Physical viscosity numerical methods for non-conservative hyperbolic systems

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The design of numerical methods for nonconservative hyperbolic systems of the form

$$w_t + A(w)w_x = 0$$

is a very active front of research, as PDE systems of this nature arises in many flow models. One of the main difficulties both from the theoretical and the numerical point of view is that, for discontinuous solutions, the nonconservative products $A(w)w_x$ cannot be defined within the framework of the distributions. Nevertheless, there are different mathematical theories that allow one to define these products as Borel measures: sea [1]. Unfortunately, these theories lead to different notion of weak solutions: in particular, the jump conditions across a discontinuity depend on the chosen notion. Therefore, an important issue is to choose the definition of weak solutions in such a way that the jump conditions are in good agreement with the physics of the problem.

When the hyperbolic system is the vanishing-viscosity limit of the parabolic problems

$$w_t^\epsilon + \mathcal{A}(w^\epsilon)\, w_x^\epsilon = \epsilon(\mathcal{R}(w^\epsilon)w_x^\epsilon)_x, \tag{1}$$

where $\mathcal{R}(w)$ is a positive semi-definite matrix, the physical jump conditions should be related to the viscous profiles of the regularized problem (1). While in the case of conservative systems the usual Rankine-Hugoniot conditions are recovered independently of the choice of $\mathcal{R}(w)$, this is not the case for non-conservative systems. Therefore, the definition of weak solutions should be such that the corresponding jump conditions are consistent with the viscous profiles related to the physical viscosity.

Once the definition of weak solution has been chosen, a second difficulty is related to the design of numerical methods providing approximations whose limits are weak solutions of the system: see [2]. The difficulty comes from the fact that most of the usual discretization methods for P.D.E. involve some numerical diffusion and, due to this, the numerical approximations converge to functions that are classical solutions where they are smooth but whose discontinuities are consistent with the numerical viscosity but not with the physical one. This phenomenon has been first studied by T. Hou and P. LeFloch [3] when a non-conservative numerical scheme is applied to a system of conservation laws.

The strategy proposed here is to consider numerical schemes which are obtained by adding a physical viscosity term to a second order Lax-Wendroff method. The numerical viscosity coincides thus with the physical one so that the convergence to the correct weak solutions is expected to be improved. This improvement will be illustrated by applying this strategy to he strategy to a non-conservative coupled Burgers system.

References:

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