Nonlinear Models in Partial Differential Equations

An international congress on occasion of Jesús Ildefonso Díaz's 60th birthday

Toledo (Spain), June 14-17, 2011

Talks of June 15th

(IN ELLABORATION)

On some singular parabolic equations

Herbert Amann *

We discuss a class of diffusion equations whose diffusion coefficient vanishes on a lower-dimensional submanifold S of the closure of the underlying domain Ω. In the particular case where $S = \partial\Omega$, we are led to study problems without boundary conditions. Such equations generate analytic semigroups possessing maximal regularity in appropriate weighted L_p-Sobolev spaces.

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Pattern formation: The oscillon equation

Roger Temam *

In this lecture, we will consider the oscillon equation which is used in cosmology to model and represent some transient persistent structures. We will discuss questions of existence and uniqueness of solutions and of long time behavior of solutions

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Apocalypse now: considerations of past and future climate using generalised energy balance models

A. C. Fowler*

One of the lessons one learns from the EPICA Antarctic Ice Sheet ice core, which describes climatic history over the last 800,000 years, is that atmospheric $\mathrm{CO_2}$ has fairly faithfully followed proxy (δD) temperatures over the entire period. In turn, this tells us that carbon variation in the atmosphere is a major cause of palaeoclimatic ice ages. In order to understand this, we need to include a description of the carbon cycle. This includes the production of $\mathrm{CO_2}$ by volcanoes, its loss from the atmosphere by acid rain and consequent weathering and run off to the ocean, and the buffering of carbon in the ocean between the reservoirs of bicarbonate, carbonate and dissolved $\mathrm{CO_2}$.

The simplest possible model for the interaction of these components is a compartment (o.d.e.) model for the concentrations of $\mathrm{HCO_3^-}$, $\mathrm{CO_3^{2-}}$, $\mathrm{CO_2}$, as well as calcium ion $\mathrm{Ca^{2+}}$, calcium carbonate $\mathrm{CaCO_3}$ and acidity $\mathrm{H^+}$ ($\mathrm{pH} = -\log_{10}[\mathrm{H^+}]$) in the ocean, together with a conservation law for atmospheric $\mathrm{CO_2}$. To this set we add the simplest (o.d.e.) energy balance model describing the dependence of temperature on cloud and ice albedo, and on $\mathrm{CO_2}$ and cloud greenhouse effect, as well as an ice sheet growth model which allows for nucleation and the ice sheet elevation effect, thus providing for bistability in the growth of the Pleistocene ice sheets.

The simplest asymptotic reduction of this generalised energy balance model is to two equations for ice sheet volume I and $\mathrm{CaCO_3}$ concentration N, and in certain circumstance the model exhibits self-sustained oscillations, which represent a candidate for the 100,000 year periodic ice ages of the last half a million years.

In passing, we note the implication of the model for anthropogenic warming: century scale equilibration to an atmosphere where the ice sheets melt, as they are apparently doing, with consequent sea level rise of a metre or more per century.

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The Partial Differential Equations of Finance

Olivier Pironneau*

The object of financial mathematics is to price financial object and evaluate the risks.

To minimize risks self financing portfolios has been a great source of models. Very simple - yet unrealistic - assumptions such as brownian randomness and market completeness (no arbitrage) lead to stochastic differential equations and Itô calculus then gives the partial differential equations for pricing the financial objects.

Even in this restricted setting a large class of partial differential equations can be generated, mostly parabolic in nature, linear but with non constant coefficients. Variational methods work in a slightly adapted Hilbert space setting (weighted Sobolev spaces).

At first one dimensional in space, the partial differential equations are now multidimentional because of stochastic volatility models or because of stochastic interest rates or because the object depends on several assets. For these existence is not always guaranteed, the so called Feller condition is necessary in the case of Stein-Stein models. They can also be degenerate as in the case of Asian options.

New boundary conditions are needed sometimes such a for Bermuda options.

Non-linear systems are not used often so far; but an important class of objects do need nonlinear models, when early exercise is allowed such as for American options. Then the problem is best solved as a variational inequality.

When the Brownian assumption is relaxed, like when jump processes are added, a partial integro differential system is obtained, similar to the equations for radiative transfers in engineering. In several dimensions these problems are challenging and not solved yet.

Finally options on compound objects or basket options lead to partial differential equations in many dimensions as much as 50 or more. These are numerically extremely challenging.

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Travelling waves in a convection-diffusion equation

Peter Takáč *

Abstract: We will discuss existence and stability of travelling waves for a nonlinear convection diffusion equation in the 1-D Euclidean space. The diffusion coefficient depends on the gradient in analogy with the p-Laplacian and may be degenerate or singular. We establish unconditional stability with respect to initial data perturbations in $L^1(\mathbb{R})$. Although our solutions typically do not belong to $L^1(\mathbb{R})$, their difference usually does belong there; therefore, the $L^1(\mathbb{R})$ metric is of crucial importance in our approach.

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Asymptotic behavior of solutions of elliptic and parabolic problems blowing up at the boundary

Catherine Bandle *

Semi linear problems whose nonlinearities grow strongly at infinity give rise to large solutions which blow up at the boundary. The geometry of the boundary has in general no effect to the first order asymptotic behavior of these solutions, which is expressed as a function of the distance to the boundary. It appears only in the higher order terms. In this talk we shall give a survey on the behavior of large solutions near the boundary and discuss some mechanism such as a Hardy potential or additional nonlinear gradient terms, which can have a considerable effect and determine the boundary behavior of the large solutions.

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On the entropy conditions for some flux limited diffusion equations

Vicent Caselles *

In this talk we give a characterization of the notion of entropy solutions of some flux limited diffusion equations for which we can prove that the solution is a function of bounded variation in space and time. This includes the case of the so-called relativistic heat equation and some generalizations. For them we prove that jump set consists in fronts that propagate at the speed given by Rankine-Hugoniot condition and we give on it a geometric characterization of the entropy conditions. Since entropy solutions are functions of bounded variation in space once the initial condition is, to complete this program we study the time regularity of solutions of the relativistic heat equation under some conditions on the initial datum. An analogous result holds for some other related equations without additional assumptions on the initial condition.

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An example of functional which is weakly lower semicontinuous on $W_0^{1,p}$ for every $p > 2$ but not on H_0^1

François Murat *

In this work in collaboration with Fernando Farroni and Raffaella Giova, we give an example of functional which is defined and coercive on $H_0^1(\Omega)$, which is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega)$ for every $p > 2$, but which is not sequentially lower semicontinuous on $H_0^1(\Omega)$. This functional is non local.

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A variational approach to Navier-Stokes

Pablo Pedregal *

We introduce a variational approach to treat the Navier-Stokes equations both in dimensions 2 and 3. Though the method allows the full treatment in dimension 2, we seek to precisely stress where it breaks down for dimension 3. The basic feature of the procedure is to look directly for strong solutions, by minimizing a suitable error functional that measures the departure of feasible fields from being a solution of the problem. By considering the divergence-free property as part of feasibility, we are able to avoid the explicit analysis of the pressure. Two main points in our analysis are:
1. Coercivity for the error functional is achieved by looking at scaling.
2. Zero is the only critical value: global minimizers of the error are shown to have zero error (and thus they are solutions of the problem) by looking at optimality conditions, which lead to investigate the linearized problem.

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POSTER SESSION

Numerical analysis of a climate model

Arturo Hidalgo [*] **and Lourdes Tello** [†]

In this work we consider a coupled model surface/deep ocean, which was first proposed by Watts-Morantine (1990). It is a diagnostic model which can be used to understand the long-term climate evolution. The unknown is the temperature over each parallel and the effect of the deep ocean on the temperature of the Earth surface is considered. One of the main difficulties of this problem is the dynamic and diffusive boundary condition. The purpose of this work is to develop a numerical scheme to obtain an approximate solution of the coupled model. The numerical technique used is based on the finite volume method together with WENO reconstruction and a Runge-Kutta TVD scheme for time discretization. As an important consequence, we analyze the behaviour of the solution of the energy balance model with and without the effect of deep ocean. This sort of climate models has been extensively studied by J.I. Díaz.

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Nonlinear models in partial differential equations arising in nuclear fusion

J.F. Padial *

We carry out mathematical analysis of some bidimensional nonlinear problems satisfied by the averaged poloidal flux of the magnetic field in the magnetic confinement of a plasma in the nuclear fusion. We show some mathematical models related to the stationary and evolution regime of a plasma in Tokamak and Stellarotor devices. The models can be formulated as an inverse problems since several nonlinear terms of the partial differential equation are not a priori known (non local terms). Using the current balance within each flux magnetic and the notion of relative rearrangement we can reformulate the problem as a non local one but having a direct formulation. We review some models concerning to Stellarator devices and we review some results about existence, uniqueness and regularity of solutions.

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Modelling a High-Pressure Shift Freezing Process

Stephen Peppin[1], Ángel Manuel Ramos[2] and Nadia Smith[2] *

Freezing is a widespread food preservation technology, as it ensures high food quality with long storage duration, and also because it has an extended implementation area (meat, fish, fruit and vegetables, dairy and egg products, etc.). Despite the benefits, freezing of foods can also cause undesirable changes in their texture and organoleptic properties, and its main drawback is the risk of food damage due to the formation of ice crystals. The size and location of the crystals formed during the freezing process depend on the freezing rate (slow freezing produces large crystals, whilst rapid freezing promotes intensive nucleation and the formation of small ice crystals) and the final temperature of the process. The general purpose of food technologists working on this area has been to create a homogeneous matrix of small ice crystals. Improvement of known freezing methods and development of new techniques are important research goals for the food industry at present. With the recent increasing impact of High-Pressure technology on Food Processing, there has been a lot of research dealing with the potential applications of High-Pressure effects on ice-water transitions, given that pressure decreases the freezing and melting point of water to a minimum of $-22°$C at 207.5 MPa, namely High-Pressure Freezing and Thawing.

One particular case of High-Pressure Freezing is High-Pressure Shift Freezing (HPSF), in which phase transition occurs due to a pressure change that promotes metastable conditions and instantaneous ice production. On expansion, pressure release occurs instantaneously throughout the product (Pascal principle), and subsequently, its temperature decreases. Large-scale supercooling takes place throughout the sample, which implies high ice nucleation velocities. Different authors have proved experimentally that ice nucleation occurs homogeneously throughout the whole volume of the product and not only on the surface, as they have found small granular shape ice crystals disperse throughout the resulting sample for several products. When comparing HPSF to classical freezing processes, important reductions of freezing times have been reported.

In this work we derive a model for a HPSF process of a solid type food, with a big and small filling sample vs pressurizing media ratio. We present a heat transfer model derived from an enthalpy formulation based on volume fractions dependent on temperature and time, that simulates the temperature profile during a HPSF process, calculating also the amount of ice instantaneously formed after expansion and the supercooling of the sample.

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On the Lipschitz Property of the Relative Rearrangement

María Luisa Seoane *

Since its introduction as the derivative of the decreasing rearrangement u_*, several properties of the relative rearrangement b_{*u} have been established, in particular the Lipschitz condition considering the relative rearrangement like a function of b. Here, our interest will be focused on the relative rearrangement considered as a function of u which is a new feature as far as we know.

We give some useful estimations for the numerical solution of differential equations involving relative rearrangements of a data function b with respect to the solution u, when fixed point techniques are used. This kind of nonlocal problems is widely discussed in the literature dealing the plasma physics. In order to prove the convergence of iteration schemes it is helpful to have a Lipschitz condition (even more, a contractive property) of nonlocal terms. We shall just suggest a way to obtain them and analyse some suitable hypothesis.

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Mathematical treatment of a climatological model with deep ocean effect

Lourdes Tello *

This poster is a review of some results obtained in collaboration with J.I. Díaz in the last years. We have studied the existence and multiplicity of solutions to several transient and equilibria simple climate models.

In the last decades, many authors have studied the so-called global climate energy balance models (EBM) dealing with the evolution of the mean surface temperature of the Earth. Among them we can mention D. Arcoya, R. Bermejo, J.I. Díaz, M. Ghil and S. Childress, J. Hernández, G. Hetzer, G.R. North, B. Schmidt, X. Xu, etc.

In many of the previous models, the effect of the oceans is only considered in an implicit and empirical way in the spatial dependence of the coefficients. However, some works about the rapid climatic change in Glacial-Holocene transition (see Berger et al) show that it could be related to the past changes in deep water formation. In this work we study a model including the effect of the deep ocean based on the model proposed by Watts - Morantine.

The simplified model represents the evolution of the temperature U in a global ocean Ω with constant depth H. The upper boundary of Ω simulates the Earth surface. The governing equation for the ocean interior is parabolic and the upper boundary condition comes from an energy balance for the mean surface temperature of the Earth. In such energy balance, the absorbed energy depends on the planetary coalbedo β (which is discontinuous on U). The coalbedo is treated in the general class of multivalued graphs. More precisely, we shall assume that β is a bounded maximal monotone graph of $\mathbb{R}^2$. Other nonlinearity at the boundary concerns the surface diffusion proposed by P.H. Stone: its diffusion coefficient depends on the temperature gradient in order to include the negative feedback of the eddy fluxes. So, the energy balance involves in this way the p-Laplacian surface operator.

This kind of models is very sensitive to small fluctuations of Solar and terrestrial parameters. We also analyze how the Solar constant is related to the number of equilibrium solutions. One of the main difficulties in the mathematical treatment of these models comes from the presence of a nonlinear dynamic and diffusive boundary condition in its formulation.

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