is isomorphic to the discrete group of the integers.*

The usual topology on $\mathbb{Z}(\mathbf{b}^\infty)$.

Proposition

Let $(x_n+\mathbb{Z}) \subset \mathbb{Z}(\mathbf{b}^\infty)$. *Write* $x_n = \sum_{k\geq 1} \frac{\beta_k^{(n)}}{b_k}$. *Then the following assertions are equivalent:*

1. $x_n+\mathbb{Z} \to 0+\mathbb{Z}$ *in* τ_U.
2. $|x_n| \to 0$ *in* $\mathbb{R}$.
3. *For any* $k \in \mathbb{Z}$ *there exists* n_k *such that* $\beta_1^{(n)} = \cdots = \beta_k^{(n)} = 0$ *if* $n \geq n_k$.

$\mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T})$.

- We want to find a topology of uniform convergence on $\mathbb{Z}(\mathbf{b}^\infty)$.
- To that end we need a subset of $\mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T})$; i. e, of $\mathbb{Z}_{\mathbf{b}}$, the group of **b**-adic integers.
- Describe the action of an element of $\mathbb{Z}_{\mathbf{b}}$ on $\mathbb{Z}(\mathbf{b}^\infty)$. Let $k = \sum_{n\in\mathbb{N}} k_n b_n \in \mathbb{Z}_{\mathbf{b}}$. We define $\chi_k : \mathbb{Z}(\mathbf{b}^\infty) \to \mathbb{Z}$ where
$$\chi(x) := xk + \mathbb{Z} = \lim_{N\to\infty} \sum_{n=0}^{N} k_n b_n x + \mathbb{Z}.$$
- Since $x \in \mathbb{Z}(\mathbf{b}^\infty)$ implies that $\beta_n = 0$ for $n \geq n_0$ for some n_0 we have that $xb_n \in \mathbb{Z}$ if $n \geq n_0$ and $\lim_{N\to\infty} \sum_{n=0}^{N} k_n b_n x + \mathbb{Z}$ stabilizes.

The topology $\eta_{\mathbf{b}}$ on $\mathbb{Z}(\mathbf{b}^\infty)$.

Definition

Let **b** be a D-sequence we define
$W_{\mathbf{b},m} := \{x + \mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty) : xb_n + \mathbb{Z} \in \mathbb{T}_m \text{ for all } n\}$.

Proposition

- *The family* $\{W_{\mathbf{b},m}\}_{m\in\mathbb{N}}$ *is a neighborhood basis for a locally quasi-convex group topology on* $\mathbb{Z}(\mathbf{b}^\infty)$. *We will call this topology* $\eta_{\mathbf{b}}$.
- $x \in W_{\mathbf{b},m}$ *if and only if* $\left| \sum_{k=n+1}^{\infty} \frac{b_n \beta_k}{b_k} \right| \leq \frac{1}{4m}$ *for all* $n \in \mathbb{N}$.
- *If* $|\beta_k| \leq \frac{q_k}{8m}$ *for all* $k \in \mathbb{N}$ *then* $x = \sum_{k\geq 1} \frac{\beta_k}{b_k} \in W_{\mathbf{b},m}$.
- *If* $x \in W_{\mathbf{b},m}$ *then* $|\beta_k| \leq \frac{3q_k}{8m}$ *for all* $k \in \mathbb{N}$.

Convergence on $\eta_{\mathbf{b}}$.

Proposition

Let $x_n + \mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$. Write $x_n = \sum_{k\geq 1} \frac{\beta_k^{(n)}}{b_k}$. Then, the following assertions are equivalent:

1. $x_n + \mathbb{Z} \to 0 + \mathbb{Z}$ *in* $\eta_{\mathbf{b}}$.
2. *For all $m \in \mathbb{N}$ there exists n_m such that* $\left|\beta_k^{(n)}\right| \leq \frac{q_k}{8m}$ *if* $n \geq n_m$.

Comparing τ_U and $\eta_{\mathbf{b}}$

Corollary

- *Let* $\mathbf{b} \in \mathscr{D}_\infty$*. Then* $\frac{1}{b_n} + \mathbb{Z} \to 0 + \mathbb{Z}$ *in* $\eta_{\mathbf{b}}$*.*
- *There exists L such that* $q_n \leq L$ *for all n if and only if the topology* $\eta_{\mathbf{b}}$ *in* $\mathbb{Z}(\mathbf{b}^\infty)$ *is discrete.*

Proposition

$\tau_U < \eta_{\mathbf{b}}$.

Theorems on duality.

Theorem

If $\mathbf{b} \in \mathscr{D}_\infty$ *then* $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge = (\mathbb{Z}(\mathbf{b}^\infty), \eta_{\mathbf{b}})^\wedge$.

Theorem

If $\mathbf{b} \in \mathscr{D}$ *then* $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)$ *is not a Mackey group.*

Theorem

The group $\mathbb{Q}$ *of rational numbers endowed with the usual topology is not a Mackey group.*

The topology $\Lambda_{\mathbf{b}}$.

Definition

Let **b** be a *D*-sequence. Define

$$U_n := \{\mathbf{x} \in \mathbb{Z}_{\mathbf{b}} : x_0 = \cdots = x_{n-1} = 0\}$$

for $n \geq 1$. Put $U_0 = \mathbb{Z}_{\mathbf{b}}$.

Proposition

Let **b** *be a D-sequence. Then the family* $\{U_n\}_{n \in \mathbb{N}_0}$ *is a neighborhood basis of* 0 *for a linear metrizable group topology on* $\mathbb{Z}_{\mathbf{b}}$*. We will denote this topology by* $\Lambda_{\mathbf{b}}$*.*

Theorem

Let **b** *be a D-sequence. Then* $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ *is the completion (up to isomorphism) of* $(\mathbb{Z}, \lambda_{\mathbf{b}})$*.*

Thank you for your attention.