The 'how much' war. A numerical method to solve a duopolistic differential game in a closed-loop equilibrium

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

MsC Thesis. Mathematical Engineering
UCM, Facultad de Matemáticas. Madrid
27-09-2012

In a nutshell

The “how much” war:

Marketing departments have to decide how much to invest in advertising to maximize sales. Also, they have to set a price. Both variables are relevant key drivers on sales.

A duopolistic differential game:

Usually, companies are in a competitive world (generally oligopolistic or duopolistic). We need a theoretical framework in order to incorporate it to study the optimal decision

A numerical method:

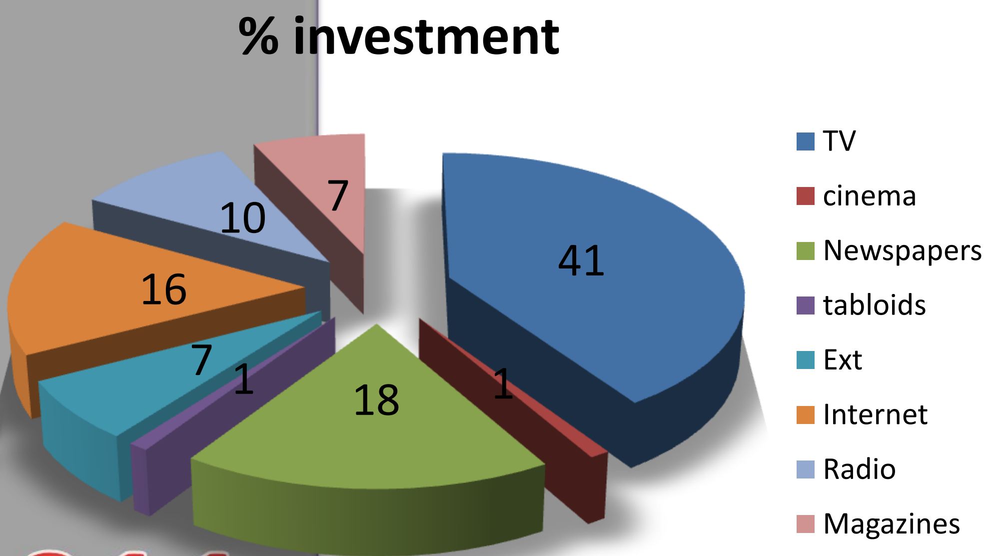
Differential games are hard to solve analytically, so we have to rely on numerical solutions as we will present here.

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

Motivation I

12.000 M. € advertising expenditure in Spain in 2011

(Infoadex)



Motivation II

Most of these investors are in an oligopolistic or duopolistic war

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

Motivation III

We believe it is important to develop methods to optimize their expenses.

These methods should have to deal with this idea: "all competitors in the market are rational"

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

The question(s)

How much to spend in advertising per week in a fixed period of time…

What should be our weekly price…

…in a competitive environment?

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

In this presentation...

- We will show a theoretical method in a differential game form to answer these key questions
- We will estimate parameter values for this theoretical model
- We will develope a numerical method inspired in dynamic programming methods to solve the game
- We will show the results of the game, and we will compare it versus the real data.

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

We present two novel ideas

1. We use LOTKA VOLTERRA models of species competition in a Differential Marketing Game.

The literature uses Lanchaster-war models as benchmark

2. We present an algorithm to solve **closed-loop** equilibrium adapted from dynamic programming literature.

The literature in differential games usually obtains analytic closed-loop equilibrium from simple models or obtains **open-loop** equilibrium

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

Theoretical Model

Theoretical Model (I): a differential game

There are i=1,2 players

The functional is a dynamic discounted benefit function:
Discounting factor **r,** Price **p,** sales **x** and C(u) is a cost function.

$$\max_{u_i \in U} J_i = \int_0^{\infty} e^{-r_i t}[p_i(t)x_i(t) - C_i(u_i(t))]\,dt$$

Each player seeks to maximize this functional with respect to control variables (u)

St :

$$\dot{x}_i = f(x_i, x_j, u_i, u_j)$$

Theoretical Model (II)

$$\max_{u_i \in U} J_i = \int^{\infty} \qquad (u_i(t))]\, dt$$

The players coexist in a dynamical system that reflects the relationship among :

-State variables: *sales* [x]

-Controls: *advertising and price* [u]

St :

$$\dot{x}_i = f(x_i, x_j, u_i, u_j)$$

Solving the model

This model is a **differential game.** We can derive two equilibrium (solutions):

Open-loop

Optimal Controls are a function of time
$Ui^*=Ui(t)$

closed-loop

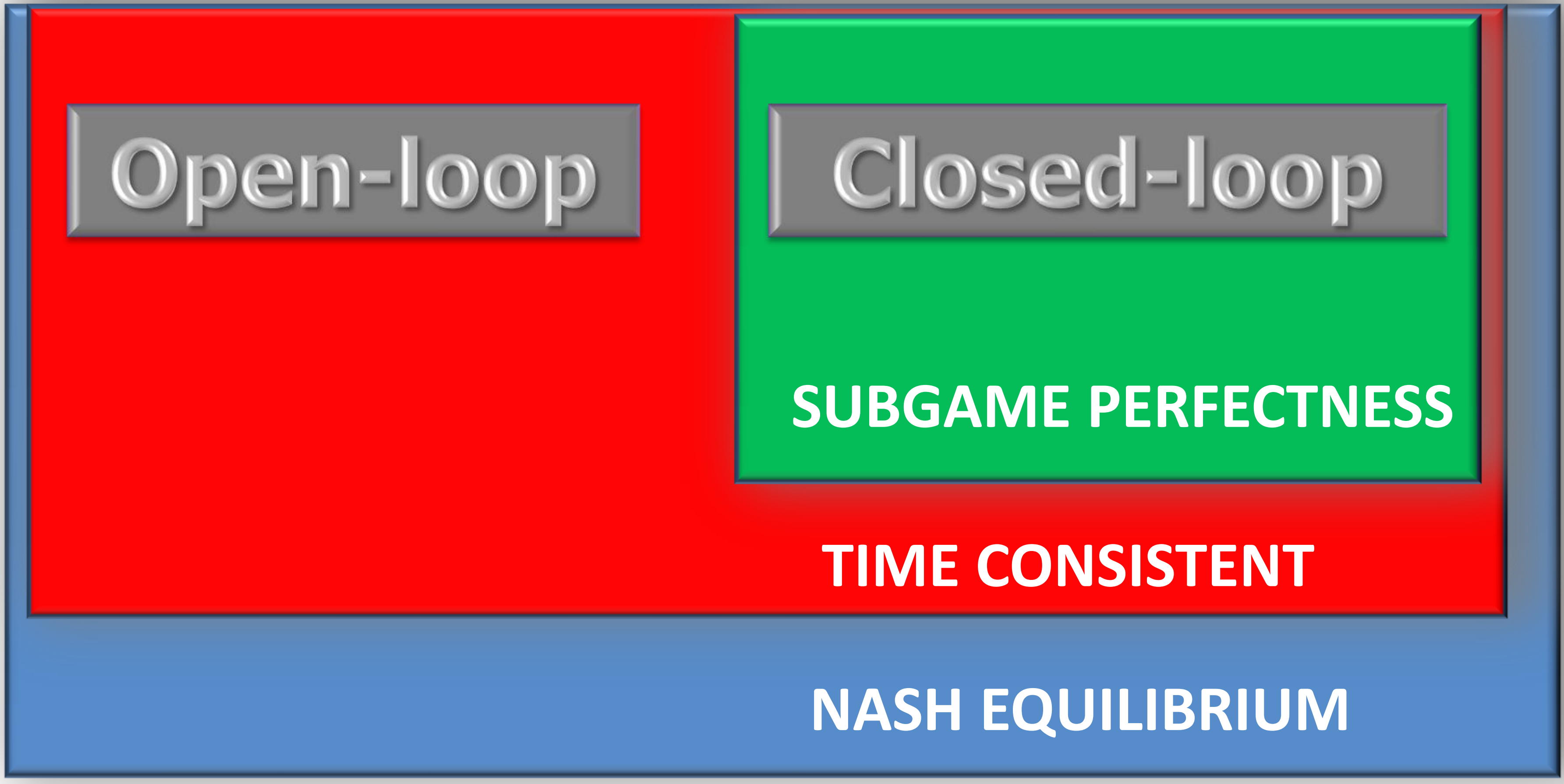
Optimal Controls are a function of time and state variables
$Ui^*=Ui(t,x_i(t))$

Open-loop are well documented in the literature, because they involve to solve boundary differential equations.
Closed-loop are more difficult, but more realist. They involve to solve two coupled PD equation called **Hamilton-Jacobi-Bellman:**

$$r_i V_i = \max_{u_i \in U}\left[p_i(t)x_i(t) - C_i(u_i(t)) + \nabla_x V_i f(x_i, x_j, u_i, u_j)\ \right]$$

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

Properties of equilibrium



Parameter estimation

What to estimate?

$$\max_{u_i \in U} J_i = \int_0^{\infty} e^{-r_i t}\left[p_i(t)x_i(t) - \boldsymbol{C_i(u_i(t))}\right] dt$$

St :

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f(x_i, x_j, u_i, u_j)}$$

We have to estimate parameters for **Cost function** and the **market dynamic system**

A little bit of notation:

x_i Sales of player i

M_i Market Share

v_i TV advertising (GRP is the standard unit)

p_i Price of good or service

N_i Market size of player i

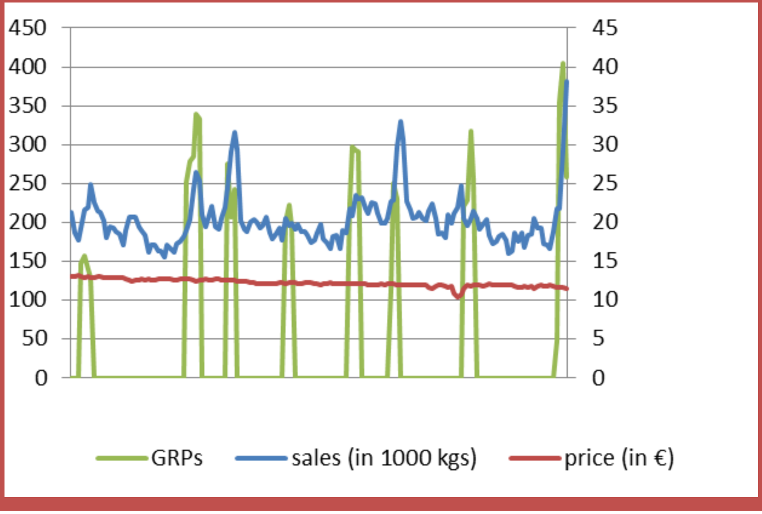
$\alpha_i, \beta_i, \rho_i \ldots$ We will also use several parameters to connect these variables into equations

Our data set:

We will use a data set of a well-known company in Spain. This company competes with a withe label, so:

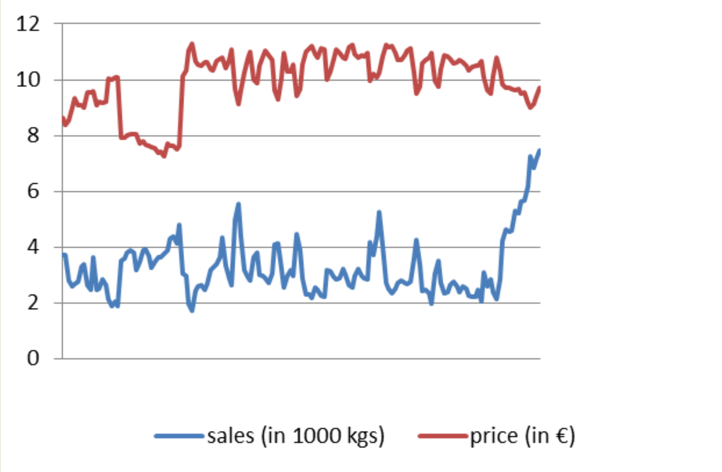
Our company needs to maximize:

p_i v_i



The competitor needs to maximize:

p_i



Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

Market Dynamic System

$$\dot{x}_i = f(x_i, x_j, u_i, u_j)$$

$$\dot{M} = \rho_1\sqrt{v_1}(1 - \mathrm{M}) - \rho_2\sqrt{v_2}M$$

LANCHESTER Model of COMBAT

$$\dot{x}_i = \rho_i v_i \sqrt{\left(N_1 + N_2 - x_i - x_j\right) D_i(p_i)}$$

LOTKA-VOLTERRA Species competition

$$\dot{x}_i = \left[\alpha_1\left(1 - \frac{\alpha_1 x_i}{N_i}\right) - \beta_{ij}x_j\right]x_i$$

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

Discrete time LOTKA VOLTERRA models WITH EXOGENOUS INPUTS

$$x_i\widetilde{(t+1)} = \frac{x_i(t+1) - x_i(t)}{x_i(t)}$$

LV-1

$$x_i\widetilde{(t+1)} = \alpha_i\left(1 - \frac{\alpha_i x_i(t)}{N_i}\right) - \beta_{ij}x_j(t) + \rho_i v_i(t+1) - w_1 p_1(t+1) + \varepsilon_i(t+1)$$

LV-2

$$x_i\widetilde{(t+1)} = \alpha_i\left(1 - \frac{\alpha_1 x_i(t)}{N_i}\right) - \beta_{ij}x_j(t) + \rho_i v_i(t+1) - \rho_{ii}{v^2}_i(t+1) - w_1 p_1(t+1) + w_{ii}{p^2}_i(t+1) + \varepsilon_i(t+1)$$

VARX

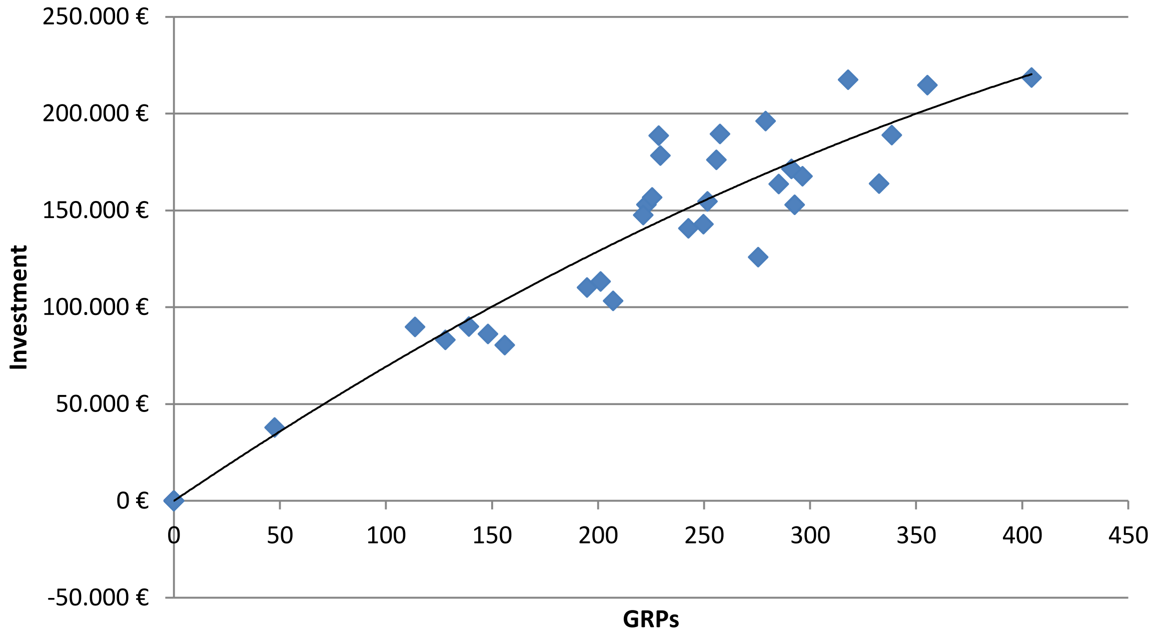
$$x_i\widetilde{(t+1)} = \alpha_{0i} + \alpha_i\widetilde{x_i(t)} - \beta_{ij}\widetilde{x_j(t)} + \rho_i v_i(t+1) - w_1 p_1(t+1) + \varepsilon_i(t+1)$$

Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

OLS estimations

	LV1 model		LV2 model		VAR MODEL	
	Equation 1	Equation 2	Equation 1	Equation 2	Equation 1	Equation 2
Dependent Variable	$\frac{x_1(t+1)-x_1(t)}{x_1(t)}$	$\frac{x_2(t+1)-x_2(t)}{x_2(t)}$	$\frac{x_1(t+1)-x_1(t)}{x_1(t)}$	$\frac{x_2(t+1)-x_2(t)}{x_2(t)}$	$\frac{x_1(t+1)-x_1(t)}{x_1(t)}$	$\frac{x_2(t+1)-x_2(t)}{x_2(t)}$
α_{01}					0.695	1.285
std deviation					0.201	0.557
α_{02}						
std deviation						
α_1	0.690		5.410		-0.200	-0.042
std deviation	0.146		2.070		0.096	0.267
α_2		1.540		1.540		
std deviation		0.348		0.348		
$\frac{\alpha_1}{N}$	-0.022		-0.022			
std deviation	0.003		0.003			
$\frac{\alpha_2}{M}$		-0.083		-0.083		
std deviation		0.017		0.017		
β_1	-0.032		-0.034		-0.016	-0.115
std deviation	0.008		0.007		0.037	0.102
β_2		-0.029		-0.029		
std deviation		0.009		0.009		
ρ_1	0.0004		0.000		0.0002	-0.0003
std deviation	0.0001		0.000		0.0001	0.0002
ρ_{11}						
std deviation						
ω_1	-0.023		-0.800		-0.032	-0.061
std deviation	0.010		0.342		0.013	0.037
ω_2		-0.047		-0.047	-0.014	-0.037
std deviation		0.014		0.014	0.006	0.016
ω_{11}			0.032			
std deviation			0.014			
ω_{21}				0.000		
std deviation				0.000		
R-squared	0.775034	0.538344	0.79	0.53834	0.72	0.48
Dwatson	1.899407	1.833568	1.9	1.83356	1.9	1.83
AIC	-423.0484	-73.21995	-435.76	-73.2199		
SC	-252.6166	61.49	-262.289	61.4		

Cost Function



- Theoretical models usually purpose a quadratic function to derive maximum policies.

- Data shows a high probability of being a logarithmic process.

Numerical Method

Dynamic Programming (I)

- From continuous to discrete, using h>0:

$$\max_{u_i \in U} J_i = \int_0^{\infty} e^{-r_i t} f_i(x(t), u(t), \alpha)\, dt$$

$$\max_{u_i \in U} J_i = h \sum_{t=0}^{\infty} \beta^t f_i(x_h(t), u(t); \alpha)$$

$$\dot{x}_i = g_i(x_i, x_j, u_i, u_j)$$

$$x_i(t+1) = x_i(t) + h g_i(x_h(t), u(t); \alpha)$$

Dynamic Programming (II)

- Discrete Hamilton-Jacobi-Bellman (we simplify notation)

$$V_{i,h}(x,\alpha) = \max_{u}\{h \times f_i\,(x(t), u(t), \alpha) + \beta \times V_{i,h}(x(t) + hg_i(x(t), u(t); \alpha))\}$$

The algorithm

1
- Let G a 2D grid with x1 and x2, define dx1,dx2 and h
- Initialize control vectors to 0, $u(t)$=0

2
- Start with a guess estimate for V_{ih} and calculate its gradient (gradVi)
- Actualize x1 and x2 according to $x(t) + hg_i(x(t), u(t); \alpha)$

3
- **VALUE ITERATION** (subrutine). Taking $u(t)$ fixed, iterate in HJB till convergence. Use interpolation

4
- **POLICY ITERATION** (subrutine). Taking V_{ih} fixed, explicitly obtain a new $u(t)$ iterating in HJB till convergence.

5
- Alternate **VALUE** and **POLICY** till tolerance criteria is achieved.

Dynamic Programming (III)

- Discrete Hamilton-Jacobi-Bellman (we simplify notation)
- To solve it we will iterate in time and in space.

$$V_{i,h}(x) = \max_{u}\{h \times f_i\,(x(t), u(t), \alpha) + \beta \times V_{i,h}(x(t) + hg_i(x(t), u(t); \alpha))\}$$

- To solve this fixed point [V is in both terms] we iterate supported by contracting mapping theorem

- We use linear interpolation to obtain the values of V out of discrete grid of x

- This maximum is explicitly calculated (so we need the Gradient of $V_{i,h}$)

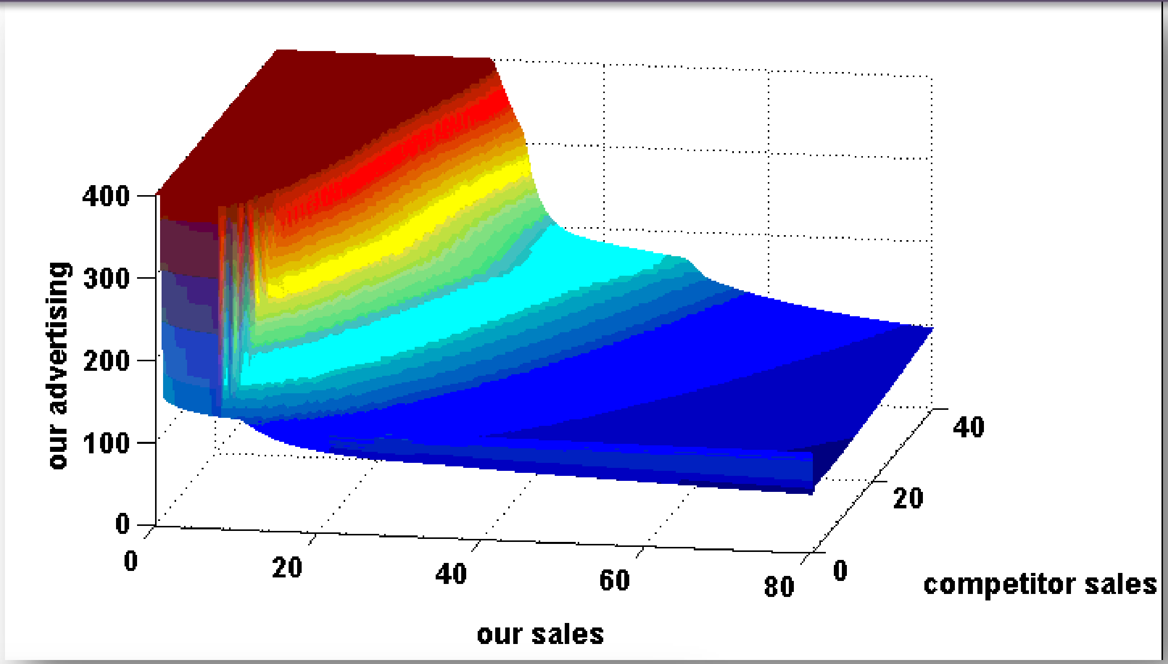
Jorge Herrera de la Cruz
Coordinators: Benjamín Ivorra, Ángel M. Ramos

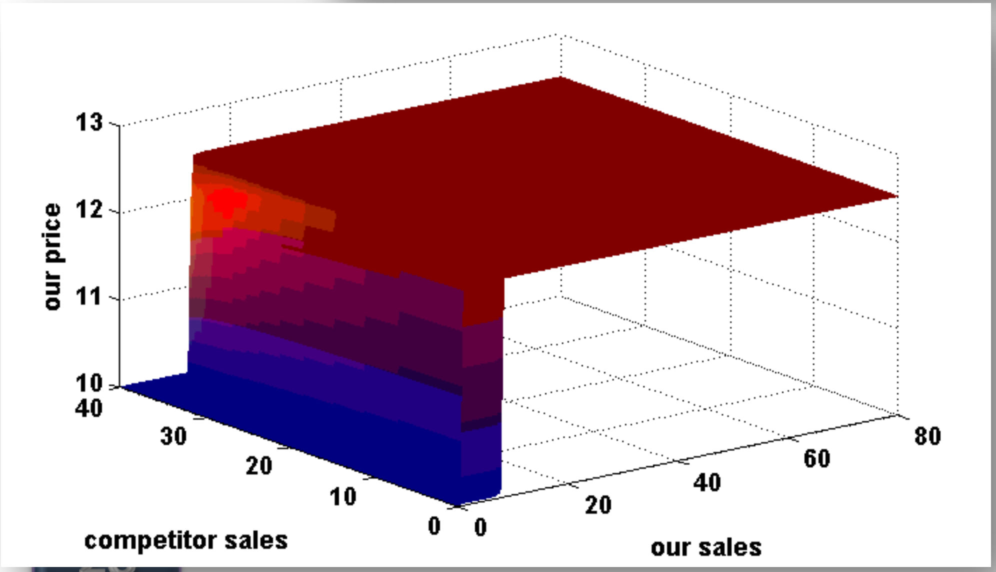
Dynamic Programming (II)

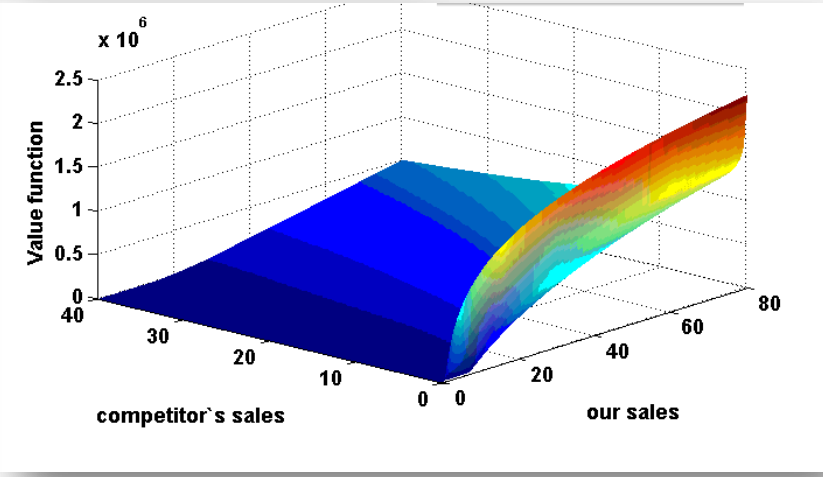
- We have run numerical experiments to validate our algorithm:
    - We have solved a 1D model with explicit solution [Brock-Mirman]
    - We have also solved a 2D benchmark model in the literature, without explicit solution, but reaching similar results as those published.

The ‘how much’ war. Numerical Solution

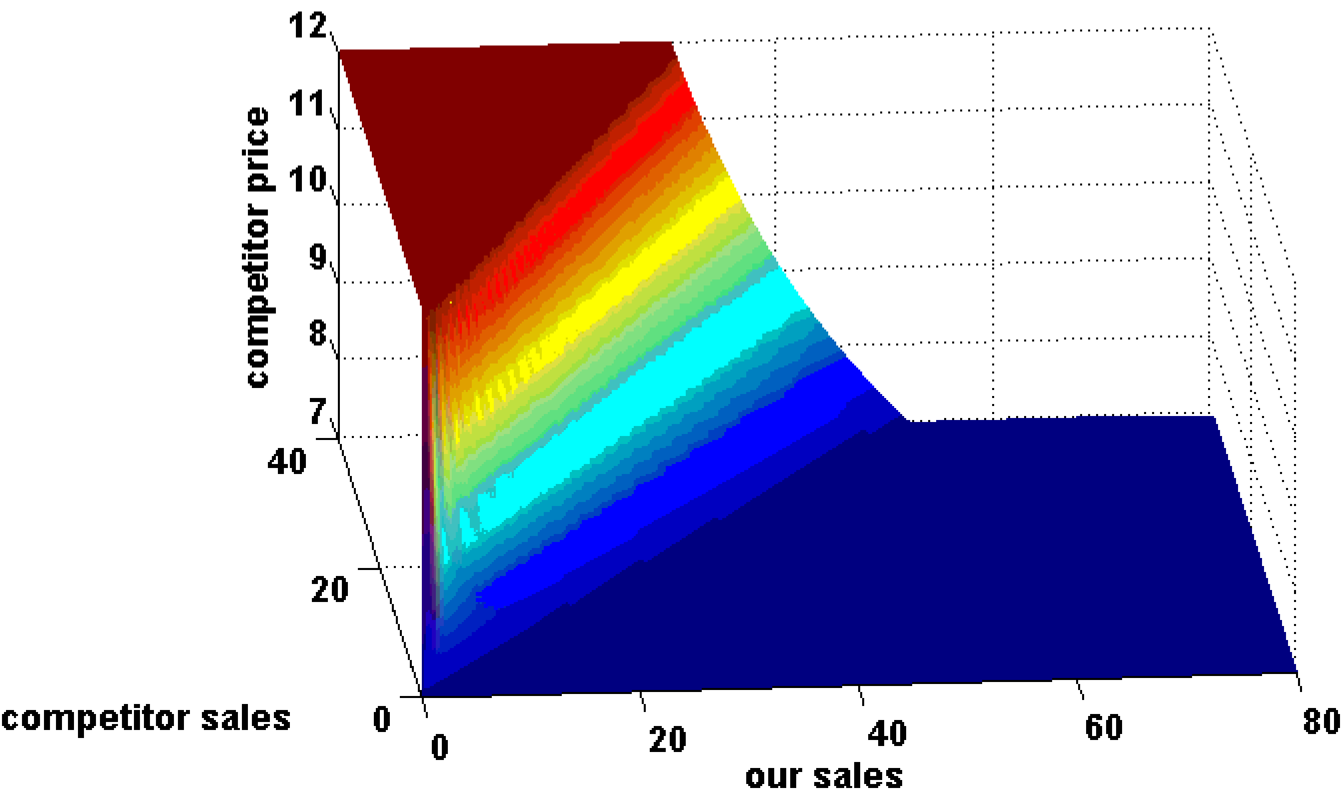
Our solution



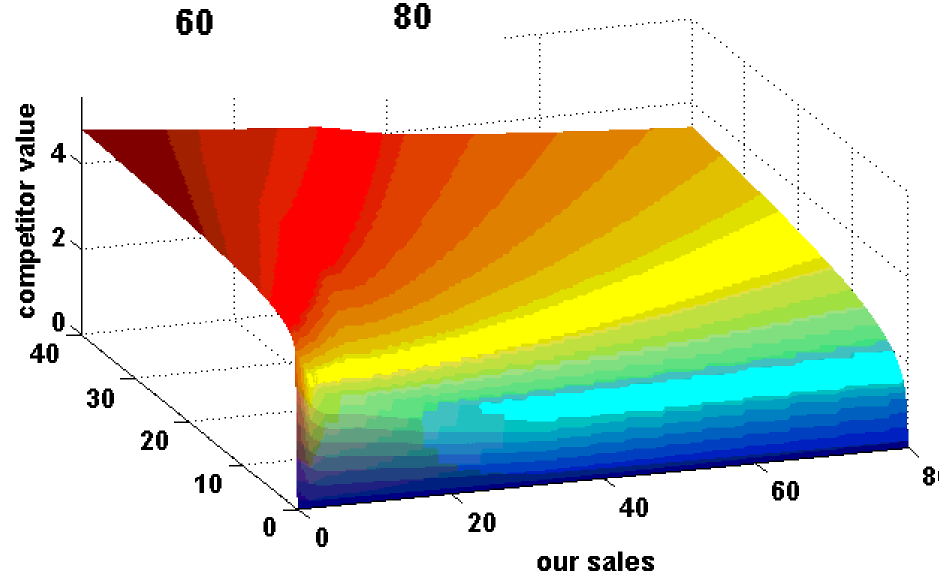




Competitor

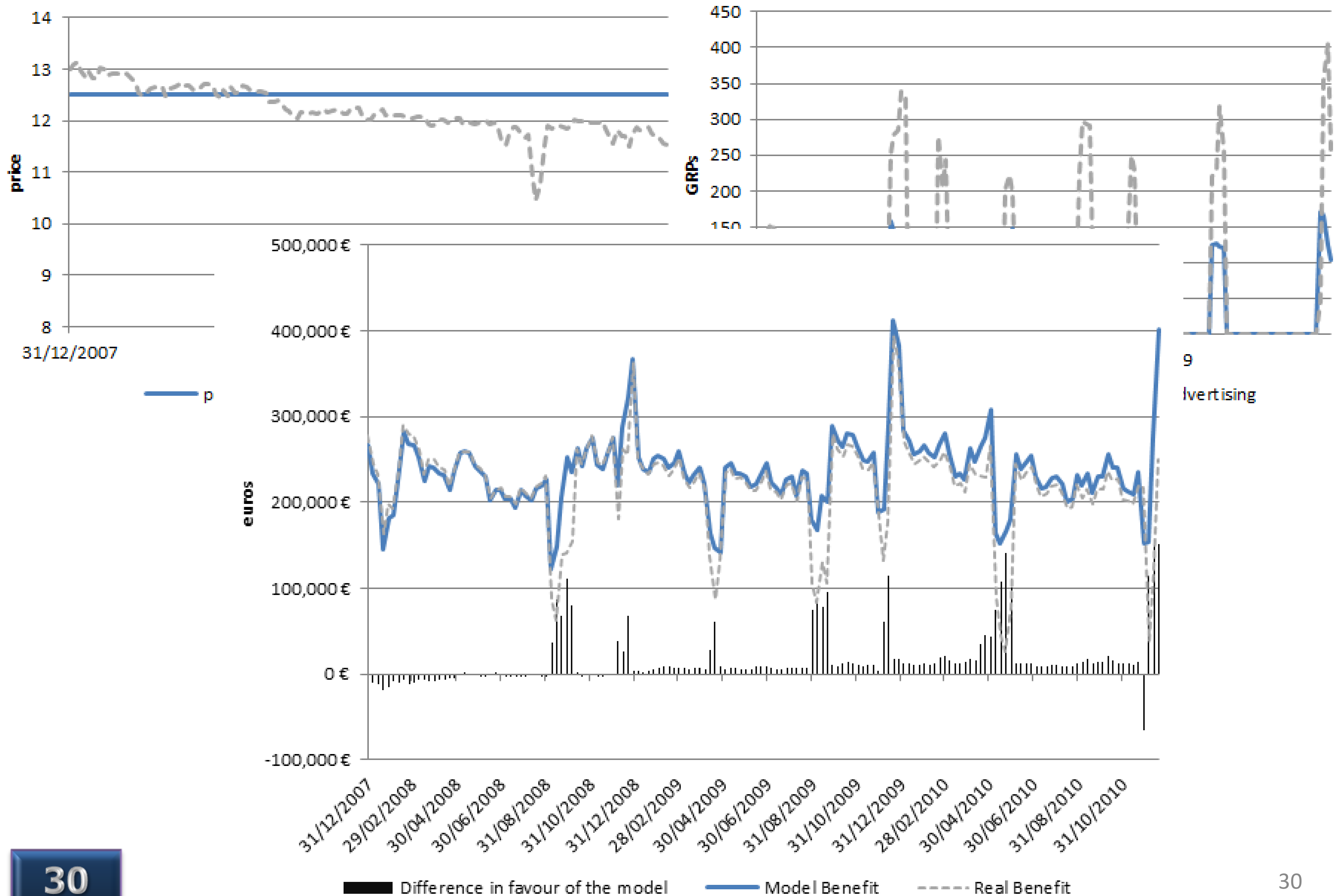


competitor price
12
11
10
9
8
7
40
20
0
competitor sales
0
20
40
60
80
our sales



competitor value
4
2
0
40
30
20
10
0
0
20
40
60
80
our sales

We could have won a 7% more



30

Conclusions and Further Research

Conclusions and Further Research

We have shown a differential game algorithm in a closed-loop equilibrium to solve a duopolistic problem with real world data:

1. Obtaining optimal advertising
2. Obtaining optimal price

In our experiment, we show that we could have won a 7% more than with the adopted strategy.

In future works, we propose:

1. Research on Stackelberg equilibrium (leader-follower)
2. Research on Stochastic Games taking into account estimated values of error sizes with statistical methods
3. Incorporate richer lag schemes in dynamical system as statistical tests suggest.