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Algorithms and Applications in Real Algebraic and Tropical Geometry

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Introduction

The research of the members of the project MTM2011-25816-C02-02 includes the study of hypercircles and ultra-cuadrics, tropical geometry, reparametrization of rational curves, the computation of the topology of a curve given by values, the parametrization of bisectors of curves and surfaces, the Voronoi diagram of a family of half-lines, algebraic geodesy and GPS modeling, and some others. Part of this work has been done in collaboration with the researchers of Project [3].

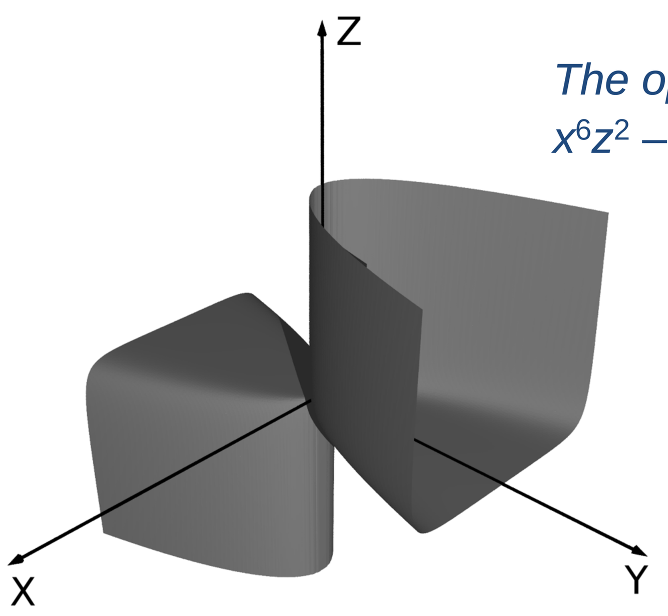
For this poster, we have chosen four topics which have applications in industrial problems: Reparametrization of Swung surfaces, Transformation of coordinates, from Cartesian to hyperboloidal, Voronoi diagram of a family of parallel half-lines, and Bisectors of low degree surfaces.

Swung surfaces

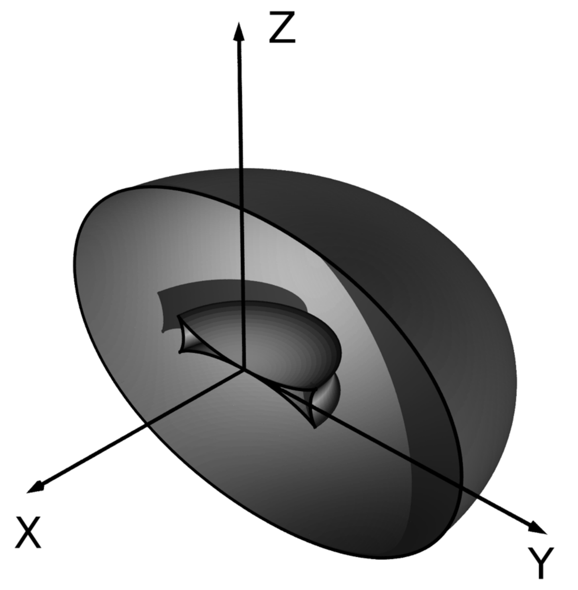
A surface of revolution is a surface globally invariant by rotations around a certain line (the axis of revolution). The intersection of the surface with planes containing the revolution axis yields the so called profile curves. We studied a natural extension of surfaces of revolution called *swung surfaces,* of relevance in CAGD. They are produced by sweeping around the z-axis a profile curve in the yz-plane along a trajectory curve in the xy-plane.

- Profile curve (rational): $(0, \phi_1(t), \phi_2(t))$
- Trajectory curve (rational): $(\psi_1(s), \psi_2(s), 0)$
- Parametrized swung surface:

$$P(s,t) = (\phi_1(t)\,\psi_1(s), \phi_1(t)\,\psi_2(s), \phi_2(t))$$



The open "doble cencerro"
$x^6z^2 - 8y^3z + 6x^4zy + x^6 = 0$



The offset of an ellipsoid (half)

$$-240\,y^2z^2x^2 + 66\,y^2z^4x^2 + 30\,y^4z^2x^2 + 30\,y^2z^2x^4 + 450\,z^2y^2 - 120\,y^4z^2 - 210\,y^2z^4 - 30\,y^4x^2 - 30\,y^2x^4 - 120\,z^2x^4 - 210\,z^4x^2 + 450\,z^2x^2 + 18\,x^2y^2 + 40\,y^2z^6 + 10\,y^6z^2 + 33\,y^4z^4 + 4\,y^6x^2 + 6\,y^4x^4 + 4\,y^2x^6 + 33\,z^4x^4 + 40\,z^6x^2 + 10\,z^2x^6 - 207\,z^4 - 324\,z^2 + 9\,x^4 + 9\,y^4 + 8\,z^6 - 10\,y^6 - 10\,x^6 + 16\,z^8 + y^8 + x^8 = 0$$

Surfaces of revolution and all quadrics are swung surfaces.

Assume that the coefficients of the parametrization are in **K**($\imath$), where **K** is a computable subfield of **R** (e.g. **Q**) and $\imath$ is the imaginary unit.

Does every swung surface admit a reparametrization with real coefficients?

Theorem. *A swung surface admits a real reparametrization, if and only if, the surface is "real", in the sense of having a two dimensional piece in* **R**^{3}.

From 3D Cartesian to hyperboloidal coordinates

Hyperboloidal coordinates (λ, ϕ, h) are used in Geodesy. In particular, in hyperboloidal building and cooling tower construction.

$$X = (\nu + h)\cos\phi\cos\lambda\,,$$
$$Y = (\nu + h)\cos\phi\sin\lambda\,,$$
$$Z = (\nu(e^2-1) - h)\sin\phi\,,$$

ϕ: latitude, λ: longitude, h: heigth

Prime normal section curvature

$$\nu = \frac{a}{\sqrt{1-e^2\sin^2\phi}},$$

Eccentricity of the hyperboloid

$$e = \sqrt{1+b^2/a^2}$$

a is the semi-axis

The transformation problem is reduced to find the smallest positive root of a polynomial of degree 4 (see [6]). The analysis of the roots is performed by an algebraically complete stra-tification, based on symbolic techniques (Sturm-Habicht sequences), of a planar region situated in the positive quadrant. We propose two approaches:

- using the Merriman method,
- using the Computer Algebra System `Maple`.

Voronoi diagram of n parallel half-lines

The Voronoi diagram (VD) is a fundamental data structure in computational geometry with various applications in theoretical and practical areas (see, for example [4]). Consider the VD of a set of n parallel half-lines in

$$D_0 = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$$

of the following form

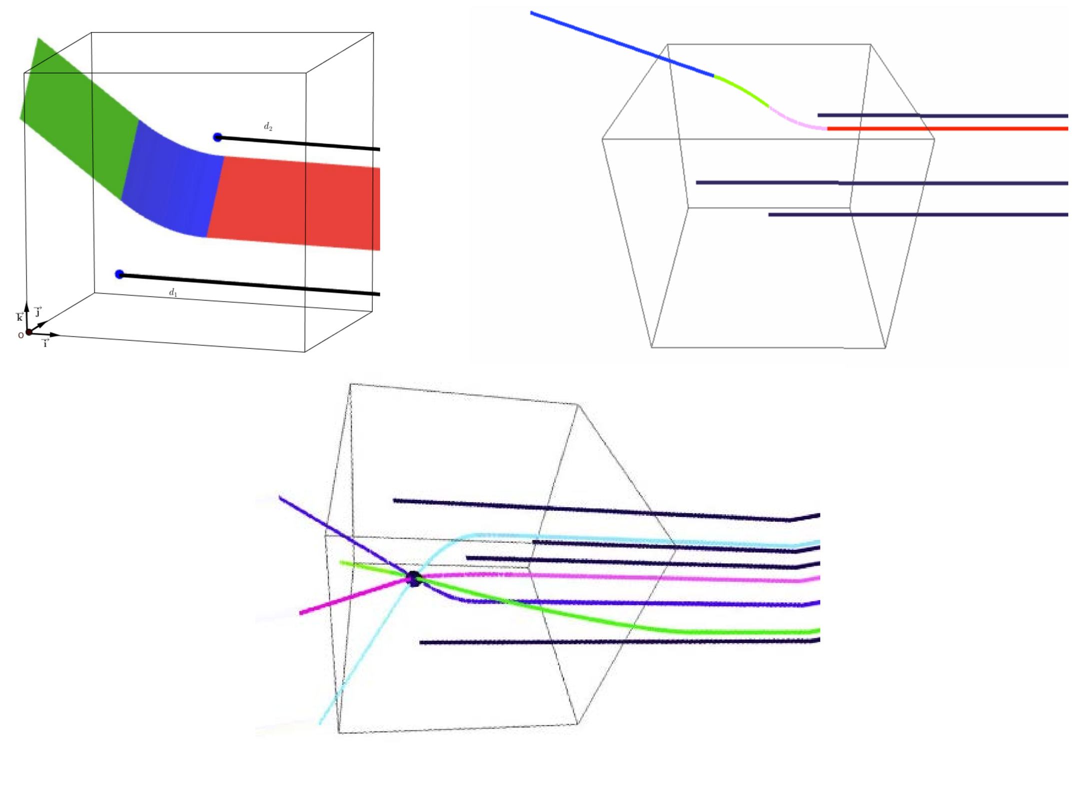
$$d_i = (x_i + t, y_i, z_i),\ t \geq 0,\ i = 1, \ldots, n,$$

$$x_l \neq x_k, (y_l, z_l) \neq (y_k, z_k), \forall l \neq k$$

This new kind of VD has applications in drilling system design, in mining, oil, and other industries (see [5], [8]).

Each cell of the VD is formed by the points in D_0 that are closer to a particular d_i than to any other d_j, $j \neq i$. The boundary of a cell is composed by:

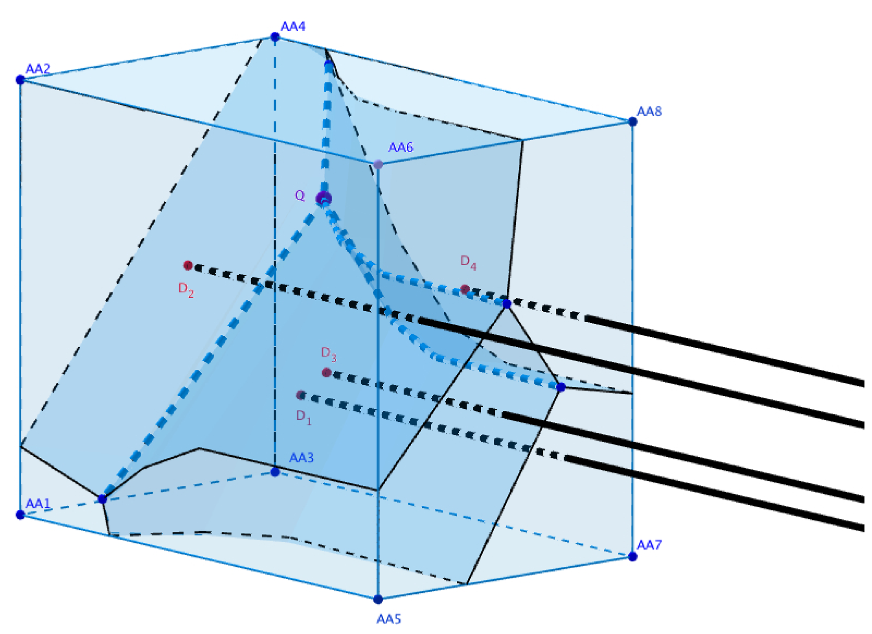
- **Bisectors**: piecewise algebraic surfaces formed by pieces of planes and parabolic cylinders.
- **Trisectors**: piecewise algebraic curves formed by straight segments and pieces of parabolas.
- **Quadrisectors**: a point, intersection of four trisectors.



The algorithm

Input: d_i, i = 1, … , n; a small constant $\varepsilon > 0$.
Output: a list of rectangular boxes, and a mesh approximating the VD; the topology in each box is correct or the box is of size < ε.

- **Subdivision**: A box is said to be *t-regular* if the topology of bisectors, trisectors and quadrisectors is determined by its intersection with the faces of the box. Starting with D_0, if a box is not t-regular is subdivided in two smaller boxes. The procedure continues, generating an adjacency graph, following a *kd-tree* structure, until all the boxes are either t-regular or of diameter < ε.
- **Meshing**: In each t-regular box, a mesh of polygonal faces is determined. The mesh is isotopic to the boundaries of the VD cells inside that box.
- **Reconstruction of approximate VD cells**: The adjacency graph is traversed using a *DFS* (*Depth First Search*) algorithm.



The bisector of two low degree surfaces

The (untrimmed) bisector of two smooth surfaces is the set of centers of spheres which are tangent to both surfaces ([10], [7]). Bisectors are geometric constructions with applications in Tool path generation, Motion planning, NC-milling, and many others.

Let $S_1(s,t)$ and $S_2(u,v)$ be two rational surfaces. A point $\mathbf{B} = (X,Y,Z)^T$ is in the bisector of S_1 and S_2 if

$$\begin{aligned}
\langle (X,Y,Z) - S_1(s,t), \partial_s S_1(s,t)\rangle &= 0,\\
\langle (X,Y,Z) - S_1(s,t), \partial_t S_1(s,t)\rangle &= 0,\\
\langle (X,Y,Z) - S_2(u,v), \partial_u S_2(u,v)\rangle &= 0,\\
\langle (X,Y,Z) - S_2(u,v), \partial_v S_2(u,v)\rangle &= 0,
\end{aligned}$$

$$\langle (X,Y,Z), 2(S_2(u,v) - S_1(s,t))\rangle + \|S_1(s,t)\|^2 - \|S_2(u,v)\|^2 = 0. \qquad (*)$$

$$\mathbf{A}\,\mathbf{B} = \mathbf{R},$$

$$\mathbf{A} = \begin{bmatrix} \partial_s S_{1x} & \partial_s S_{1y} & \partial_s S_{1z} \\ \partial_t S_{1x} & \partial_t S_{1y} & \partial_t S_{1z} \\ \partial_u S_{2x} & \partial_u S_{2y} & \partial_u S_{2z} \\ \partial_v S_{2x} & \partial_v S_{2y} & \partial_v S_{2z} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \langle S_1, \partial_s S_1\rangle \\ \langle S_1, \partial_t S_1\rangle \\ \langle S_2, \partial_u S_2\rangle \\ \langle S_2, \partial_v S_2\rangle \end{bmatrix}$$

Algorithm for computing parametrizations of the bisector:

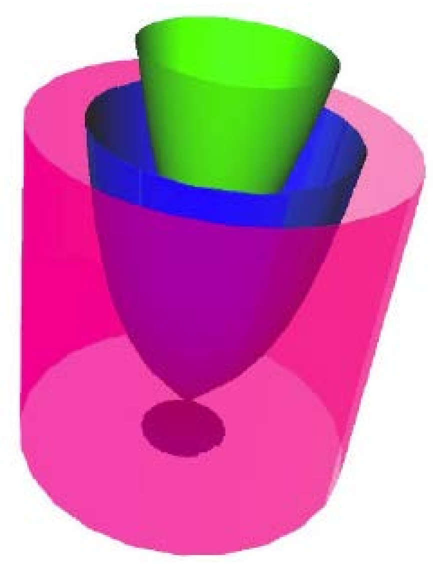
1. Solve for $\mathbf{B} = \mathbf{B}(u, v, s, t)$, using the generalized Cramer rule, or the Moore-Penrose inverse of **A**.
2. Eliminate s and t, using the rank condition det(|**A**,**R**|) = 0, and (*).
3. Substitute to obtain one or more parametrizations for the different components of the bisector: **B**(u, v).

Bisector of a circular cylinder and a quadric or a torus

$$\mathbf{C}(s,t) = \begin{bmatrix} \frac{2rs}{1+s^2} \\ \frac{r(1-s^2)}{1+s^2} \\ t \end{bmatrix}$$

$$\mathbf{S}(u,v) = \begin{bmatrix} S_x(u,v) \\ S_y(u,v) \\ S_z(u,v) \end{bmatrix}$$

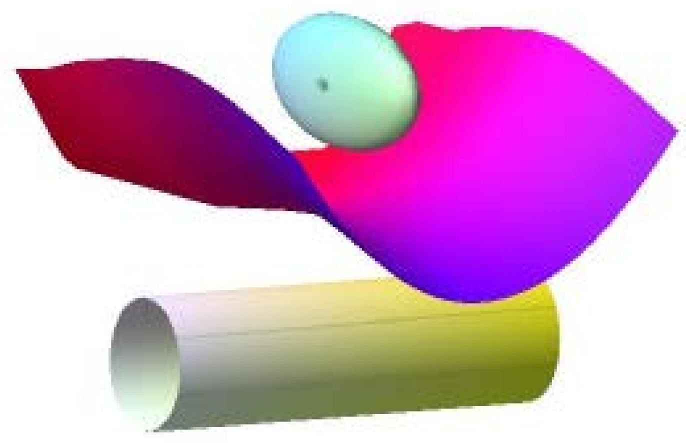
$$\sigma_s(u,v) = \|\partial_u \mathbf{S} \times \partial_v \mathbf{S}\|$$



Theorem. If the surface **S** is of revolution and share the same axis as **C**, the algorithm gives four components $\mathbf{B}_i(u,v)$, i = 1,…,4, for the parametrization of the bisector, that may contain a square root of a positive expression.
If σ_s is rational, then so are $\mathbf{B}_i$.
If $\sigma_s \in \mathbb{Q}[\delta](u,v)$ then $\mathbf{B}_i \in \mathbb{Q}[\delta](u,v), i = 1, \ldots, 4.$

The algorithm gives rational parametrizations for the bisector of a plane and any of the following quadrics:

- Parabolic cylinder
- Circular cylinder or cone
- Sphere
- Ellipsoid
- Paraboloid
- Hyperboloids



One component of the bisector of a cylinder and an ellipsoid

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