Character groups of unusual topologies for $\mathbb{R}^n$

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The subject of this research project

Topologies for $\mathbb{R}^n$ that are metrizable, weaker than the usual topology, and make $(\mathbb{R}^n, +)$ a topological group.

$$id : (\mathbb{R}^n, usual) \to (\mathbb{R}^n, new) \text{ is continuous.}$$

Goal: Understand these groups.

One strategy: Study their continuous characters.

The topologies

CHOOSE TWO SEQUENCES:

• a sequence $\{v_j\}$ of non-zero elements of $\mathbb{R}^n$ that you want to force to converge to zero
[the “converging sequence”]

• a sequence $\{p_j\}$ of positive real numbers that specifies the approximate rate at which $\{v_j\}$ will converge to zero ($p_j \to 0$ in the usual metric)
[the “rate sequence”]

An example on $\mathbb{R}$

Create a groupnorm ν for $\mathbb{R}$ such that $\nu(x) \leq |x|$ for all $x \in \mathbb{R}$ and $\nu(j!) \leq 1/j$ for all $j \in \mathbb{N}$.

Aside: A groupnorm $\nu : \mathbb{R} \to \mathbb{R}^{\geq 0}$ is similar to absolute value:

$$\begin{aligned} &\nu(x) \geq 0 \text{ for all } x \in \mathbb{R}; \\ &\nu(x) = 0 \iff x = 0; \\ &\nu(-x) = \nu(x) \text{ for all } x \in \mathbb{R} \\ &\nu(x+y) \leq \nu(x) + \nu(y) \text{ for all } x, y \in \mathbb{R}, \end{aligned}$$

but not multiplicative.

If ν is a groupnorm, then $d(x,y) = \nu(x-y)$ is a metric that makes $(\mathbb{R}, +)$ a topplogical group (where $x, y \in \mathbb{R}$).

Creating ν: Want $\nu(j!) \leq 1/j$

We need $\nu(1) \leq 1, \nu(2) \leq 1/2, \nu(6) \leq 1/3, \nu(24) \leq 1/4 \ldots$

What can we say about $\nu(70)$**? Clearly,** $\nu(70) \leq 70$**, but also:**

$70 = 3(4!) - 2!$**, so** $\nu(70) \leq \frac{3}{4} + \frac{1}{2} = 1.25$

$70 = 5! - 2(4!) - 2!$**, so** $\nu(70) \leq \frac{1}{5} + \frac{2}{4} + \frac{1}{2} = 1.2$

If $70 = \sum a_j(j!)$**, then we need to have** $\nu(70) \leq \sum |a_j|/j$**.**

Thus we define: $\nu(70) = \inf \left\{ \sum |a_j|/j : 70 = \sum a_j(j!) \right\}$, **where** $a_j \in \mathbb{Z}$ **and sums are finite.**

Creating ν: What if $x \notin \mathbb{Z}$?

If $x = \sum a_j(j!) + y$, where $y \in \mathbb{R}$,

then $\nu(x) \leq \sum |a_j|/j + |y|$, so define $\nu(x)$ by

$$\nu(x) = \inf \left\{ \sum |a_j|/j + |y| : x = \sum a_j(j!) + y \right\}.$$

PROPOSITION: ν is a groupnorm on $\mathbb{R}$.

Partial proof: If $m \in \mathbb{N}$ and $\nu(x) < 1/m$, then we can write $x = \sum a_j(j!) + y$, where $\sum |a_j|/j + |y| < 1/m$, so x is within $1/m$ (in the usual metric) of an element of the subgroup generated by $(m+1)!$. If this is true for all $m \in \mathbb{N}$, then $x = 0$.

This construction works for other sequences in $\mathbb{R}$

THEOREM: If $\{|v_j|\}$ is a non-decreasing sequence in $\mathbb{R}$; $\{p_j\}$ is a non-increasing sequence in $\mathbb{R}^{>0}$; and

$$\left\{\frac{p_{j+1}|v_{j+1}|}{|v_j|}\right\}$$

has a strictly positive lower bound, then there is a groupnorm ν on $\mathbb{R}$ that is $\leq$ than the usual norm and such that $\nu(v_j) \leq p_j$ for all j.

$(\{v_j\}, \{p_j\}, \nu)$ is a **"SNT"**

EXAMPLES: $\nu_1(j! + 1) \leq 1/\sqrt{j}$, $\nu_2(j! + \sqrt{2}) \leq 1/j$, $\nu_3(2^{j^2}) \leq 1/j^3$

This construction works for sequences in $\mathbb{R}^n$

THEOREM **(tcs, 1982): If $\{\|v_j\|\}$ is a non-decreasing sequence in $\mathbb{R}^n$; $\{p_j\}$ is a non-increasing sequence in $\mathbb{R}^{>0}$; and**

$$\left\{ \frac{p_{j+1}\|v_{j+1}\|}{\|v_j\|} \right\}$$

has a strictly positive lower bound, then there is a groupnorm ν on $\mathbb{R}^n$ that is $\leq$ than the usual norm and such that $\nu(v_j) \leq p_j$ for all j.

EXAMPLES: $\nu_4\left(j!, (-1)^j\right) \leq 1/j$, $\quad \nu_5(j, 3^{3^j}) \leq 1/j^2$

The character group

Let $(G, \mathcal{T})$ be a topological group. A character of G is a homomorphism from G into the circle group:

$$f : G \to \mathbb{T}$$

The continuous characters of $(G, \mathcal{T})$ form a group, with the operation of pointwise multiplication: If f, h are two such characters and $x \in G$, then

$$(f\,h)(x) = f(x)h(x).$$

This group is the character group or dual group of $(G, \mathcal{T})$, often denoted $G^\wedge$.

The character group of a SNT

Does $(\mathbb{R}^n, \nu)$ have non-trivial continuous characters?

Notes:

• Duality theory of locally compact abelian groups does not apply.

• There exists a metrizable group topology for $\mathbb{R}$ that is weaker than the usual topology and for which the only continuous character is the trivial one
(J.W. Nienhuys, Fund. Math., 1971/2).

Characters of SNT's

Let $(\{v_j\}, \{p_j\}, \nu)$ be a SNT for $\mathbb{R}$. Suppose that $f : (\mathbb{R}, \nu) \to \mathbb{T}$ is a continuous character.

Let $\mathcal{U}$ be the usual topology for $\mathbb{R}$. The composition

$$(\mathbb{R}, \mathcal{U}) \xrightarrow{id} (\mathbb{R}, \nu) \xrightarrow{f} \mathbb{T}$$

is continuous. Therefore f must have the form $f(x) = \exp(i\theta x)$ for some $\theta \in \mathbb{R}$.

For $\theta \in \mathbb{R}$, let f_θ denote the function $x \mapsto \exp(i\theta x)$. Let $B_\nu = \{\theta \in \mathbb{R} : f_\theta$ is ν-continuous$\}$

There are many characters

For $\theta \in \mathbb{R}$, let f_θ denote the function $x \mapsto \exp(i\theta x)$. Let $B_\nu = \{\theta \in \mathbb{R} : f_\theta$ is ν-continuous$\}$

THEOREM (tcs, 2013) : For every SNT on $\mathbb{R}$, B_ν is an uncountable dense subgroup of $(\mathbb{R}, \mathcal{U})$, and its complement is also uncountable and dense.

PROBLEM: Find the character group of $(\{j!\}, \{1/j\}, \nu)$.

First step: Show $2\pi\mathbb{Z} \subseteq B_\nu$.

Finding characters of $(\{j!\}, \{1/j\}, \nu)$

Define an invariant metric ρ on $\mathbb{T}$ by

$$\rho(e^{ix}, e^{iy}) = min\{|x - y - 2\pi m| : m \in \mathbb{Z}\}.$$

We claim that $f_{2\pi n} : x \mapsto \exp(2\pi inx)$ is ν-continuous when $n \in \mathbb{Z}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(\exp(2\pi iny), 1) < \epsilon$ whenever $|y| < \delta$. If $x \in \mathbb{R}$ and $\nu(x) < \delta$, then we can write $x = \sum a_j(j!) + y$, where $a_j \in \mathbb{Z}$, $y \in \mathbb{R}$, the sum is finite, and $\sum |a_j|/j + |y| < \delta$. Then

$$f_{2\pi n}(x) = \exp 2\pi in(\sum a_j(j!) + y) = \exp(2\pi iny),$$

and $\rho(f_{2\pi n}(x), 1) = \rho(\exp(2\pi iny), 1) < \epsilon$. So $2\pi n \in B_\nu$.

Finding more characters

Claim: $2\pi\mathbb{Q} \subseteq B_\nu$.

Proof: Write $q \in \mathbb{Q}$ as $q = m/k$, where $m \in \mathbb{Z}, k \in \mathbb{N}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $\delta < 1/k$ and $\rho(\exp(2\pi i q y), 1) < \epsilon$ whenever $|y| < \delta$. If $x \in \mathbb{R}$ and $\nu(x) < \delta$, then we have $x = \sum a_j(j!) + y$, where $a_j \in \mathbb{Z}$, $y \in \mathbb{R}$, the sum is finite, and $\sum |a_j|/j + |y| < \delta < 1/k$. Moreover, for each $j \leq k$, we have $a_j = 0$. Therefore

$$f_{2\pi q}(x) = \exp 2\pi \frac{m}{k} i(\sum_{j \geq k+1} a_j(j!) + y) = \exp(2\pi i \frac{m}{k} y),$$

and $\rho(f_{2\pi n}(x), 1) = \rho(\exp(2\pi i q y), 1) < \epsilon$. So $2\pi q \in B_\nu$.

Comments

• $2\pi\mathbb{Q}$ is dense in $(\mathbb{R}, \mathcal{U})$.

• $2\pi\mathbb{Q}$ is countable, and B_ν is uncountable. What is in B_ν but not in $2\pi\mathbb{Q}$?

• The rate sequence $\{p_j\}$ played no role in these computations.

• What about other SNT's?

  • For $(\{j!+1\}, \{1/j\}, \nu_1)$, the same proof shows that $2\pi\mathbb{Z} \subseteq B_{\nu_1}$, but not that $2\pi\mathbb{Q} \subseteq B_{\nu_1}$.

  • Find a non-trivial continuous character of $\left(\{j!+\sqrt{2}\}, \{1/j\}, \nu_2\right)$.

• The theorem says that B_ν is always uncountable.

Summary of the first lecture

$(\{v_j\}, \{p_j\}, \nu)$ is an SNT on $\mathbb{R}$: ν is a groupnorm on $\mathbb{R}$ that forces the sequence $\{v_j\}$ to converge to 0 at approximately the rate $\{p_j\}$. (With appropriate hypotheses, such a ν exists.)

A continuous character of $(\mathbb{R}, \nu)$ is a continuous homomorphism $(\mathbb{R}, \nu) \to \mathbb{T}$. Every continuous character has the form $f_\theta : x \mapsto \exp(i\theta x)$ for some $\theta \in \mathbb{R}$.

$(\mathbb{R}, \nu)$ has uncountably many continuous characters, but they are hard to find.

Example: Is $x \mapsto \exp(2\pi i x\sqrt{2})$ a continuous character of the SNT $(\{j!\}, \{1/j\}, \nu)$? I have no idea!

Today: One way to prove a character is continuous

Let $(\{v_j\}, \{p_j\}, \nu)$ be an SNT for $\mathbb{R}$.

IDEA: A homomorphism $f : \mathbb{R} \to \mathbb{T}$ should be ν-continuous if $f(v_j) \to 1$ at least as fast as $\nu(v_j) \to 0$ [Rephrase: at least as fast as $p_j \to 0$].

DEFINITION: Let $f : \mathbb{R} \to \mathbb{T}$ be a $\mathcal{U}$-continuous homomorphism. We say that f is a sequentially bounded (SB) character of $(\mathbb{R}, \nu)$ if there exists $\alpha > 0$ such that $\rho(f(v_j), 1) \leq \alpha p_j$ for all $j \in \mathbb{N}$. The set of all (SB) characters of $(\mathbb{R}^n, \nu)$ is denoted by H_ν.

THEOREM (tcs, 2014): If f is sequentially bounded, then f is ν-continuous.

f is sequentially bounded $\implies$ f is ν-continuous.

PROOF: Assume $\alpha > 0$ and $\rho(f(v_j), 1) \leq \alpha p_j$ for all $j \in \mathbb{N}$. Let $\epsilon > 0$ be given. Because f is $\mathcal{U}$-continuous, there is a $\delta > 0$ such that $\rho(f(y), 1) < \epsilon/2$ whenever $|y| < \delta$. We may assume that $\delta < \epsilon/(2\alpha)$. If $\nu(x) < \delta$, then we can write $x = \sum c_j v_j + y$, where $c_j \in \mathbb{Z}$, $y \in \mathbb{R}$, the sum is finite, and $|y| + \sum |c_j| p_j < \delta$. Since $|y| < \delta$, we know that $\rho(f(y), 1) < \epsilon/2$. Using the fact that $\rho(f(v_j), 1) \leq \alpha p_j$ for all j and the invariance of ρ, we find that

$$\begin{aligned} \rho(f(x), 1) = \rho(f(y + \sum c_j v_j), 1) &\leq \rho(f(y), 1) + \sum |c_j| \rho(f(v_j), 1) \\ &< \epsilon/2 + \alpha \sum |c_j| p_j < \epsilon/2 + \delta\alpha < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence f is ν-continuous.

$\mathcal{D}$-expansions

Assume the converging sequence $\{v_j\}$ is also a $\mathcal{D}$-sequence: $v_j \in \mathbb{Z}$ and $v_j | v_{j+1}$ for all $j \in \mathbb{N}$ (but start with $v_1 = 1$, not v_0).

PROPOSITION (de la Barrera Mayoral, 2014): Let $\theta \in \mathbb{R}$. Then θ has a $\mathcal{D}$-expansion:

$$\theta = \sum_{j=1}^{\infty} \frac{a_j}{v_j} = a_1 + \frac{a_2}{v_2} + \frac{a_3}{v_3} + \cdots ,$$

where all $a_j \in \mathbb{Z}$, $|a_j| \leq \frac{v_j}{2v_{j-1}}$ for all $j \geq 2$, and

$$-\frac{1}{2v_n} < \theta - \sum_{j=1}^{n} \frac{a_j}{v_j} \leq \frac{1}{2v_n} \qquad \text{for all } n \in \mathbb{N}.$$

Deciding whether a character is SB

THEOREM (tcs): Let $\theta = \sum_{j=1}^{\infty} \frac{a_j}{v_j} \in \mathbb{R}$**. Then** $f_{2\pi\theta}$ **is SB if and only if**

$$\left\{ \frac{a_{j+1} v_j}{p_j\, v_{j+1}} : j \in \mathbb{N} \right\}$$

is a bounded set.

EXAMPLE: $(\{j!\}, \{1/j\}, \nu)$, $\theta = e$. **Then**

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \frac{2}{1!} + \sum_{j=2}^{\infty} \frac{1}{j!} \quad \textbf{and}$$

$$\frac{a_{j+1} v_j}{p_j\, v_{j+1}} = \frac{1(j!)}{(1/j)(j+1)!} = \frac{j}{j+1} < 1.$$

So $f_{2\pi\, e}$ **is SB and therefore** ν**-continuous.**

More SB characters

For the SNT $(\{j!\}, \{1/j\}, \nu)$

$$\frac{a_{j+1} v_j}{p_j v_{j+1}} = \frac{a_{j+1}(j!)}{(1/j)(j+1)!} = \frac{j a_{j+1}}{j+1}.$$

Thus

$$\left\{ \frac{a_{j+1} v_j}{p_j v_{j+1}} : j \in \mathbb{N} \right\}$$

is a bounded set if and only if the coefficients a_j in the $\mathcal{D}$-expansion of θ are bounded.

PROPOSITION: For this SNT, if the coefficients a_j in the expansion $\theta = \sum_{j=1}^{\infty} \frac{a_j}{v_j}$ are bounded, then $f_{2\pi\theta}$ is SB, hence ν-continuous. [B_ν is uncountable.]

A computational proof

Claim: **For the SNT** $(\{j!\}, \{1/j\}, \nu)$**, the character** $f_{2\pi e} : x \mapsto \exp(2\pi iex)$ **is SB, hence** ν**-continuous.**

Proof: The $\mathcal{D}$**-expansion of** e **is** $e = 2 + \sum_{j=2}^{\infty} \frac{1}{j!}$**. First compute** $e(k!)$ **for** $k \in \mathbb{N}, k \geq 2$**:**

$$e(k!) = 2(k!) + k! \sum_{j=2}^{k} \frac{1}{j!} + k! \sum_{j=k+1}^{\infty} \frac{1}{j!} = \textbf{integer} + k! \sum_{j=k+1}^{\infty} \frac{1}{j!}.$$

Thus

$$f_{2\pi e}(k!) = \exp\left(2\pi i \left(k! \sum_{j=k+1}^{\infty} \frac{1}{j!}\right)\right).$$

Now

$$k! \sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} + \dots$$

$$= \frac{1}{k+1}\left(1 + \frac{1}{k+2} + \frac{1}{(k+2)(k+3)} + \dots\right) < \frac{e}{k+1}.$$

Therefore

$$\rho\left(\exp\left(2\pi i k! \sum_{j=k+1}^{\infty} \frac{1}{j!}\right), 1\right) < \frac{2\pi e}{k+1} < \frac{2\pi e}{k}.$$

Thus $\rho(f_{2\pi e}(k!), 1) < \frac{2\pi e}{k}$, and $f_{2\pi e}$ is SB.

Another example

For the SNT $\left(\{j!\}, \{1/\sqrt{j}\}, \nu\right)$
Let

$$\theta_1 = \sum_{j=1}^{\infty} \frac{j}{(j^2)!}.$$

The coefficents in this expansion are unbounded, but the only non-zero coefficents are $a_{j^2} = j$**. For those coefficents, we have, for all** $j \geq 2$**,**

$$\frac{a_{j^2} v_{j^2-1}}{p_{j^2-1} v_{j^2}} = \frac{j(j^2-1)!}{(1/\sqrt{j^2-1})(j^2)!} = \frac{j\sqrt{j^2-1}}{j^2} = \frac{\sqrt{j^2-1}}{j},$$

which converges to 1 **as** $j \to \infty$**. Thus the relevant set is bounded, and** $f_{2\pi\theta_1}$ **is SB and** ν_1**-continuous.**

Open questions

Does a SNT have any continuous characters that are not sequentially bounded?

Rephrase: Let G_ν denote the set of all ν-continuous characters. We know that $H_\nu \leq G_\nu$. Are these groups ever equal? Are they always equal?

Partial result: We can show that, if $2\pi\theta \in G_\nu$, then $\frac{a_{j+1}v_j}{v_{j+1}} \to 0$ as $j \to \infty$. But does

$$\left\{ \frac{a_{j+1}v_j}{p_j\, v_{j+1}} : j \in \mathbb{N} \right\}$$

have to be bounded?

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THANK YOU!

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