Equicontinuity criteria for metric-valued sets of continuous functions

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Let X and (M, d) be a topological space and a metric space respectively, and let $G \subseteq C(X, M)$.

Definition

1. A point $x \in X$ is an ***equicontinuity point*** of G when for every $\epsilon > 0$ there is a neighborhood U of x such that $diam(g(U)) < \epsilon$ for all $g \in G$.
2. G is ***almost equicontinuous (AE)*** when the subset of equicontinuity points of G is dense in X.
3. G is ***hereditarily almost equicontinuous (HAE)*** when $G|_A$ is almost equicontinuous for every subset A of X.

Lemma

Consider the following two properties:

(a) G is almost equicontinuous.

(b) For every nonempty open subset U of X and $\epsilon > 0$, there exists a nonempty open subset $V \subseteq U$ such that $diam(g(V)) < \epsilon$ for all $g \in G$.

Then (a) implies (b).

If X is a **Baire space**, then (a) and (b) are equivalent.

***Proof*:**

- Consider the open set

$$O_\epsilon := \bigcup\{U \subseteq X : \quad U \text{ is a nonempty open subset } \wedge \\ diam(g(U)) < \epsilon \quad \forall g \in G\}.$$

If X is a **Baire space**, then (a) and (b) are equivalent.

***Proof*:**

- Consider the open set

$$O_\epsilon := \bigcup \{U \subseteq X : \quad U \text{ is a nonempty open subset } \wedge \\ diam(g(U)) < \epsilon \quad \forall g \in G\}.$$

- By (b), $O_\epsilon \neq \emptyset$ and dense in X.

If X is a **Baire space**, then (a) and (b) are equivalent.

***Proof*:**

- Consider the open set

$$O_\epsilon := \bigcup\{U \subseteq X : \quad U \text{ is a nonempty open subset } \wedge \\ diam(g(U)) < \epsilon \quad \forall g \in G\}.$$

- By (b), $O_\epsilon \neq \emptyset$ and dense in X.
- Since X is Baire, taking $W := \bigcap_{n\in\omega} O_{\frac{1}{n}} \neq \emptyset$, we obtain a dense G_δ subset which is the subset of equicontinuity points of G.

Example

Let $X = 2^\omega$ be the Cantor space and let $G = \{\pi_n\}_{n\in\omega}$ be the set of all projections of X onto $\{0, 1\}$.
Then G is not AE.

Example

Let $X = \mathbb{R}$ and let $G = \{\arctan(nx)\}_{n \in \omega}$
Then G is HAE.

Recall that a ***dynamical system***, or a G-***space***, is a Hausdorff space X on which a topological group G acts continuously. For each $g \in G$ we have the self-homeomorphism $x \mapsto gx$ of X that we call g-translation.

Example (Glasner and Megrelishvili)

There are dynamical systems AE which are not HAE.

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Theorem A

Let X and (M, d) be a Čech-complete space and a **separable** metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following three properties:

(a) G is almost equicontinuous.

(b) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.

(c) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.

Then $(b) \Rightarrow (c) \Rightarrow (a)$. If X is also a **hereditarily Lindelöf** space, then all conditions are equivalent.

Theorem B

Let X and (M, d) be a Čech-complete space and a metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:

(a) G is hereditary almost equicontinuous.

(b) L is almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(c) $(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

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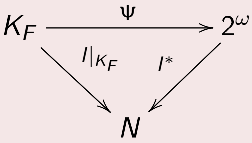
Definition

X is said to be ***hemicompact*** if there exists a sequence of compacts sets $\{X_n\}_{n\in\omega}$ such that $X = \bigcup_{n\in\omega} X_n$ and for every compact subset C of X there is $n \in \omega$ such that $C \subseteq X_n$.

Lemma 1

Let X and (M, d) be a Čech-complete space and a **hemicompact** metric space, respectively, and let G be a subset of $C(X, M)$ such that $\overline{G}^{M^X}$ is compact. **If G is not almost equicontinuous**, then for every G_δ and dense subset F of X there exists a countable subset L in G, a compact separable subset $K_F \subseteq F$, a compact subset $N \subseteq M$ and a continuous and surjective map Ψ of K_F onto the Cantor set 2^ω such that if the maps l^* are defined to make the following diagram commutative



then the subset $\boldsymbol{L^* := \{l^* : l \in L\}} \subseteq C(2^\omega, N)$ **is not almost equicontinuous on $\boldsymbol{2^\omega}$**.

Remark

Let X be a topological space, (M, d) be a metric space and G be a subset of $C(X, M)$.
Set

$$K := \{\alpha : M \to [-1, 1] : |\alpha(m_1) - \alpha(m_2)| \leq d(m_1, m_2), \quad \forall m_1, m_2 \in M\}.$$

K is a compact subspace of $[-1, 1]^M$.

Remark

Consider the evaluation map

$$\begin{aligned} \varphi : X \times G &\longrightarrow M \\ (x, g) &\longmapsto g(x) \end{aligned}$$

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which is clearly separately continuous.

The map φ has associated a separately continuous map

$$\begin{aligned} \varphi^* : X \times (G \times K) &\longrightarrow [-1, 1] \\ (x, (g, \alpha)) &\longmapsto \alpha(g(x)) \end{aligned}$$

Remark

Set

$$\begin{array}{rcl} \nu : \overline{G}^{M^X} \times K & \longrightarrow & [-1,1]^X \\ (h,\alpha) & \longmapsto & \alpha \circ h \end{array}$$

Since $G \subseteq C(X, M)$, we have that $\nu(G \times K) \subseteq C(X, [-1,1])$.

$\bar{d} : M \times M \to \mathbb{R}$ defined by $\bar{d}(m_1, m_2) := \min\{d(m_1, m_2), 1\}$ $\forall m_1, m_2 \in M$

Lemma 2

Let X and (M, d) be a topological and a metric space, respectively. If G is a subset of $C(X, M)$, then G is equicontinuous at a point $x_0 \in X$ if and only if $\nu(G \times K)$ is equicontinuous at it.

Lemma 2

Let X and (M, d) be a topological and a metric space, respectively. If G is a subset of $C(X, M)$, then G is equicontinuous at a point $x_0 \in X$ if and only if $\nu(G \times K)$ is equicontinuous at it.

***Proof*:**

$(\Rightarrow)$ Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $d(g(x_0), g(x)) < \epsilon$ for all $x \in U$ and $g \in G$.

Lemma 2

Let X and (M, d) be a topological and a metric space, respectively. If G is a subset of $C(X, M)$, then G is equicontinuous at a point $x_0 \in X$ if and only if $\nu(G \times K)$ is equicontinuous at it.

***Proof*:**
$(\Rightarrow)$ Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $d(g(x_0), g(x)) < \epsilon$ for all $x \in U$ and $g \in G$.

Let $\alpha \in K$, $x \in U$ and $g \in G$, then we have

$$|\nu(g, \alpha)(x_0) - \nu(g, \alpha)(x)| = |\alpha(g(x_0)) - \alpha(g(x))| \leq d(g(x_0), g(x)) < \epsilon.$$

***Proof*:**
($\Leftarrow$) Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $|\nu(g,\alpha)(x_0) - \nu(g,\alpha)(x)| < \epsilon$ for all $x \in U$, $g \in G$ and $\alpha \in K$.

***Proof*:**
($\Leftarrow$) Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $|\nu(g,\alpha)(x_0) - \nu(g,\alpha)(x)| < \epsilon$ for all $x \in U$, $g \in G$ and $\alpha \in K$.

Set $\alpha_0 \in [-1,1]^M$ defined by $\alpha_0(m) \stackrel{\mathrm{def}}{=} d(m, g(x_0))$ for all $m \in M$. It is easy to check that $\alpha_0 \in K$.

***Proof*:**
($\Leftarrow$) Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $|\nu(g,\alpha)(x_0) - \nu(g,\alpha)(x)| < \epsilon$ for all $x \in U$, $g \in G$ and $\alpha \in K$.

Set $\alpha_0 \in [-1,1]^M$ defined by $\alpha_0(m) \stackrel{\text{def}}{=} d(m, g(x_0))$ for all $m \in M$. It is easy to check that $\alpha_0 \in K$.

It suffices to observe that

$$|\alpha_0(g(x_0)) - \alpha_0(g(x))| = d(g(x), g(x_0))$$

for all $x \in U$ and $g \in G$.

Corollary 1

Let X and (M, d) be a topological and a metric space, respectively. If G is a subset of $C(X, M)$, then G is almost equicontinuous if and only if $\nu(G \times K)$ is almost equicontinuous.

Theorem (Cascales, Namioka and Vera)

Let X be a **compact space**, (M, d) be a **compact metric space** and let G be a subset of $C(X, M)$. If $(X, t_p(\overline{G}^{M^X}))$ is Lindelöf, then G is hereditarily almost equicontinuous.

Theorem 1

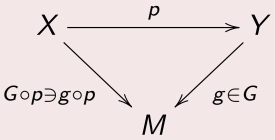
Let X be a **Čech-complete space**, (M, d) be a **metric space** and let G be a subset of $C(X, M)$ such that $\overline{G}^{M^X}$ is compact. If there exists a a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf, then G is almost equicontinuous.

Theorem (Glasner, Megrelishvili and Uspenskij)

Let X be a compact metric space, (M, d) be a metric space and let $G \subseteq C(X, M)$ and $H = \overline{G}^{M^X}$. If H is compact hereditary almost equicontinuous, then H is metrizable.

Lemma (Glasner and Megrelishvili)

Let X, Y and (M, d) be two arbitrary **compact spaces** and a metric space, respectively, and let G be a subset of $C(Y, M)$. Suppose that $p : X \to Y$ is a continuous onto map.



Then $G \circ p := \{g \circ p : g \in G\} \subseteq C(X, M)$ is HAE if and only if G is HAE.

Proposition 1

Let X be a Čech-complete space, (M, d) be a **hemicompact metric space** and $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:

(a) G is hereditary almost equicontinuous.

(b) L is almost equicontinuous on F, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

***Proof*:**
$(b) \Rightarrow (a)$ Assume that (a) does not hold $\Rightarrow \exists A \subseteq X$ such that $G|_A$ is not AE.

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$(b) \Rightarrow (a)$ Assume that (a) does not hold $\Rightarrow \exists A \subseteq X$ such that $G|_A$ is not AE.

By **Lemma 1** there exists a compact and separable subset F of X included in A, an onto and continuous map $\Psi : F \to 2^\omega$, and a countable subset L of G such that the subset $L^* \subseteq C(2^\omega, M)$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in F$ is not AE.

***Proof*:**

$(b) \Rightarrow (a)$ Assume that (a) does not hold $\Rightarrow \exists A \subseteq X$ such that $G|_A$ is not AE.

By **Lemma 1** there exists a compact and separable subset F of X included in A, an onto and continuous map $\Psi : F \to 2^\omega$, and a countable subset L of G such that the subset $L^* \subseteq C(2^\omega, M)$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in F$ is not AE.

By **Lemma GM** L is not AE on F. *Contradiction*

Corollary 2

Let X be a Čech-complete space, (M, d) be a **metric space** and $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:

(a) G is hereditary almost equicontinuous.

(b) L is almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

Proof of Theorem B

(a) G is hereditary almost equicontinuous.

(b) L is almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

$(a) \Leftrightarrow (b)$ **Corollary 2**.

Proof of Theorem B

(a) G is hereditary almost equicontinuous.

(c) $(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

$(a) \Rightarrow (c)$ Let $L \in [G]^{\leq\omega}$ and let F be a separable and compact subset of X.

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(a) G is hereditary almost equicontinuous.

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$(a) \Rightarrow (c)$ Let $L \in [G]^{\leq \omega}$ and let F be a separable and compact subset of X.

$$x \sim y \text{ if and only if } l(x) = l(y) \text{ for all } l \in L.$$

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$\widetilde{F} = F/\sim$ is the compact quotient space and $p : F \to \widetilde{F}$ is the quotient map.

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$(a) \Rightarrow (c)$ Let $L \in [G]^{\leq\omega}$ and let F be a separable and compact subset of X.

$$x \sim y \text{ if and only if } l(x) = l(y) \text{ for all } l \in L.$$

$\widetilde{F} = F/\sim$ is the compact quotient space and $p : F \to \widetilde{F}$ is the quotient map.

Each $l \in L$ has associated a map $\tilde{l} \in C(\widetilde{F}, M)$ defined as $\tilde{l}(\tilde{x}) \stackrel{\text{def}}{=} l(x)$ for any $x \in F$ with $p(x) = \tilde{x}$.

Proof of Theorem B

Each $l \in \overline{L}^{M^F}$ has associated a map $\tilde{l} \in \overline{\tilde{L}}^{M^{\widetilde{F}}}$ such that $\tilde{l} \circ p = l$.

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$(\widetilde{F}, t_p(\tilde{L}))$ is a compact metric space.

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$(\widetilde{F}, t_p(\tilde{L}))$ is a compact metric space.

L is HAE on $F \overset{GM}{\Longrightarrow} \widetilde{L}$ is HAE on $\widetilde{F}$. $\overset{GMU}{\Longrightarrow} \overline{\tilde{L}}^{M^{\widetilde{F}}}$ is metrizable.

Proof of Theorem B

Each $l \in \overline{L}^{M^F}$ has associated a map $\tilde{l} \in \overline{\tilde{L}}^{M^{\widetilde{F}}}$ such that $\tilde{l} \circ p = l$.

$(\widetilde{F}, t_p(\tilde{L}))$ is a compact metric space.

L is HAE on $F \overset{GM}{\Longrightarrow} \widetilde{L}$ is HAE on $\widetilde{F}$. $\overset{GMU}{\Longrightarrow} \overline{\tilde{L}}^{M^{\widetilde{F}}}$ is metrizable.

$\overline{L}^{M^F}$ is canonically homeomorphic to $\overline{\tilde{L}}^{M^{\widetilde{F}}} \Rightarrow \overline{L}^{M^F}$ is metrizable.

Proof of Theorem B

(c) $(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

$(c) \Rightarrow (d)$ $H := ((\overline{L}^{M^X})|_F, t_p(F))$ is compact metric.

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We may assume that M is separable without loss of generality.

Proof of Theorem B

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(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

$(c) \Rightarrow (d)$ $H := ((\overline{L}^{M^X})|_F, t_p(F))$ is compact metric.

We may assume that M is separable without loss of generality.

Every element $x \in F$ has associated an element $\hat{x} \in \overline{M}^H$ defined by $\hat{x}(h) \stackrel{\text{def}}{=} h(x)$ for all $h \in H$.

Proof of Theorem B

(c) $(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq \omega}$ and F a separable and compact subset of X.

$(c) \Rightarrow (d)$ $H := ((\overline{L}^{M^X})|_F, t_p(F))$ is compact metric.

We may assume that M is separable without loss of generality.

Every element $x \in F$ has associated an element $\hat{x} \in \overline{M}^H$ defined by $\hat{x}(h) \stackrel{\text{def}}{=} h(x)$ for all $h \in H$.

Set $\widehat{F} := \{\hat{x} : x \in F\}$.

Proof of Theorem B

Since H is compact metric $\Rightarrow$ $(F, t_\infty(H))$ is separable and metrizable $\Rightarrow$ $(F, t_\infty(H))$ is Lindelöf.

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Since H is compact metric $\Rightarrow (F, t_\infty(H))$ is separable and metrizable $\Rightarrow (F, t_\infty(H))$ is Lindelöf.

$\Rightarrow (F, t_p(H))$ is Lindelöf.

Proof of Theorem B

(b) L is almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

$(d) \Rightarrow (b)$ Apply **Theorem 1**.

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Topological Groups

Corollary 3

Let X and (M, d) be a Čech-complete topological group and a metric separable group, respectively, and let G be a subset of $CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following three properties:

(a) G is equicontinuous.

(b) G is relatively compact in $CHom(X, M)$ with respect to the compact open topology.

(c) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.

(d) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.

Then $(d) \Rightarrow (c) \Rightarrow (a) \Leftrightarrow (b)$. If X is also ω-narrow, then all conditions are equivalent. Furthermore (c) and (d) are also true for $F = X$.

Topological Groups

Definition

A topological group G is said to be ω-***narrow*** if for every neighborhood V of the neutral element, there exists a countable subset E of G such that $G = EV$.

Topological Groups

Corollary 3

Let X and (M, d) be a Čech-complete topological group and a metric separable group, respectively, and let G be a subset of $CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following three properties:

(a) G is equicontinuous.

(b) G is relatively compact in $CHom(X, M)$ with respect to the compact open topology.

(c) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.

(d) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.

Then $(d) \Rightarrow (c) \Rightarrow (a) \Leftrightarrow (b)$. If X is also ω-narrow, then all conditions are equivalent. Furthermore (c) and (d) are also true for $F = X$.

Topological Groups

Corollary 4

Let X and (M, d) be a Čech-complete topological group and a metric group, respectively, and let G be a subset of $CHom(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:

(a) G is equicontinuous.

(b) L is equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(c) $((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

Dynamical Systems

Corollary 5

Let X be a Polish G-space such that $\overline{G}^{X^X}$ is compact. The following properties are equivalent:

(a) X is almost equicontinuous.

(b) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{X^X})|_F$ is metrizable.

(c) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{X^X}))$ is Lindelöf.

Dynamical Systems

Corollary 5

Let X be a complete metrizable G-space such that $\overline{G}^{X^X}$ is compact. Then the following conditions are equivalent:

(a) X is hereditary almost equicontinuous.

(b) L is almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(c) $((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.

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Thank you for your attention!

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5 Proof of Theorem 1

6 Proof of ★

***Sketch of the Proof*:**

Consider the (continuous) map $\nu_F : (\overline{G}^{M^X})|_F \times K \to [-1,1]^F$ defined by $\nu_F(h,\alpha) \stackrel{\text{def}}{=} \alpha \circ h$ for all $h \in (\overline{G}^{M^X})|_F$ and $\alpha \in K$.

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Consider the (continuous) map $\nu_F : (\overline{G}^{M^X})|_F \times K \to [-1,1]^F$ defined by $\nu_F(h,\alpha) \stackrel{\text{def}}{=} \alpha \circ h$ for all $h \in (\overline{G}^{M^X})|_F$ and $\alpha \in K$.

$\nu_F(G|_F \times K)$ is a subset of $C(F,[-1,1])$ such that:

(i) $\nu_F(G|_F \times K) = \nu(G \times K)|_F$

(ii) $(\overline{\nu(G \times K)}^{[-,1,1]^X})|_F = \nu_F((\overline{G}^{M^X})|_F \times K) = \nu(\overline{G}^{M^X} \times K)|_F$

***Sketch of the Proof*:**

Consider the (continuous) map $\nu_F : (\overline{G}^{M^X})|_F \times K \to [-1,1]^F$ defined by $\nu_F(h,\alpha) \stackrel{\text{def}}{=} \alpha \circ h$ for all $h \in (\overline{G}^{M^X})|_F$ and $\alpha \in K$.

$\nu_F(G|_F \times K)$ is a subset of $C(F,[-1,1])$ such that:

(i) $\nu_F(G|_F \times K) = \nu(G \times K)|_F$

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Reasoning by contradiction, suppose that $\nu(G \times K)$ is not AE.

By **Lemma 1** there exists a compact and separable subset K_F of F, an onto and continuous map $\Psi : K_F \to 2^\omega$, and a countable subset L of $\nu(G \times K)$ such that the subset $L^* \subseteq C(2^\omega,[-1,1])$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in K_F$ is not AE.

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By **Corollary 1**, G is AE.

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5 Proof of Theorem 1

6 Proof of ★

Let L be a countable subset of $G \subseteq C(X, M)$. We denote by X_L the topological space $(\widetilde{X}, t_p(\tilde{L}))$, which is metrizable because $\tilde{L}$ is countable. Consider the map $p^* : (M^{\tilde{X}}, t_p(\tilde{X})) \to (M^X, t_p(X))$ defined by $p^*(\tilde{f}) = \tilde{f} \circ p$, for each $\tilde{f} \in M^{\tilde{X}}$.

Proposition

p^* is a homeomorphism of $\overline{\tilde{L}}^{M^{\tilde{X}}}$ onto $\overline{L}^{M^X}$.

***Proof*:**

We observe that p^* is continuous, since a net $\{\tilde{f}_\alpha\}_{\alpha \in A}$ $t_p(\tilde{X})$-converges to $\tilde{f}$ in $\overline{\tilde{L}}^{M^{\tilde{X}}}$ if and only if $\{\tilde{f}_\alpha \circ p\}_{\alpha \in A}$ $t_p(X)$-converges to $\tilde{f} \circ p$ in $\overline{L}^{M^X}$.

Let's see that $p^*(\overline{\tilde{L}}^{M^{\tilde{X}}}) = \overline{L}^{M^X}$. Indeed, since p^* is continuous we have that $p^*(\overline{\tilde{L}}^{M^{\tilde{X}}}) \subseteq \overline{p^*(\tilde{L})}^{M^X} = \overline{L}^{M^X}$. We have the other inclusion because $\overline{L}^{M^X}$ is the smaller closed set that contains L and $L \subseteq p^*(\overline{\tilde{L}}^{M^{\tilde{X}}})$.

***Proof*:**

Let $\tilde{f}, \tilde{g} \in \bar{\tilde{L}}^{M^{\tilde{X}}}$ such that $\tilde{f} \neq \tilde{g}$, then there exist $\tilde{x} \in \tilde{X}$ such that $\tilde{f}(\tilde{x}) \neq \tilde{g}(\tilde{x})$. Let $x \in X$ an element such that $\tilde{x} = p(x)$, then $(\tilde{f} \circ p)(x) \neq (\tilde{g} \circ p)(x)$. So, p^* is injective because $\tilde{f} \circ p \neq \tilde{g} \circ p$. Finally, we arrive to the conclusion that $p^*|_{\bar{\tilde{L}}^{M^{\tilde{X}}}}$ is a homeomorphism because it is defined between compact spaces.