# Algorithms and Applications in Geometry of Curves and Surfaces


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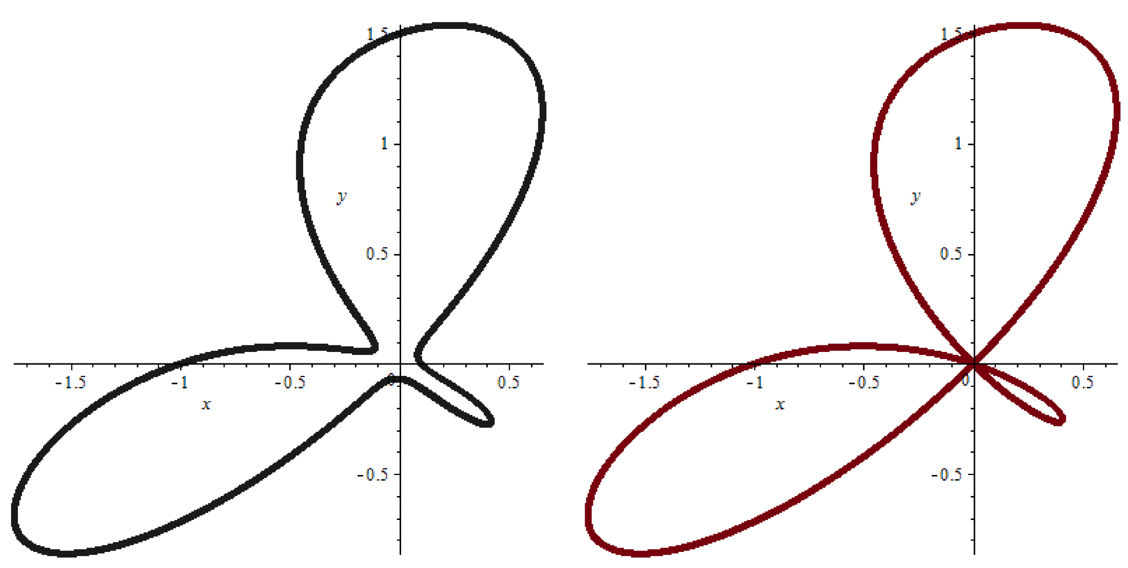
## Introduction

Our group has worked on various problems for curves and surfaces dealing with algorithmic questions and with their potential applications. We have, among others, investigated the following problems: the development of new approaches for computing differential resultants (see [12], [13], [14]), the approximate parametrization problem for curves (see below), the algorithmic study of topological aspects of curves given in polar coordinates, jointly with T. Recio's group (see the poster [1]), briefly addressed here, the detection of symmetries between curves (see below), the analysis of algebraic properties and parametrization algorithms for conchoidal constructions of curves and surfaces as well as their relationship with offset and convolutions of curves and surfaces (see [9], [10], [11]), and the problem of computing (affine) surjective rational parametrizations of surfaces (see [8], [19]). In addition, other treated questions, as the theoretical and algorithmic analysis of ultraquadrics and their applications, have been developed jointly with T.Recio's group, and the computation of rational and radical solutions of algebraic differential equations can be seen, respectively, in the posters [1] and [6]. In this poster, we have chosen two of these lines of research to illustrate our work: approximate parametrizations, and algorithmic-topological aspects of curves.

## Approximate parametrization of curves

Many applications use algebraic curves and surfaces in their development. Examples of this situation can be found in computer aided design, computer graphics, geometric modeling, computer numerical control or pattern recognition, solving differential equations and diophantine equations, modeling lenses for cameras, shape symmetry detection, etc. Moreover, depending on the problem one uses different representations of the curve or the surface, namely: implicit or parametric. However, although implicit representations are always available, rational parametric representations are not always possible. One may extend the class of rational curves and surfaces to the class of curves and surfaces parametrizable by fractions of nested radicals of polynomials (see [17], [18]). Nevertheless, this extension is still underdeveloped. So the use of alternative approaches, like approximate techniques, is unavoidable. The approximate parametrization problem for curves is as follows:

**Problem**
Given *the implicit equation of a non-rational real plane curve* $\mathcal{C}$ *and a tolerance* $\epsilon > 0$, decide *whether there exists a rational real plane curve* $\overline{\mathcal{C}}$ *at finite small Hausdorff distance (i.e. small related to the tolerance* $\epsilon$*) to the input curve* $\mathcal{C}$ *and, in the affirmative case,* compute *a real rational parametrization of* $\overline{\mathcal{C}}$.



For instance, consider the curves

$$\mathcal{C}: x^4 + 2y^4 + \frac{1001}{1000}x^3 + 3x^2y - y^2x - 3y^3 + \frac{1}{100000}y^2 - \frac{1}{1000}x - \frac{1}{1000}y - \frac{1}{1000} = 0,$$
$$\overline{\mathcal{C}}: x^4 + 2y^4 + \frac{1001}{1000}x^3 + 3x^2y - y^2x - 3y^3 = 0,$$

(see in the figure above: $\mathcal{C}$ on the left and $\overline{\mathcal{C}}$ on the right) The curve $\mathcal{C}$ cannot be rationally parametrized, indeed its genus is 3, while $\overline{\mathcal{C}}$ admits the rational parametrization

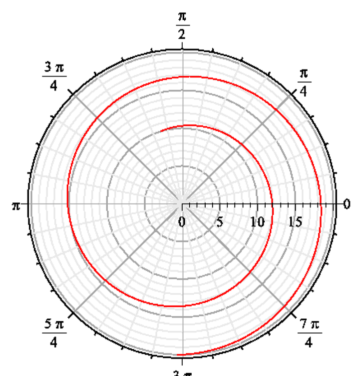
$$\left(\frac{3000\,t^3 + 1000\,t^2 - 3000\,t - 1001}{1000(2\,t^4+1)}, \frac{t\,(3000\,t^3 + 1000\,t^2 - 3000\,t - 1001)}{1000(2\,t^4+1)}\right).$$

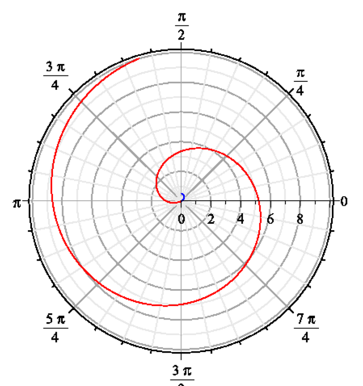
In [15] we show how to solve the problem for the case of **space curves**. For this purpose, the curve is projected birationally on an $\epsilon$**-rational plane curve**. This plane curve is parametrized approximately with the algorithm in [7]. Then, using **Chinese remainder techniques**, the approximate plane curve is lifted to an approximate space curve. In [16], we introduce the notion of **Hausdorff rational divisor**, and we prove that all curves in the linear systems of curves associated to these divisors are solutions to problem. This allows one to analyze optimal solutions under some fixed criteria.

## Algorithmic-topological aspects of curves

Two different topics have been considered in this context. On the one hand, we have studied **curves which are rational in polar coordinates**, i.e. planar curves described by means of a parametrization $\psi(t) = (r(t), \theta(t))$, where the usual polar coordinates $r, \theta$ are rational functions of a parameter $t$. Some well-known **spirals**, for instance, belong to this kind. One can prove that the only real algebraic curves admitting such a parametrization are lines and circles [3]; as a result, many of these curves show properties that are impossible in the algebraic realm. For instance, they can have infinitely many self-intersections, and they can wind infinitely around a circle centered at the origin, or the origin itself. In [3] we analyze these phenomena, we give algorithms to detect them in advance (i.e. just from the parametrization) and we provide an algorithm, implemented in Maple, that takes an input parametrization and outputs a collection of plottings around the "most interesting" parts of the curve. An example follows: the curve $\psi(t) = (t, (t^3+1)/(t^2-3t+2))$.

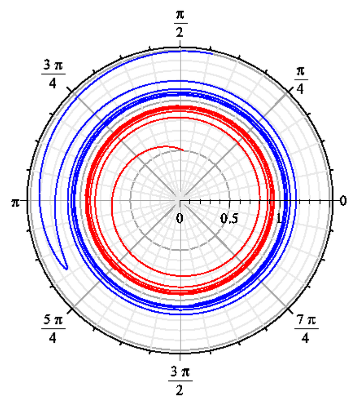
Plot around $t = -\infty$

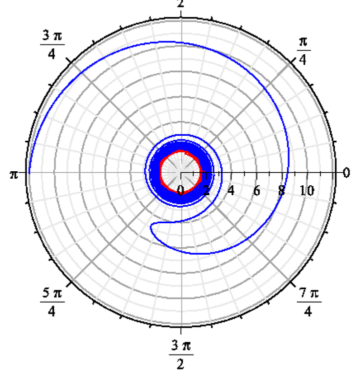




Plot around $t = 0$

Plot around $t = 1$





Plot around $t = 2$

On the other hand, we have considered **symmetry detection and similarity detection in rational curves**. The key idea is to reduce all the computations to the parameter space by making use of the following theorem [5].

**Theorem**
Let $\mathcal{C}$ be a rational plane or space curve properly parametrized by $\psi(t)$. Then $\mathcal{C}$ is symmetric if and only if there exist a symmetry $f$ and a Möbius transformation $\varphi$ for which we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C} \\ \psi\uparrow & & \uparrow\psi \\ \mathbb{R} & \xrightarrow[\varphi]{} & \mathbb{R} \end{array}$$

This theorem leads to a polynomial system whose consistency characterizes the existence of symmetry. In principle, solving this system is not efficient; however, one can write all the parameters in terms of just one, in fact one of the parameters of the Möbius transformation, so that only a univariate gcd is required. In the special case when the curve can be polynomially parametrized [2], one eventually obtains closed formulae for the symmetry elements of the curve, if any, that allow us to analyze curves of degree, say, 100 in just a few seconds. The approach can be generalized to checking whether two given planar curves are similar, an important problem in **Pattern Recognition** [4].

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