An Hybrid Optimization Method For Credit Portfolio Optimization Under Constraints

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Outlines

- **Problem definition:**
  - Financial product description
  - Product modeling + evaluation
  - Example of optimization problem

- **Optimization algorithm:**
  - Semi-deterministic algorithm
  - Genetic algorithm
  - Hybdridation

- **Problems resolution:**
  - Results
  - Financial analysis
Part I: Problems Definition
Historical context

**Object:** Credit derivative products:
- Development of those products (In 2004: 2300 $M$)

**Main problems:**
1-Risk management:
   - What is the risk? Risk of a trading partner (here company) not fulfilling his obligations (here repay its credit) on the due date $\rightarrow$ Loss.
   - Example: Bankruptcy (Enron/Worldcom in 2001/2002).
   - Works: Loss evaluation method (Merton 1974...), Risk measure (Rockafellar, Artzner 1998...)

2-Income/Profitability improvement
Considered derivative credit product

Collateralized Debt Obligations (CDO)

-Objective: buy security on facilities (credits).

N.B: For each facility we can only buy security on a part of it’s nominal (amount of money).
Considered derivative credit product

$CDO^2$ / Master CDO

-Objective: Resilient to low losses.
-Main default: When Losses: Fast and highly severe.
We consider a portfolio $PORT$ compound by $I$ facilities. For each facility $i$, market gives:
- A nominal ($N_i$).
- A maturity ($T_i$) date.
- A spread ($Sp_i$) (rate of interest).
- A loss given default ($LGD_i$).
- Sector / Geographical Zone / Stability Coefficients.

We can directly compute:

**Income:** $IC = \sum_{i=1}^{I} Sp_i \times N_i$
In order to compute more complex indicator. We consider:

$L$ the CLO\(^2\)’s amount of losses: Random variable.

→ We need to determine is Loss density function $\beta_L$:

We use here a Default Time’s Model
Default times model

Input: Facilities’ Data

For \( i=1:M \)

- Generate a default time vector (using normally distributed random variables)
- Compute Loss amount of the scenario \( i \).
- Complete the discrete CLO\(^2\)’s Loss density function \( \beta_L \)

EndFor

Output: \( \beta_L \)
Density function $\beta_L$
Risk measures

- **VaR**: The smallest nominal loss of the $\alpha$ % worst of losses:
  \[ VaR_\alpha(L) = \inf \{ L' \mid \int_0^{L'} \beta_L(x) dx > (1 - \alpha) \} \]

- **C-VaR (Expected Shortfall)**: The average of the worst $\alpha$ % of losses:
  \[ CVaR_\alpha(L) = \frac{1}{\alpha} \int_0^\alpha VaR_p(L) dp \]

CVaR is often preferred: In some case it’s coherent and convex risk measure.
Optimize **Facilities’ nominal** of a BNP-Paribas´ portfolio in order to:

- Reduce risk measure keeping the income higher than the initial value (0.1 -VaR).
Part II: Hybrid optimization method
Recursive Semi-Deterministic methods

\[
\min_{x \in \Omega_{ad}} J(x)
\]

Where:
- \(x\) is the optimization parameter
- \(\Omega_{ad} \in \mathbb{R}^N\) is the admissible space

Assumptions:
- \(J \in C^2(\Omega_{ad}, \mathbb{R})\)
- \(J\) coercive
- \(J_m\) denotes: the minimum of \(J\) or a low value
Recursive Semi-Deterministic methods

Many minimization algorithms can be seen as discretizations of dynamical systems with initial conditions.

Solve numerically the optimization problem with one of those algorithms (core optimization method) ⇔ Solve:

\[
\begin{aligned}
&\min_{t \in [0, Z]} \left( |J(x(t)) - J_m| \right) < \epsilon \\
\end{aligned}
\]

where \( Z \in \mathbb{R} \) and \( \epsilon \) are given.

This BVP is over-determined.

Idea: Remove over-determination: One of the initial condition is considered as a variable \( v \)

Then apply BVP techniques: Example: SHOOTING METHOD.
Hybrid Genetic Algorithm

General overview:

We want to minimize: \( J(x) = x_1^2 + x_2^2 \)

\[
\begin{align*}
(0.8,0.3) \\
(1.0,0.5) \\
(4.0,0.6) \\
(2.0) \\
(5.5) \\
(0.7,3) \\
(0.8,0.3)
\end{align*}
\]

Initial Population

\[
\begin{align*}
(1,0.5) \\
(0.7,3) \\
(2.0)
\end{align*}
\]

Selection

\[
\begin{align*}
(1,3) \\
(2,0.5) \\
(0.7,0)
\end{align*}
\]

Cross-Over

\[
\begin{align*}
(0.8,0.3) \\
(4,0.6) \\
(1,3) \\
(2,0.5) \\
(0.7,0) \\
(2.3)
\end{align*}
\]

Intermediate Population

\[
\begin{align*}
(0.8,0.5) \\
(4,0.6) \\
(1.6,3) \\
(2.2,0.5) \\
(0.7,0.1)
\end{align*}
\]

Final Population

Reproduction

We iterate the process
Hybrid Genetic Algorithm

Matrix representation: With Laurent Dumas (Paris VI)

$i^{th}$ Population: $X^i = \{x^i_l \in \Omega_{ad}, l = 1, ..., N_p\}$

Selection: $S^i$, Crossover: $C^i$, Mutation: $E^i$

The new population can be written as:

$$X^{i+1} = C^i S^i X^i + E^i$$

Thus genetics algorithms can be associated to:

$$\dot{X}(t) = \Lambda_1 X(t) \Lambda_2 - X(t)$$

where:

- $X$ of the form: $X = \{x_i/ i = 1, ..., N_p \ x_i \in \Omega_{ad}\}$
- $\Lambda_i : \Lambda_i(t, X(t), P)$
- $P$ are a set of fixed parameters
Hybrid Genetic Algorithm

BVP formulation:

\[
\begin{align*}
\dot{X}(t) &= \Lambda_1 X(t) \Lambda_2 - X(t) \\
X(0) &= X^0 \\
\min_{t \in [0,Z]} (|\hat{J}(X(t)) - J_m|) &< \epsilon
\end{align*}
\]

where: \( \hat{J}(X) = \min_i (J(x_i) | x_i \in X) \)

Inconveniences of GA:
- Slow convergence
- Computational complexity
- Lack of precision

Idea: Solve BVP: \( X^0 \) Considered as a new variable (admissible)
Hybrid Genetic Algorithm

Example of algorithmic implementation:
Part III: Application to portfolio optimization
Considered Portfolio

Characteristics:

- $\text{CLO}^2$ structure
- Compound by 500 facilities dispatched into 40 Sub-CLOs’ tranches and 54 independant credits.
- Portfolio nominal: $2.0 \times 10^9$ Euros (E)
- Income: $2.1 \times 10^7$ E.
- VaR: $1.9 \times 10^8$ E.
Parameterization

**Parameters:** nominal of each ICDO’s tranches and SN included in the entire Asset Management’s universe (1500 facilities)

For versatility and respecting the BNP investment guideline, we consider constraints on facility:

- **Avoid too much concentration in one facility:** maximum nominal 1.e8 €.
- **Minimum facility investment:** if nominal <5e6 € → nominal set to 0.
- **Facility quality:** Depending on quality coefficients: The nominal can be raised, decreased or unmodified.

Thus the Parameter number: 65.
Cost function

Optimization problem is of the form:

\[
\min_{x \in \Omega} J(x)
\]

\[
l_{c1} \leq C_1(x) \leq u_{c1} \\
\vdots \\
\]

\[
l_{c_{ncons}} \leq C_{ncons}(x) \leq u_{c_{ncons}}
\]

We rewrite the cost function using wall functions:

\[
\tilde{J}(x) = J(x) + \vartheta \sum_{j=1}^{ncons} \left( \max(uc_j - C_j(x), 0) + \max(C_j(x) - lc_j, 0) \right)
\]
Computational time reduction

Sensitivity analysis is difficult:
- Time for one evaluation + high dimensional problem.
- Start from the boundary of the admissible space + pathogen gradient directions $\rightarrow$ slow convergence.

$\Rightarrow$ we use HSGA + computational reduction techniques:
- Default times are computed once
- When constraint is not satisfied: intend to use projection.

One optimization: 2000 functional evaluations $\rightarrow$ 6H computation.
<table>
<thead>
<tr>
<th></th>
<th>Nominal</th>
<th>N. S-CLOs</th>
<th>N. Ind. Cr.</th>
<th>IC</th>
<th>VaR$_{0.1%}$</th>
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<td>2.0e9</td>
<td>7.5e8</td>
<td>1.25e9</td>
<td>2.1e7</td>
<td>1.9e8</td>
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<td><strong>Sen.</strong></td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>7%</td>
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<tr>
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<td>1.45e9</td>
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<td>1.3e8</td>
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<tr>
<td><strong>Evo.</strong></td>
<td>0%</td>
<td>-26%</td>
<td>11%</td>
<td>0%</td>
<td>-31%</td>
</tr>
<tr>
<td><strong>Sen.</strong></td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1%</td>
<td>1%</td>
</tr>
</tbody>
</table>
Var min/ IC constraint

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- Part I: Problems Definition
- Part II: Hybrid optimization method
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  - Considered Portfolio
  - Parameterization
  - Cost function
  - Computational time reduction
  - Var min/ IC constraint

Conclusions and perspectives
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- Results are in adequacy with financial intuition.
- Hybrid method + simplification techniques: New efficient and fast tool (To be compared with linear search methods.)
- Perspectives: Try with more complete models, extension to other kind of facilities (hedge)...
Conclusion and perspectives

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Conclusions and perspectives

!!! Thank You !!!