Nonlinear Models in Partial Differential Equations

An international congress on occasion of Jesús Ildefonso Díaz's 60th birthday

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Talks of June 14th

(IN ELLABORATION)

Jacobians revisited

Haim Brezis[*]

I will present joint results with H.-M. Nguyen concerning the study of the Jacobian determinant of maps from $\mathbb{R}^N$ into $\mathbb{R}^N$ (and also $\mathbb{S}^N$ into $\mathbb{S}^N$). Surprisingly, we are able to give a robust definition of a Jacobian determinant for a class of maps which do not admit derivatives (for example a Holder condition suffices). New estimates illuminate classical results of Y. Reshetnyak and J.Ball concerning the behavior of the distributional Jacobian under weak convergence in Sobolev spaces

References

H. Brezis and H.-M. Nguyen; On the distributional Jacobian of maps from $\mathbb{S}^N$ into $\mathbb{S}^N$ in fractional Sobolev and Holder spaces, *Annals of Math.* (to appear).

H. Brezis and H.-M. Nguyen; The Jacobian determinant revisited, *Inventiones* (to appear).

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A free boundary problem arising in the analysis of pressure swirl atomizers

Amable Liñán*

Pressure swirl atomizers are used to inject liquid fuels in combustion chambers, in the form of a hollow, conical, high velocity liquid film, so that it penetrates deeply while disintegrating into small droplets. The atomizers have an axisymmetric chamber where the liquid is fed, away from the axis, with an azimuthal circulating velocity; the swirling liquid flows out of the injector chamber through a cylindrical orifice of radius R_I and length L_I, small compared with those of the chamber.

It is possible to show that the axisymmetric flow can be described using an irrotational solution of the inviscid, Euler, equations, with a velocity field derived from a potential that satisfies the Laplace equation. The axisymmetric liquid domain is hollow inside and also outside the injector chamber, limited by two free boundaries that separate the liquid from the gas of the combustion chamber, of negligible density compared with that of the liquid.

We formulate the problem of determining, in terms of the volumetric flow rate and the circulating velocity, the two free boundaries of the liquid fuel, and the main characteristics of the solution, which are given explicitely for moderately large values of the ratio $\frac{\mathrm{L}_I}{\mathrm{R}_I}$.

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Free boundary problems arising in cancer models

Avner Friedmam*

Tumor growth can be modeled by a system of PDEs in a the tumor region which is continuously changing in time. The ”free boundary ” of the tumor is held together by cell-to-cell adhesion, which is assumed to be proportional to the mean curvature. The dependent variables are tumor cells densities and nutrient concentration. In addition one needs to provide a constituent law for the tissue, e.g., assuming it to follow Darcy’s law, Stokes equation, etc. In this talk I will describe such models,state existence theorems, show the existence of families of stationary spherical solutions, discuss their stability, and then proceed to describe the existence of non-radially symmetric family of solutions as bifurcation branches of the spherical ones. I will end by stating several open problems

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Energy Balance Climate Models with Bio-Feedback

Georg Hetzer *

Professor Díaz has made fundamental contributions to the study of energy balance climate models. Some of them serve as starting point for the topic of this talk, incorporating a bio-feedback in a 2-dimensional energy balance climate model.

It is well-known that there is a multitude of interactions between the climate system and the biosphere. E.g., the rise of temperature accompanied by increased rainfall may lead to a richer vegetation which in turn causes an higher absorption of solar radiation and therefore a further increase in temperature. On the other hand, higher temperatures associated with drier conditions may accelerate desertification and lead to a higher albedo with lower average temperatures as consequence.

Energy balance climate models describe the evolution of a long-term mean of temperature by employing the relevant balance equations for the heat fluxes involved. The horizontal heat flux is parameterized by a diffusion operator, and a bio-feedback can, e.g., be introduced by a Volterra map $V = V(u)\phi$ which is, say, the solution family (climate indicator u as parameter, ϕ a fixed initial vegetation) of a two-species competition system. A typical example for the resulting reaction-diffusion problem is

$$\begin{cases} c(x)\partial_t u - \nabla\cdot\left[k(x)\left|\nabla u\right|^{p-2}\nabla u\right] + g(u, V(u)\phi)(t)) \\ \qquad\qquad \in F(t, x, u, \overline{u}, V(u)\phi)(t)) \qquad t > 0,\ x \in M, \\ \overline{u}(t,x) := \int_{-T}^{0} \beta(s,x) u(t+s,x)\, ds,\ t > 0,\ x \in M, \\ u(s,x) = u_0(s,x), \qquad\qquad\qquad -T \le s \le 0, x \in M. \end{cases}$$

One is interested in nonnegative solutions $u = u(t,x)$ (temperature in Kelvin). M is a closed, compact, oriented Riemannian surface representing the Earth's surface, the positive functions c and k represent the thermal inertia and the diffusivity of the system, respectively, F stands for the absorbed solar radiation flux, and g represents the emitted terrestrial radiation flux.

I plan to discuss some models for V and the basic dynamics of the resulting problem.

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A parabolic-elliptic system of drift-diffusion type with subcritical mass in $\mathbb{R}^2$

Toshitaka Nagai *

We consider the Cauchy problem to the following parabolic-elliptic system in $\mathbb{R}^2$:

$$\text{(CP)} \quad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), & t > 0,\ x \in \mathbb{R}^2, \\ -\Delta \psi = u, & t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2. \end{cases}$$

Here ψ is specified as

$$\psi(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} u(t,y)\, dy,$$

and $u(t,x)$ and $u_0(x)$ are nonnegative. This system is a simplified version of chemotaxis system derived from the original Keller-Segel model in $\mathbb{R}^2$, and also a model of self-attracting particles in $\mathbb{R}^2$.

The total mass of the nonnegative solution u to (CP) is conserved, namely $\int_{\mathbb{R}^2} u(t,x)\, dx = \int_{\mathbb{R}^2} u_0(x)\, dx$, and the global existence of the solution heavily depends on the total mass $\int_{\mathbb{R}^2} u_0(x)\, dx$. It is known that the nonnegative solution may blowup in finite time in the supercritical case $\int_{\mathbb{R}^2} u_0(x)\, dx > 8\pi$. In this talk, we consider the subcritical case $\int_{\mathbb{R}^2} u_0(x)\, dx < 8\pi$ and discuss the global existence and large time behavior of the nonnegative solution.

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Effective Chemical Processes in Porous Media

Carlos Conca *

This lecture deals with the homogenization of nonlinear models for chemical reactive flows through the exterior of a domain containing periodically distributed reactive solid grains (or reactive obstacles). The mathematical modeling involves diffusion, different types of adsorption rates and chemical reactions which take place at the boundary of the periodic medium

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A nonlinear parabolic-hyperbolic system for contact inhibition of cell-growth

Michiel Bertsch[*], **Danielle Hilhorst**[†], **Hirofumi Izuhara**[‡] **and Masayasu Mimura** [§]

We consider a tumor growth model involving a nonlinear system of partial differential equations which describes the growth of two types of cell population densities with contact inhibition. In one space dimension, it is known that global solutions exist and that they satisfy the so-called *segregation property*: if the two populations are initially segregated - in mathematical terms this translates into disjoint supports of their densities - this property remains true at all later times. In this paper we apply recent results on transport equations and regular Lagrangian flows to obtain similar results in the case of arbitrary space dimension.

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Some asymptotic issues for variational inequalities

Michel Chipot*

We would like to study the asymptotic behaviour in $\ell \to +\infty$ of the solution to some elliptic variational inequalities set in cylinders of the type $\ell\omega_1 \mathrm{x} \omega_2$ where ω_1, ω_2 are bounded open subsets of $\mathbf{R^p}$, $\mathbf{R^{n-p}}$ respectively.

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Optimal shapes with convexity constraint

Michel PIERRE *

We will discuss some questions arising when optimizing shapes in the class of convex bodies. Since a convexity constraint provides compactness properties, it is generally easy to prove existence of optimal shapes in this class. On the other hand, the qualitative study of optimal shapes is hard, due to the difficulty of writing the Euler-Lagrange equation. Extracting information from optimality conditions is a serious issue. We will give some recent results concerning convex optimal shapes in two dimensions where it is possible to analytically write complete first and second order optimality conditions. We may deduce a description of a subclass of problems for which optimal shapes are always polygons, and another class for which no corner can appear at the boundary. Other situations surprising lead to $C^{1,1/2}$- regularity at the junction between flat and stricrly convex parts of the boundary.

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POSTER SESSION

Numerical solutions of an elastic-gravitational model by the finite element method

A. Arjona Almodóvar, T. Chacón Rebollo, M. Gómez Marmol, *

Elastic-gravitational model allow the computation of gravity, deformation, and gravitational potential changes in order to investigate crustal deformation Earth (see [2] and [3]). This model can be represented by a coupled system of linear parabolic (for the deformations) and elliptic PD equations (for gravitational potential changes) (see for instance [4], [5] and [1]).

We have considered the internal source as response to the effect of a pressurized magma reservoir into a multilayered, elastic-gravitational earth model.

We present the numerical analysis of such a coupled model by means of the finite element method for the steady model. Finally, we present some numerical tests in meaningful situations confirm the predictions theoretical order of convergence.

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Eigenvalue Problems With Fully Discontinuous Operators

Houssam Chrayteh[1]

We introduce the notion of $\overrightarrow{\rho}$–multivoque Leray-Lions operator

$$Au = -\operatorname{div}_{\overrightarrow{\rho}} \left(\partial \Phi_i \left(x, \frac{\partial u}{\partial x_i} \right) \right), \qquad \overrightarrow{\rho} = (\rho_0, ..., \rho_N)$$

that are strongly monotonic on a Banach-Sobolev function space V and we study the generalized eigenvalue problem $Au = \lambda \partial_j(u)$. Here $\partial \Phi_i$ and ∂_j denote the subdifferential in the sense of convex analysis or more generally in the sense of H. Clarke.

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Nodal and multiple solutions for a prescribed mean curvature problem

Ann Derlet *

We present some existence and multiplicity results for *sign-changing* solutions of the nonlinear mean curvature problem

$$(P)\begin{cases} -\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $2 < p < 2^*$. For instance, if the parameter λ is sufficiently large then (P) possesses at least one nodal solution u_λ with exactly two nodal domains. In addition, it is possible to prove the existence of arbitrarily many nodal solutions of (P), again for large values of λ.

Unlike many authors, we do not look for solutions of bounded variation. Instead we consider a perturbation of the degenerate part $1/\sqrt{1+s^2}$ and work in the Sobolev space $H_0^1(\Omega)$. This allows us to use classical variational techniques such as the Nehari manifold, the nodal Nehari set, and a symmetric version of the mountain pass theorem.

The above results have been obtained in collaboration with D. Bonheure (Brussels) and S. de Valeriola (Louvain-la-neuve).

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Numerical Analysis of a closed loop Thermosyphon model with a viscoelastic fluid

Ángela Jiménez-Casas, Mario Castro and Justine Yasappan *

We analyze the motion of a viscoelastic fluid in the interior of a closed loop thermosyphon under the effects of natural convection and a given external heat flux. Numerical experiments are performed in order to describe the behavior of the solution for different ranges of the relevant parameters.

$$\begin{cases} \varepsilon \frac{d^2v}{dt^2} + \frac{dv}{dt} + G(v)v = \oint Tf, v(0) = v_0, \frac{dv}{dt}(0) = w_0 \\ \frac{\partial T}{\partial t} + v\frac{\partial T}{dx} = h(x, v, T) + \gamma \frac{\partial^2 T}{\partial x^2}, T(0, x) = T_0(x) \end{cases} \tag{1}$$

Where $v(t)$ is the velocity, $T(t, x)$ is the distribution of the temperature of the viscoelastic fluid in the loop, γ is the temperature diffusion coefficient,$G(v)$ is the friction law at the inner wall of the loop, the function f is the geometry of the loop and the distribution of gravitational forces, $h(x)$ is the heat flux and ε is the viscoelastic parameter. Suitable parameters are chosen to carry out the different numerical analysis. The numerical experiments are summarized for a detailed analysis of the behaviour of the system. The experiments made in this poster come to verify the complex nature of the behavior of the models of the thermosyphon system.

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Mathematical analysis of a thermoelastic problem

P. Barral, M.C. Naya-Riveiro and P. Quintela *

In this work we carry out a mathematical analysis of the coupling between the motion and energy conservation equations for thermoelastic materials. Specifically, the existence and uniqueness of a quasistatic problem with mixed displacement-traction and Robin boundary conditions is obtained.

In the literature, there exist several existence results for thermoelastic problems, although not many deal with quasistatic problems with Robin boundary condition. A similar problem was studied by Viaño in [1], and by Figueiredo and Trabucho in [2] considering the dynamic motion equation. We use the same methodology in order to prove the existence of a solution of the problem, although we study the quasistatic problem without contact and for thermoelastic materials, taking into account that our reference temperature depends on the material point, our stress tensor has a particular thermal part and we consider a Robin boundary condition for the temperature. These contributions cause some difficulties to calculate *a priori* estimates. Furthermore, in order to prove the uniqueness their technique was not useful to us and the result is obtained following the methodology of Gawinecki [3].

Let $\Omega \subset \mathbb{R}^n$ (with $n = 2, 3$) be an open and bounded set with smooth boundary and let $[0, t_f]$ be the time interval of interest. Then, the problem we are going to study is finding a displacement field $\mathbf{u}(p, t)$ and a temperature field $\theta(p, t)$ in $\Omega \times (0, t_f]$, verifying:

- the equilibrium equations in $\Omega \times (0, t_f]$

$-\text{Div}\,\boldsymbol{\sigma}(\theta, \mathbf{u}) = \mathbf{b}$	$\boldsymbol{\sigma}(\theta, \mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - \alpha(\theta - \theta_r)(3\lambda + 2\mu)\mathbf{I}$ Λ^{-1}: The inverse elasticity tensorial function. $\boldsymbol{\varepsilon}(\mathbf{u})$: The deformation tensor. α: The coefficient of thermal expansion. θ_r, $\mathbf{b}$: The reference temperature and the body forces. λ, μ: The Lamé's parameters.
$\rho_0 c_F \dot{\theta} = -\theta_r \alpha(3\lambda + 2\mu)\text{Div}\,\dot{\mathbf{u}} + \text{Div}\,(k\nabla\theta) + f$	ρ_0: The reference density. c_F: The specific heat at constant deformation. k: The thermal conductivity of the material. f: The body heat.

- the mixed displacement-traction conditions and Dirichlet, Neumann and Robin boundary conditions in temperature.

Under appropriate assumptions we prove the existence and uniqueness of solution using the Galerkin's method.

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Vortices in a cylindrical annulus

M. C. Navarro and H. Herrero *

Thermal convection has been shown to be determinant in the formation and intensity of some meteorological events such as dust devils and cyclones: dust devils are likely to form in the presence of large horizontal temperature gradients [1], and the evolution of hurricane intensity depends, among other factors, on the heat exchange with the upper layer of the ocean under the core of the hurricane [2, 3].

We consider a fluid in a cylindrical annulus heated from below with a Gaussian profile, the governing equations are the incompressible Boussinesq Navier-Stokes equations. For the numerical implementation, non-linearities are treated with Newtons method. For the discretization (for basic state and linear stability analysis) we use a spectral method by expanding the fields in Chebyshev polynomials and evaluating at the Gauss-Lobatto points [4, 5]. Convergence properties of the numerical method depend on the parameters present in the problem.

Under certain thermal and geometrical conditions, a stable vortex, very similar to a dust devil, can be generated from a convective instability. The horizontal temperature gradient at the bottom of the annulus and the vertical temperature gradient, determine the intensity of the vortex formed and its behavior can be controlled thermally by cooling or heating adequately the bottom boundary [6]. These results connect with that observed for the evolution of the intensity of cyclones and dust devils.

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