Dirichlet problems with singular convection terms and applications with and without Ildefonso

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In the first hour of the lecture, I will recall some classical results due to G. Stampacchia: let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$ and M be a bounded, elliptic, measurable matrix, $|E| \in L^N(\Omega)$, $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, $\mu > 0$ large enough, then the boundary value problem

$$(1) \quad -\operatorname{div}(M(x)\nabla u - u\,E(x)) + \mu u = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a unique weak solution u with some summability properties. If $\mu = 0$, it is possible to prove the coercivity of the differential operator if $\|E\|_{L^N(\Omega)}$ is small enough.

Nevertheless, *in the second hour*, I will show that if $f \in L^m(\Omega)$, $m \geq 1$, existence and summability properties (depending on m, but independent of the size of $\|E\|$) of weak or distributional solutions [L.B.: Boll. UMI 2008, dedicated to G. Stampacchia] can be proved. I will give examples of explicit radial solutions which show how existence and summability results can be lost in the borderline cases when f and E are smooth enough but instead of the assumption $E \in (L^N(\Omega))^N$ the weaker condition $E \in (L^q(\Omega))^N$ for any $q < N$ is imposed.

One of the main parts of the talk (*third hour*) deals with equations with $E \in (L^2(\Omega))^N$. The starting point here is the definition of solution since the distributional definition does not work. It is possible to give meaning to the solution thanks to the concept of *entropy solutions*, introduced some years ago in collaboration with Philippe+Thierry+Juan Luis. Then I wil discuss some existence results about the system

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + u = -\operatorname{div}(u\,M(x)\nabla z) + f(x) & \text{in } \Omega, \\ -\operatorname{div}(M(x)\nabla z) = u^\theta & \text{in } \Omega, \\ u = z = 0 & \text{on } \partial\Omega. \end{cases}$$

In the last hour, I will recall existence results in collaboration with Ildefonso+Daniela+François for Dirichlet problems of the type

$$(2) \quad -\operatorname{div}(M(x)\nabla u - \Phi(u)) = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and existence results in collaboration with Haim for Dirichlet problems of the type

$$(3) \quad -\operatorname{div}\Big(\frac{M(x)\nabla u}{(1+|u|)^\gamma}\Big) + u = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

in order to present the last existence results.

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