Metastable Solutions for the Thin-Interface Limit of a p-Laplacian Phase Field Model

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We consider a generalization of the semilinear phase field model from [1, 4, 5], using a non-linear diffusion operator (p-Laplacian) for the phase field function; in particular, the following one-dimensional nonlinear parabolic system

$$\left\{ \begin{array}{rcll} \tau\varphi_t & = & \xi^p(|\varphi_x|^{p-2}\varphi_x)_x - g'(\varphi) + ch'(\varphi)u, & x \in (a,b) \\ u_t & = & ku_{xx} - \frac{l}{2}(h(\varphi))_t, & x \in (a,b) \\ \varphi'(a) & = & \varphi'(b) = 0 & \\ u'(a) & = & u'(b) = 0 & \\ \varphi(0,x) & = & \varphi_0(x) \in H^1(a,b) & \\ u(0,x) & = & u_0(x) \in L^2(a,b) & \end{array} \right. \qquad (1)$$

Here $u(t,x)$ represents the temperature of the point x at time t of a substance which may appear in two different phases, (for example liquid-solid) and $\varphi(t,x)$ is the phase field function or order parameter, which represents the local phase average. The functions $g(\varphi)$ and $h(\varphi)$ are positive , such that $g'(\varphi)$ is typically $\frac{1}{2}(\varphi^3 - \varphi)$, and $h'(\varphi) = (1-\varphi^2)^a$ with $a = 0, 1$ (see[6]).The positive constants l and k refer to latent heat and diffusivity, whereas ξ (interface width) and τ are positive parameters related to time and length scales. Finally c is also a positive constant which measures the strenth of the coupling between the fields (see [1]) and the enthalpy function given by $v = u + \frac{l}{2}h(\varphi)$, allows to study more general couplings between a diffusion field and a phase-field. For instance, the phase-field can be seen as the density of bacterial collony or the mass of growing tumor. Analogously, the diffusion field can stand for the density of nutrient.

We study the behavior of the solutions with $\tau = \tau(\xi)$, when ξ goes to zero (thin-interface limit [6]). The main objective of this work is to prove the existence of the metastable solutions of the generalized system (1),($p \neq 2$) that evolve very slowly in time (see [2, 3, 4, 5].)

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