Splittings and cross-sections in topological groups

Madrid, December 3-4, 2015

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(Based on a joint work with M. J. Chasco, X. Dominguez and M. Tkachenko)

Throughout this work I will prove that every extension of topological abelian groups of the form $0 \to K \to X \to A(Y) \to 0$ splits when K is compact and $A(Y)$ is a free abelian topological group group generated by a zero-dimensional k_ω-space Y.

This result is related with the splitting problem. The splitting problem consists in finding conditions on two topological abelian groups G and H so that every extension of topological abelian groups of the form $0 \to H \to X \to G \to 0$ splits.

Contents

- We will work on topological abelian groups
- A topological abelian group G is zero-dimensional if there exists a system of neighbourhoods of the neutral element consisting of clopen sets.
- A topological space X is k_ω if it can be represented as the direct limit of an increasing sequence of compact spaces.

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Extensions

- *An extension* of topological abelian groups:
  $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ short exact sequence [i, π relatively open continuous homomorphisms] E *splits* it is equivalent to the trivial extension i. e. if there is a topological isomorphism $T : X \to H \times G$ making commutative the diagram

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\begin{tikzcd}
 & & X \arrow[dd, "T"] \arrow[dr, "\pi"] & & \\
0 \arrow[r] & H \arrow[ur, "i"] \arrow[dr, "\imath_H"'] & & G \arrow[r] & 0 \\
 & & H \times G \arrow[ur, "\pi_G"'] & &
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Free abelian topological groups

- Let Y be a topological space. The free abelian topological group $A(Y)$ is the unique topological abelian group containing Y such that, given any topological abelian group H and any continuous map $f : Y \to H$, there exist a continuous homomorphism $\widetilde{f} : A(Y) \to H$ extending f i. e. making commutative the following diagram:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "f"] & A(Y) \arrow[dl, "\widetilde{f}"] \\ H & \end{tikzcd}$$

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- It is known that free abelian groups are projective in the category of abelian groups i. e. every extension of abelian groups $0 \to H \to X \to G \to 0$ splits if G is free.
- Consequently, it is natural to ask ourselves in which conditions every extension of topological abelian groups of the form $0 \to H \to X \to A(Y) \to 0$ splits when $A(Y)$ is a free abelian topological group.

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The following lemma contains the technical part of the proof of the main result

Lemma

Let $\pi : X \to G$ *be a continuous open and onto homomorphism such that* $\ker \pi$ *is compact and metrizable. Suppose that a subspace* $Y \subset G$ *is zero-dimensional and* k_ω*-space, then there exists an embedding* $s : Y \hookrightarrow X$ *satisfying* $\pi \circ s = Id_Y$ *(this function is called a continuous cross-section)*

$$\begin{tikzcd} X \arrow[r, "\pi"] & G \\ & Y \arrow[u, hook] \arrow[ul, bend left, "s"] \end{tikzcd}$$

Proof.

Since Y is a k_ω-space, we can represent it as the direct limit of an increasing sequence $\{Y_n : n \in \mathbb{N}\}$ of compact subspaces. Consider $X_n = \pi^{-1}(Y_n)$.

Step 1. For every $n \in \mathbb{N}$ call $X_n = \pi^{-1}(Y_n) \subset X$ and construct a continuous map $s_n : X_n \to Y_n$ such that $\pi \circ s_n = \mathrm{Id}_{Y_n}$.

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[d, "P"'] \arrow[drr, "j"] & & G \\ M & & G \times M \arrow[ll, "p_M"] \arrow[u, "p_G"'] \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[r, "\pi_{|X_n}"] \arrow[dr, "j_{|X_n}"'] & Y_n \arrow[d, bend left, "t_n"] \\ & j(X_n) \arrow[u, "p_G"] \end{tikzcd}$$

$$s_n = j^{-1} \circ t_n$$

Step 2. Consider the limit $s = \lim_{\rightarrow} s_n$. Since $s_{|X_n} = s_n$, for every $n \in \mathbb{N}$, we obtain that $\pi \circ s = \mathrm{Id}_Y$.

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Main Result

Theorem

Let K be a compact abelian group and $A(Y)$ be the free abelian topological group on a zero-dimensional k_ω-space Y. Then every extension of topological abelian groups of the form $0 \to K \to X \xrightarrow{\pi} A(Y) \to 0$ splits.

Proof.

Step 1. *Find a continuous map* $s: Y \to X$ *such that* $\pi \circ s = Id_Y$

How?

First, we represent X as the limit of an inverse system $\mathcal{P} = \{X_\alpha, \pi_{\alpha,\beta} : X_\alpha \to X_\beta : \beta < \alpha < \tau\}$ such that $X_0 = A(Y)$ and $\ker \pi_{\alpha+1,\alpha}$ is compact and metrizable for every α.

We are going to use the previous Lemma to construct a family of embeddings $\{s_\alpha : Y \hookrightarrow X_\alpha : \alpha < \kappa\}$ such that $\pi_{\alpha+1,\alpha} \circ s_{\alpha+1} = s_\alpha$.

$$X_0 = A(Y) \longleftarrow \cdots \xleftarrow{\pi_{\alpha,\alpha-1}} X_\alpha \xleftarrow{\pi_{\alpha+1,\alpha}} X_{\alpha+1} \longleftarrow \cdots \longleftarrow \lim_{\leftarrow} \mathcal{P} = X$$

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From the definition of free abelian topological group follows that $s : Y \to X$ can be extended to a continuous homomorphism $S : A(Y) \to X$ such that $S_{|Y} = s$. It follows trivially that $\pi \circ S = \mathrm{Id}_{A(Y)}$.

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Thank you for your attention!