Nonlinear Models in Partial Differential Equations

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(IN ELLABORATION)

Wave Equation with p(x,t) - Laplacian and Damping Term: Existence and Blow-up

S. Antontsev *

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary Γ and $Q_T = \Omega \times (0,T]$. We consider the following boundary value problem

$$\begin{cases} u_{tt} = \operatorname{div}\left(a\,|\nabla u|^{p(x,t)-2}\,\nabla u + \alpha \nabla u_t\right) + b\,|u|^{\sigma(x,t)-2}\,u + f, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega;\ u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega. \end{cases} \tag{1}$$

The coefficients $a(x,t),\ \alpha(x,t),\ b(x,t),$ exponents $p(x,t), \sigma(x,t)$ and the source term $f(x,t)$ are given functions of their arguments satisfying

$$0 < a_- \le a(x,t) \le a_+ < \infty,\ 0 < \alpha_- \le \alpha(x,t) \le \alpha_+ < \infty,\ |b(x,t)| \le b_+ < \infty, \tag{2}$$

$$1 < p_- \le p(x,t) \le p_+ < \infty, 1 < \sigma_- \le \sigma(x,t) \le \sigma_+ < \infty, \tag{3}$$

$$f \in L^2(Q_T),\ u_1 \in L^2(\Omega),\ u_0 \in L^2(\Omega) \cap L^{\sigma(\cdot,0)}(\Omega) \cap W^{1,p(\cdot,0)}(\Omega). \tag{4}$$

Problem (1) appears in models of nonlinear viscoelasticity. The local and global existence theorems and blow- up effects for solutions hyperbolic equations of the type (1) with constant exponents of nonlinearity have been studied in many papers , (see, e.g., [5]). However, only papers [6, 7] are devoted to the study of hyperbolic equations of the type (1) with variable nonlinearities. In the present communication, we discuss how the variable character of nonlinearity influences the existence and blow-up theory for the EDPs of the type (1). The analysis is based on the methods developed in [1]-[4].

References

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Competition of subspecies and structural stability. Survival of the best adapted or coexistence?

Evariste Sanchez–Palencia *

Equations of population dynamics allow us to consider the competition of two species in an environment. According to the specific properties of their interaction, competition may lead to extinction of one of the species, but also in certain cases to coexistence of both.

In our work, we start from the dynamics of the population of a species; we split it into the populations of two subspecies (of the same specie) with the same properties. It appears that the corresponding dynamical system is structurally unstable, $i.e.$ small perturbations modify drastically the structure of the phase portrait of the dynamics. In that context, we introduce a small perturbation of the properties of one of the subspecies ("mutation"). It clearly appears that, according to the instability of the starting state, the issue of he mutation depends drastically on the very shape of the (even very small) perturbation. Specifically, when the modification may be clearly understood (whatever the circumstances) as an "advantage" of one of the subspecies on the other, the dynamics leads, generally speaking, to the extinction of the less adapted subspecies. But, *when the modification is "rather complex", containing features which are "advantages" or "disadvantages" according to the circumstances (in particular initial conditions), the dynamics leads generally to another equilibrium with coexistence of both subspecies*.

Analogous results follow from a starting situation mainly described by two species interacting in a predator / prey framework with a stable periodic cycle. The splitting of one of the species into two subspecies constitutes a structurally unstable system, and the perturbation (mutation) of one of the subspecies may have very different issues. In "rather complex" situations, the dynamics often leads to one or several periodic cycles with coexistence of the three new species.

On account of the fact that genetic mutations imply modifications *of a specific protein*, which, in its turn has in general an influence on *several behavioral properties*, it clearly appears that *the population dynamics lead to the extinction of the less adapted only in very elementary cases; in usual, more complex cases, it rather leads to the preservation of the variety*.

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Localization properties of solutions of parabolic equations with variable anisotropic nonlinearity

Sergey Shmarev *

We are interested in the character of propagation of disturbances from the data in solutions of parabolic equations with variable and anisotropic nonlinearity. The prototype of such equations is furnished by the equation

$$u_t = \sum_{i=1}^{n} D_i \left(|D_i u|^{p_i(x,t)-2} D_i u \right) + c_0 |u|^{\sigma(x,t)-2} u + f.$$

Anisotropy and variable nonlinearity lead to certain properties intrinsic for the solutions of equations of this type. We prove that unlike the case of isotropic diffusion the solutions vanish in a finite time even in the absence of absorption (i.e. if $c_0 = 0$), provided that the diffusion is fast in only one direction. It is shown that in the case of slow anisotropic diffusion the supports of solutions display a behavior typical for the solutions of equations with strong absorption terms: the support does not expand in the direction corresponding to the slowest diffusion. For certain ranges of the nonlinearity exponents the supports are localized both in space and time. We also discuss the influence of anisotropy on the blow-up of solutions and show that for equations with variable nonlinearity the effects of finite time vanishing and blow-up may happen even if the equation becomes linear as $t \to \infty$. The results were obtained in collaboration with S. Antontsev. The presentation follows the papers [2, 3, 4]; the main tool is the method of local energy estimates developed in [1].

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Nonlinear fractional diffusion equations of porous medium type

Juan Luis Vázquez *

The study of diffusion involving fractional Laplacian operators is a topic of much current research. We present two models for flow in porous media including nonlocal (long-range) diffusion effects of such type.

The first one is based on Darcys law and the pressure is related to the density by an inverse fractional Laplacian operator. We prove existence of solutions that propagate with finite speed, which is unexpected in fractional diffusion models. The model has also the very interesting property that mass preserving self-similar solutions can be found by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density. We use entropy methods to show that the asymptotic behaviour is described after renormalization by these solutions which play the role of the Barenblatt profiles of the standard porous medium model.

The second model comes from statistical mechanics considerations, generalizes the well-known linear fractional heat equation, generates a nice nonlinear contractive semigroup, and has infinite speed of propagation for all powers of the nonlinearity.

Some references

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Dirichlet problems with singular convection terms and applications with and without Ildefonso

Lucio Boccardo *

In the first hour of the lecture, I will recall some classical results due to G. Stampacchia: let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$ and M be a bounded, elliptic, measurable matrix, $|E| \in L^N(\Omega)$, $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, $\mu > 0$ large enough, then the boundary value problem

$$(1) \quad -\operatorname{div}(M(x)\nabla u - u\,E(x)) + \mu u = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a unique weak solution u with some summability properties. If $\mu = 0$, it is possible to prove the coercivity of the differential operator if $\|E\|_{L^N(\Omega)}$ is small enough.

Nevertheless, *in the second hour*, I will show that if $f \in L^m(\Omega)$, $m \geq 1$, existence and summability properties (depending on m, but independent of the size of $\|E\|$) of weak or distributional solutions [L.B.: Boll. UMI 2008, dedicated to G. Stampacchia] can be proved. I will give examples of explicit radial solutions which show how existence and summability results can be lost in the borderline cases when f and E are smooth enough but instead of the assumption $E \in (L^N(\Omega))^N$ the weaker condition $E \in (L^q(\Omega))^N$ for any $q < N$ is imposed.

One of the main parts of the talk (*third hour*) deals with equations with $E \in (L^2(\Omega))^N$. The starting point here is the definition of solution since the distributional definition does not work. It is possible to give meaning to the solution thanks to the concept of *entropy solutions*, introduced some years ago in collaboration with Philippe+Thierry+Juan Luis. Then I wil discuss some existence results about the system

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + u = -\operatorname{div}(u\,M(x)\nabla z) + f(x) & \text{in } \Omega, \\ -\operatorname{div}(M(x)\nabla z) = u^\theta & \text{in } \Omega, \\ u = z = 0 & \text{on } \partial\Omega. \end{cases}$$

In the last hour, I will recall existence results in collaboration with Ildefonso+Daniela+François for Dirichlet problems of the type

$$(2) \quad -\operatorname{div}(M(x)\nabla u - \Phi(u)) = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and existence results in collaboration with Haim for Dirichlet problems of the type

$$(3) \quad -\operatorname{div}\Big(\frac{M(x)\nabla u}{(1+|u|)^\gamma}\Big) + u = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

in order to present the last existence results.

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On nodal domains of u solution to $-\Delta u = \mu u + f$, $u \in H_0^1(\Omega)$ and μ close to an eigenvalue

Jacqueline Fleckinger -in collaboration with J.P.Gossez and F. de Thélin- *

Consider the Dirichlet problem

$$\begin{cases} -\Delta u = \mu u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω a smooth bounded domain in $I\!R^N$ and $f \in L^q(\Omega)$, $q > N$. Let $\hat{\lambda}$ be an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, with $\hat{\varphi}$ an associated eigenfunction. We compare the nodal domains of u with those of $\hat{\varphi}$. In particular, under suitable assumptions on f (in particular $\int_\Omega f\hat{\varphi} > 0$) and on the nodal domain of $\hat{\varphi}$, for μ sufficiently close to $\hat{\lambda}$, then the solution u of the problem has the same number of nodal domains as $\hat{\varphi}$, and the nodal domains of u appear as small deformations of those of $\hat{\varphi}$. But this is not always the case and we exhibit some examples and counterexamples for various hypotheses.

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New and old problems in Applied Mathematics

Miguel Ángel Herrero*

Ever since its foundation in 1984, the Department of Applied Mathematics of Universidad Complutense has been a meeting point for applied -minded mathematicians with an interest in real -world problems , and eager to become part of the international scientific community, thus breking a tradition of isolation. This trend was fully developed during Ildefonso Diaz's tenure as Head of Department , from 1984 to 1994, and this lecturer particularly benefited from it.

As a short illustration of ongoing research, I will briefly describe recent work on issues as formation of early vasculature, optimization problems in radiotherapy and the search for earliest fossils on Earth, as well as the role of Mathematics in addressing such issues will be discussed.

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On some singular differential equations arising in the modeling of non-elastic shocks in unilateral mechanics and decision sciences

H. Attouch *

We consider the second-order differential system with Hessian-driven damping $\ddot{u} + \alpha\dot{u} + \beta\nabla^2\Phi(u)\dot{u} + \nabla\Phi(u) + \nabla\Psi(u) = 0$, where $\mathcal{H}$ is a real Hilbert space, $\Phi, \Psi : \mathcal{H} \to \mathbb{R}$ are scalar potentials, and α, β are positive parameters. A striking property of this system is that, after introduction of an auxiliary variable y, it can be equivalently written as a first-order system involving only the time derivatives $\dot{u}$, $\dot{y}$, and the gradient operators $\nabla\Phi$, $\nabla\Psi$. This allows us to extend the previous analysis to the case of a convex lower semicontinuous function $\Phi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, and so introduce constraints in our model. When $\Phi = \delta_K$ is equal to the indicator function of a closed convex constraint set $K \subseteq \mathcal{H}$, the subdifferential operator $\partial\Phi$ takes account of the contact forces, while $\nabla\Psi$ takes account of the driving forces. In this setting, by playing with the geometrical damping parameter β, we can describe non elastic shock laws with restitution coefficient. Taking advantage of the infinite dimensional framework, we introduce a nonlinear hyperbolic PDE describing a damped oscillating system with obstacle. We complete this study by an asymptotic Lyapunov analysis. We show that this system is dissipative, each trajectory weakly converges to a minimizer of $\Phi + \Psi$, provided that Φ and $\Phi + \Psi$ are convex functions. Exponential stabilization is obtained under strong convexity assumptions. This study opens up interesting perspectives for the modeling of shocks in mechanics and decision sciences (economical shocks, stress and burnout...).

This presentation draws heavily on recent results obtained with P.E. Mainge and P. Redont.

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Some contributions about degenerate parabolic equations

Jacques Giacomoni and Monique Madaune-Tort *

The first part of the talk will trace the history of the relationship and the past and current cooperation between Ildefonso Díaz and the University of Pau.

The second part of the talk will concern the following quasilinear and singular parabolic equation:

$$\begin{cases} u_t - \Delta_p u = \dfrac{1}{u^\delta} + f(x,u) \text{ in } (0,T)\times\Omega, \\ u = 0 \text{ on } (0,T)\times\partial\Omega, \quad u > 0 \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{P$_t$}$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^{\mathrm{N}}$, $1 < p < \infty$, $0 < \delta$ and $T > 0$. We assume that $(x,s) \in \Omega \times \mathbb{R}^+ \to f(x,s)$ is a bounded below Caratheodory function, locally Lipschitz with respect to s uniformly in $x \in \Omega$ and asymptotically sub-homogeneous.

We present in this talk recent results concerning the following issues:

1) existence and uniqueness of weak solutions to (P_t),

2) regularity of weak solutions to (P_t),

3) long-time behaviour and stabilization.

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