The Baire property on precompact abelian groups

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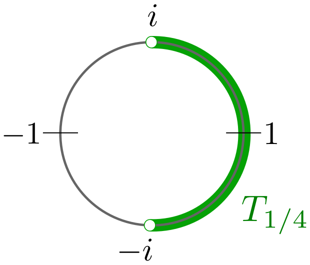
Some notations and preliminary facts

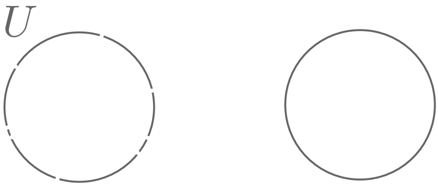
$\mathbb{T}$ is the subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$ formed by all complex numbers with modulus 1, and endowed with the usual topology.

We consider on $\mathbb{T}$ the arc-length group norm ρ, normalized in such a way that $\rho(-1) = 1/2$. For every $\varepsilon > 0$ we denote by T_ε the neighborhood of 1 defined by $\{t \in \mathbb{T} : \rho(t) < \varepsilon\}$.



Some notations and preliminary facts

Let U be an open, dense subset of $\mathbb{T}$.



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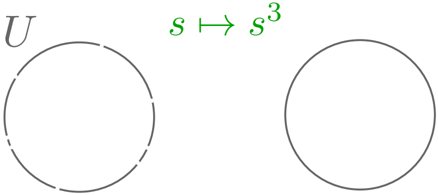

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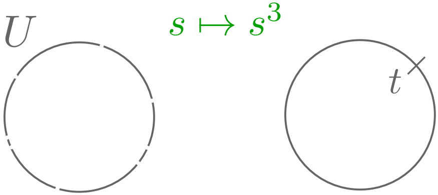

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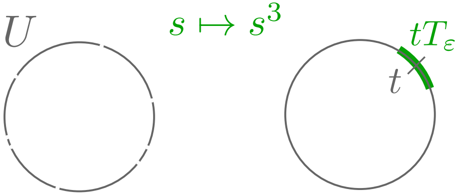

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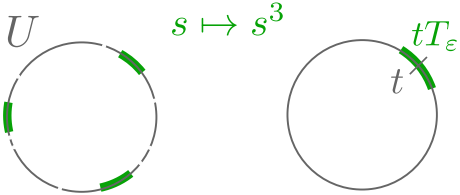

For every $k \in \mathbb{N}$, there exists a basic neighborhood tT_ε such that U contains the set of all kth roots of tT_ε.

Some notations and preliminary facts

If G is an abelian group, we call any element of $\mathrm{Hom}(G, \mathbb{T})$ a *character* of G.

Let (G, τ) be a Hausdorff *topological* abelian group. We denote by $(G, \tau)^\wedge$ the subgroup of $\mathrm{Hom}(G, \mathbb{T})$ formed by all τ-continuous characters of G. We say that (G, τ) is MAP if the elements of $(G, \tau)^\wedge$ separate the points of G.

Some notations and preliminary facts

We say that (G, τ) is *precompact* if G can be covered by finitely many translates of any neighborhood of zero. Equivalently, if the completion $\varrho(G, \tau)$ of (G, τ) is a compact group.

We say that (G, τ) is *pseudocompact* if every τ-continuous real function defined on G is bounded. Equivalently, if (G, τ) is precompact and G_δ-dense in its completion.

Some notations and preliminary facts

A *duality* of the abelian groups G and H is defined by a bihomomorphism $\langle \cdot, \cdot \rangle : G \times H \to \mathbb{T}$. We will consider *separated* dualities: for every $g \in G \setminus \{0\}$ and every $h \in H \setminus \{0\}$ the characters $\langle g, \cdot \rangle$ and $\langle \cdot, h \rangle$ are not identically 1.

Given any duality $\langle G, H \rangle$ the inverse duality $\langle H, G \rangle$ is defined in the obvious way.

We denote by $\sigma(G, H)$ the initial topology on G with respect to all characters of the form $\langle \cdot, h \rangle$ where $h \in H$.

A basis of neighborhoods of 0 for $\sigma(G, H)$ is given by the sets $\{g \in G : \langle g, \Delta \rangle \subset T_\varepsilon\}$ where Δ runs over all finite subsets of H and $\varepsilon > 0$.

$\sigma(G, H)$ is a Hausdorff, precompact group topology, and $(G, \sigma(G, H))^\wedge = H$ in the natural way.

Some notations and preliminary facts

If (G, τ) is MAP, there is a natural duality $\langle G, (G, \tau)^\wedge \rangle$. It turns out that (G, τ) is precompact if and only if $\tau = \sigma(G, (G, \tau)^\wedge)$.

Similarly, $\sigma((G, \tau)^\wedge, G)$ is the topology on $(G, \tau)^\wedge$ of pointwise convergence on the elements of G.

Some notations and preliminary facts

The duality $\langle G, H\rangle$ is *bounded* if for some $m \in \mathbb{N}$ we have $mG = \{0\}$, equivalently $mH = \{0\}$.

If $\langle G, H\rangle$ is bounded, a basis of neighborhoods of 0 for $\sigma(G, H)$ is given by the *subgroups* $\{g \in G : \langle g, \Delta\rangle = \{1\}\} =: \Delta^{\perp}$ where Δ runs over all finite subsets of H.

Baire spaces

A topological space X *has the Baire property,* or *is a Baire space*, if the intersection of any countable family of open dense subsets of X is dense in X.

Equivalently, if the only open subset in X which is expressable as a countable union of nowhere dense subsets of X is the empty set.

Every locally compact space has the Baire property.

Every completely metrizable space has the Baire property.

Baire groups

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A topological group G *has the Baire property,* or *is a Baire group*, if the intersection of any countable family of open dense subsets of G is dense in G. *It suffices that the intersection of any countable family of open dense subsets of G is nonempty.*

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The weaker sufficient conditions are consequences of Banach's Category Theorem.

Baire groups

The Baire property plays a role in

- the open mapping/closed graph theorems
- joint continuity of bi-homomorphisms
- Klee's theorem on complete metrics
- Mackey-type properties for topological abelian groups
- ...

Some known facts on precompact, Baire groups

- Every pseudocompact group is a Baire space.
- If G is infinite abelian, then $(G, \sigma(G, \mathrm{Hom}(G, \mathbb{T})))$ is *not* a Baire space.
- The class of precompact Baire groups is closed with respect to taking continuous homomorphic images and arbitrary direct products (M. Bruguera, M. Tkachenko, 2012).
- If the precompact group $(G, \sigma(G, H))$ is Baire, all convergent sequences in $(H, \sigma(H, G))$ are stationary (same reference).

A characterization of the Baire property

Let $\langle G, H\rangle$ be a bounded duality of abelian groups. Fix $g \in G$ and a finite subset Δ of H. The set $g + \Delta^{\perp}$ is an open $\sigma(G,H)-$neighborhood of g. (It is the set of all $g' \in G$ which agree with g on Δ.)

Fix a sequence $\{g_n\}$ in G and a sequence $\{\Delta_n\}$ of finite subsets of H. Consider the sets $\bigcup_{k\geq n} g_k + \Delta_k^{\perp}$, where $n \in \mathbb{N}$. They are clearly $\sigma(G,H)-$open.

Moreover they are $\sigma(G,H)-$dense in G if $\langle\Delta_n\rangle \cap \langle\Delta_k\rangle = \{0\}$ whenever $n \neq k$.
(We will check this later.)

So, if $(G, \sigma(G,H))$ is a Baire group, then

$\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} g_k + \Delta_k^{\perp}$ is nonempty: there is some g such that $g \in g_k + \Delta_k^{\perp}$ for *infinitely many* $k \in \mathbb{N}$.

A characterization of the Baire property

Let $\langle G, H\rangle$ be a bounded duality of abelian groups.

If $(G, \sigma(G, H))$ is a Baire group, then

for every sequence $\{g_n\}$ in G and every sequence $\{\Delta_n\}$ of finite subsets of H with $\langle\Delta_n\rangle \cap \langle\Delta_k\rangle = \{0\}$ whenever $n \neq k$, there is $g \in G$ such that $g \in g_n + \Delta_n^{\perp}$ for *infinitely many* $n \in \mathbb{N}$. (That is, g agrees with g_n on Δ_n for infinitely many n.)

This necessary condition is actually also sufficient!

A characterization of the Baire property

- Let $\langle G, H\rangle$ be a bounded duality of abelian groups. $(G, \sigma(G,H))$ is a Baire group if and only if

  for every sequence $\{g_n\}$ in G and every sequence $\{\Delta_n\}$ of finite subsets of H with $\langle \Delta_n\rangle \cap \langle \Delta_k\rangle = \{0\}$ whenever $n \neq k$, there is $g \in G$ such that $g \in g_n + \Delta_k^{\perp}$ for *infinitely many* $n \in \mathbb{N}$. (That is, g agrees with g_n on Δ_n for infinitely many n.)

A characterization of the Baire property

Let $\langle G, H\rangle$ be a bounded duality of abelian groups. Let $\{g_n\}$ be a sequence in G and $\{\Delta_n\}$ a sequence of finite subsets of H with $\langle\Delta_n\rangle \cap \langle\Delta_k\rangle = \{0\}$ whenever $n \neq k$. Then for every $n \in \mathbb{N}$ the set $\bigcup_{k\geq n} g_k + \Delta_k^{\perp}$ is $\sigma(G, H)-$dense in G.

Proof: Fix $n \in \mathbb{N}$ and a basic $\sigma(G, H)-$open set $g_0 + \Delta_0^{\perp}$ in G. We need to find some $g \in (g_0 + \Delta_0^{\perp}) \cap (\bigcup_{k\geq n} g_k + \Delta_k^{\perp})$. This means
$\langle g, h\rangle = \langle g_0, h\rangle$ for every $h \in \Delta_0$
$\langle g, h\rangle = \langle g_k, h\rangle$ for every $h \in \Delta_k$ (we can choose $k \geq n$.)
Pick $k \geq n$ so that $\langle\Delta_0\rangle \cap \langle\Delta_k\rangle = \{0\}$. Consider the character on $\langle\Delta_0\rangle \oplus \langle\Delta_k\rangle$ acting as $\langle g_0, \cdot\rangle$ on Δ_0 and as $\langle g_k, \cdot\rangle$ on Δ_k. Since $\langle\Delta_0\rangle \oplus \langle\Delta_k\rangle$ is finite, this character can be extended to a continuous character of the precompact group $(H, \sigma(H, G))$, that is, to some element of G as required.

Compact subsets of a precompact, bounded Baire group

By using similar techniques one can prove that

- Let $\langle G, H\rangle$ be a bounded duality of abelian groups. Suppose that $(G, \sigma(G, H))$ is a Baire space. Then every $\sigma(H, G)-$compact subset of H is finite.

  This was known for pseudocompact (not necessarily bounded) groups.

  It is not true with "countably compact" instead of "compact", even in the Boolean case.

Compact subsets of a precompact, bounded Baire group

Let $\langle G, H \rangle$ be a bounded duality of abelian groups. Suppose that $(G, \sigma(G, H))$ is a Baire space. Then every $\sigma(H, G)-$compact subset of H is finite.

Some consequences:

- If G is a bounded, precompact abelian group which is a Baire space, then its topology is the only locally quasi-convex one with its group of continuous characters.
- If G is a bounded, precompact abelian group which is a Baire space and has only finite compact subsets then G is reflexive.

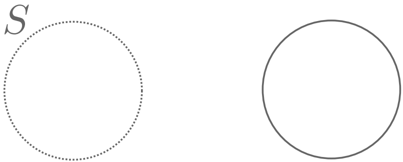
Baire subgroups of $\mathbb{T}$

Consider a dense subgroup S of $\mathbb{T}$ with its usual topology. When is S a Baire group?

Note that S is only pseudocompact in the case $S = \mathbb{T}$.

Baire subgroups of $\mathbb{T}$

- Let S be a dense subgroup of $\mathbb{T}$. Then S has the Baire property iff



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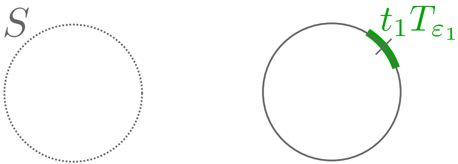

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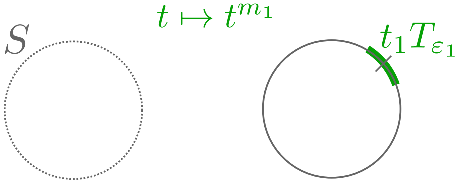

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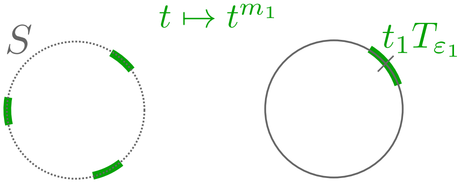

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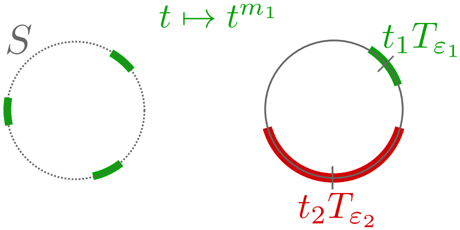

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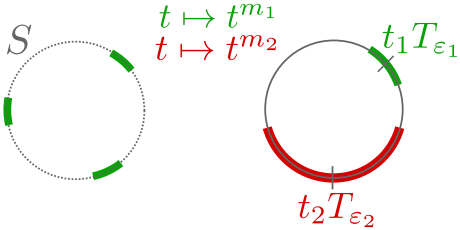

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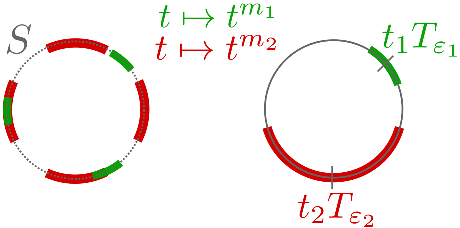

for every sequence $(t_k T_{\varepsilon_k})$ of basic neighborhoods in $\mathbb{T}$ and every faithfully indexed sequence of exponents $(m_k) \in \mathbb{Z}^{\mathbb{N}}$ there is some $t \in S$ with $t^{m_k} \in t_k T_{\varepsilon_k}$ for infinitely many k.

Baire subgroups of $\mathbb{T}$

Let S be a dense subgroup of $\mathbb{T}$. Then S has the Baire property iff for every $(\varepsilon_k) \in (0,\infty)^{\mathbb{N}}$, every $(t_k) \in \mathbb{T}^{\mathbb{N}}$ and every faithfully indexed $(m_k) \in \mathbb{Z}^{\mathbb{N}}$ there is some $t \in S$ with $t^{m_k} \in t_k T_{\varepsilon_k}$ for infinitely many k.

Sketch of the proof of $\Leftarrow$: Fix a decreasing sequence (U_n) of open, dense subsets of $\mathbb{T}$; let us show that $\cap_n U_n \cap S \neq \emptyset$. Any open, dense subset of $\mathbb{T}$ contains the set of all kth roots of a convenient basic neighborhood tT_ε. Find $t_k,\ \varepsilon_k$ $(k \in \mathbb{N})$ with $u^k \in t_k T_{\varepsilon_k} \Rightarrow u \in U_k$. Apply our hypothesis with $m_k = k$: we find $t \in S$ with $t^k \in t_k T_{\varepsilon_k}$ for infinitely many k. We deduce that $t \in U_k$ for infinitely many k, thus actually for all k.

Baire subgroups of $\mathbb{T}$

Let $\langle G, H\rangle$ be a bounded duality of abelian groups. $(G, \sigma(G, H))$ is a Baire group if and only if for every sequence $\{g_n\}$ in G and every sequence $\{\Delta_n\}$ of finite subsets of H with $\langle\Delta_n\rangle \cap \langle\Delta_k\rangle = \{0\}$ whenever $n \neq k$, there is $g \in G$ such that g agrees with g_k on Δ_k for infinitely many k.

Note that the topology on S is exactly $\sigma(S, \mathbb{Z})$.

[S is precompact, hence its topology is $\sigma(S, S^\wedge)$. But $S^\wedge = \mathbb{T}^\wedge = \mathbb{Z}$.]

We have just seen that

$S = (S, \sigma(S, \mathbb{Z}))$ is a Baire group iff for every sequence (t_n) in $\mathbb{T}$ and every faithfully indexed sequence $\{m_n\}$ in $\mathbb{Z}$ there exists some $t \in S$ such that t "almost agrees" with t_n on $\{m_n\}$ for infinitely many n.