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Spectral convergence and nonlinear dynamics of reaction–diffusion equations under perturbations of the domain

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Abstract

In this paper we obtain the continuity of attractors for semilinear parabolic problems with Neumann boundary conditions relatively to perturbations of the domain. We show that, if the perturbations on the domain are such that the convergence of eigenvalues and eigenfunctions of the Neumann Laplacian is granted then, we obtain the upper semicontinuity of the attractors. If, moreover, every equilibrium of the unperturbed problem is hyperbolic we also obtain the continuity of attractors. We also give necessary and sufficient conditions for the spectral convergence of Neumann problems under perturbations of the domain.

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Keywords: Parabolic equations; Attractors; Lower semi-continuity; Upper semi-continuity; Domain perturbation

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1. Introduction

In this paper we consider reaction–diffusion equations of the form

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\varepsilon, \end{cases} \tag{1.1}$$

where $\Omega_\varepsilon,\ 0 \leqslant \varepsilon \leqslant \varepsilon_0$, are bounded Lipschitz domains in $\mathbb{R}^N,\ N \geqslant 2$. We analyze how the asymptotic dynamics of the evolutionary problem (1.1) changes when we vary the domain. In particular, we are interested in studying how the behavior of the spectral properties of the linear operator $-\Delta$ under variations of the domain, determines the behavior of the nonlinear dynamics of (1.1).

The nonlinearity f is assumed to be defined in $\mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, it is continuous in both variables (x,u) and for fixed $x \in \mathbb{R}^N$, $f(x,\cdot) \in C^2(\mathbb{R})$. Moreover, f satisfies the dissipativeness assumption

$$\limsup_{|s|\to\infty} \frac{f(x,s)}{s} < 0 \quad \text{uniformly in } x \in \mathbb{R}^N. \tag{1.2}$$

It has been shown (see [6]) that problem (1.1) is well-posed in $W^{1,q}(\Omega_\varepsilon),\ q > N$, without any restriction on the growth of f. Moreover, under assumption (1.2) problem (1.1) has a global attractor $\mathscr{A}_\varepsilon$, which is essentially independent of q and that the attractors $\mathscr{A}_\varepsilon$ are bounded in $L^\infty(\Omega_\varepsilon)$, uniformly in ε. This enables us to cut the nonlinearity f in such a way that it becomes bounded with bounded derivatives up to second order without changing the attractors. After these considerations, we may assume, without loss of generality, $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a $C^2(\mathbb{R})$ function satisfying (1.2) and

$$\left|\frac{\partial f}{\partial u}(x,u)\right| \leqslant c_f, \quad \left|\frac{\partial^2 f}{\partial u^2}(x,u)\right| \leqslant \tilde{c}_f \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R} \tag{1.3}$$

for some $c_f,\ \tilde{c}_f$ positive constants. The fact that now the nonlinearity is globally Lipschitz allows us to study the problem in the space $H^1(\Omega_\varepsilon)$. The attractors will lie in more regular spaces, like $W^{1,q}(\Omega_\varepsilon)$ for any $1 < q < \infty$, but their continuity properties will be analyzed in the topology of the spaces H^1.

We will regard Ω_ε as a perturbation of the fixed domain Ω_0 and we will assume the following condition:

$$\left\{ \begin{array}{l} \text{For each } 0 \leqslant \varepsilon \leqslant \varepsilon_0, \quad \Omega_\varepsilon \text{ is bounded and Lipschitz and} \\ \text{for all } K \subset\subset \Omega_0, \text{ there exists } \varepsilon(K), \text{ such that } K \subset \Omega_\varepsilon, \quad 0 < \varepsilon \leqslant \varepsilon(K) \end{array} \right\}. \tag{1.4}$$

Notice that we do not require a priori that $|\Omega_\varepsilon \backslash \Omega_0| \to 0$ as $\varepsilon \to 0$.

One of the main difficulties when treating domain perturbation problems is that the solutions live in different spaces (say $u_\varepsilon \in H^1(\Omega_\varepsilon)$ and $u_0 \in H^1(\Omega_0)$) and therefore statements of the type $u_\varepsilon - u_0$ should be stated clearly. In this paper we will consider, for each $0<\varepsilon\leqslant\varepsilon_0$, the space

$$H^1_\varepsilon = H^1(\Omega_\varepsilon\cap\Omega_0)\oplus H^1(\Omega_\varepsilon\backslash\bar{\Omega}_0)\oplus H^1(\Omega_0\backslash\bar{\Omega}_\varepsilon) \tag{1.5}$$

that is $H^1_\varepsilon = \{\phi\in L^2(\Omega_0\cup\Omega_\varepsilon)$, such that $\phi_{|\Omega_0\cap\Omega_\varepsilon}\in H^1(\Omega_0\cap\Omega_\varepsilon)$, $\phi_{|\Omega_0\backslash\bar{\Omega}_\varepsilon}\in H^1(\Omega_0\backslash\bar{\Omega}_\varepsilon)$, $\phi_{|\Omega_\varepsilon\backslash\bar{\Omega}_0}\in H^1(\Omega_\varepsilon\backslash\bar{\Omega}_0)\}$ with the norm $||u||^2_{H^1_\varepsilon}=||u||^2_{H^1(\Omega_\varepsilon\cap\Omega_0)}+||u||^2_{H^1(\Omega_\varepsilon\backslash\bar{\Omega}_0)}+||u||^2_{H^1(\Omega_0\backslash\bar{\Omega}_0)}$. Notice that extending by zero outside Ω_0 we have $H^1(\Omega_0)\hookrightarrow H^1_\varepsilon$, with embedding constant 1 and extending by zero outside Ω_ε we have $H^1(\Omega_\varepsilon)\hookrightarrow H^1_\varepsilon$, with embedding constant also 1. Hence, if $u_\varepsilon\in H^1(\Omega_\varepsilon)$, $u_0\in H^1(\Omega_0)$ we can write $||u_\varepsilon - u_0||_{H^1_\varepsilon}$. Moreover with certain abuse of notation we will say that $u_\varepsilon\to u_0$ in H^1_ε if $||u_\varepsilon - u_0||_{H^1_\varepsilon}\to 0$ as $\varepsilon\to 0$.

Also, with an extension by zero outside Ω_ε or Ω_0, $L^2(\Omega_\varepsilon)\hookrightarrow L^2(\mathbb{R}^N)$ and $L^2(\Omega_0)\hookrightarrow L^2(\mathbb{R}^N)$. Hence, for functions $V_\varepsilon\in L^2(\Omega_\varepsilon)$, $V_0\in L^2(\Omega_0)$, statements of the type $V_\varepsilon\to V_0$ in $L^2(\mathbb{R}^N)$ or $w-L^2(\mathbb{R}^N)$ make perfect sense. Moreover, if we have an operator T acting on $L^2(\Omega_\varepsilon)$ we may also regard this operator as acting on $L^2(\Omega_0)$ by just viewing any element $u_0\in L^2(\Omega_0)$ as an element of $L^2(\Omega_\varepsilon)$ by extending first u_0 outside Ω_0 by zero and then making the restriction to Ω_ε. Similarly we can do with operators defined in $L^2(\Omega_0)$.

In this paper, we give conditions on the behavior of Ω_ε as $\varepsilon\to 0$ and on the unperturbed problem, (1.1) with $\varepsilon=0$, that guarantee the continuity (upper and lower semicontinuity) of the attractors $\mathscr{A}_\varepsilon$ in H^1_ε as $\varepsilon\to 0$. More precisely, we show the following two results:

(i) *The upper semicontinuity of the attractors $\mathscr{A}_\varepsilon$ in H^1_ε*, which is obtained just requiring the spectral convergence in H^1_ε of the Neumann Laplacian as $\varepsilon\to 0$; that is, requiring that the eigenvalues and eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions behave continuously in H^1_ε as $\varepsilon\to 0$.

(ii) *The lower semicontinuity of the attractors $\mathscr{A}_\varepsilon$ in H^1_ε*. Once upper semicontinuity is attained, lower semicontinuity in H^1_ε is obtained by requiring that every equilibrium of the unperturbed problem is hyperbolic.

By upper semicontinuity of the attractors in H^1_ε we mean that

$$\sup_{u_\varepsilon\in\mathscr{A}_\varepsilon}\ \inf_{u_0\in\mathscr{A}_0} ||u_\varepsilon - u_0||_{H^1_\varepsilon}\to 0 \quad \text{as } \varepsilon\to 0.$$

By lower semicontinuity of the attractors in H^1_ε we mean that

$$\sup_{u_0\in\mathscr{A}_0}\ \inf_{u_\varepsilon\in\mathscr{A}_\varepsilon} ||u_\varepsilon - u_0||_{H^1_\varepsilon}\to 0 \quad \text{as } \varepsilon\to 0.$$

It is important to mention that we do not require any geometrical condition whatsoever on the perturbation Ω_ε, apart from the general assumption (1.4). The condition we require on the perturbation is that the eigenvalues and eigenfunctions of the Laplace operator behave continuously as $\varepsilon\to 0$. In particular, once the Laplace operator behaves continuously, in the sense of the spectra described above, the attractors of (1.1) will behave upper semicontinuously. If moreover all the equilibria of the limiting problem are hyperbolic then the attractors will behave continuously. In this respect, we can say that the behavior of the nonlinear dynamics of (1.1) is dictated by the behavior of the linear operator Δ.

There are several references in the literature that study linear and nonlinear, elliptic and parabolic problems under Neumann boundary conditions when the domain undergoes certain perturbations.

One of the important points to consider is that, in general, Neumann problems are much more difficult to treat than Dirichlet ones. This point was already brought to light in the pioneer work of [12]. Actually, they provide a first and important example of a domain perturbation where the Neumann spectra does not behave continuously while the same perturbation does not produce any irregular behavior for the Dirichlet eigenvalues. This example was further developed in [8]. Actually, we could say that the main difference between Dirichlet and Neumann problems resides in the fact that for Dirichlet problems the underlying space is $H_0^1(\Omega)$, which admits a extension operator to $H^1(\mathbb{R}^N)$ of norm one, independent of the domain Ω, while for Neumann problems, the underlying space is $H^1(\Omega)$, which does not have this property and the norm of the extension operators depends drastically on the smoothness and geometry of Ω. This fact makes Neumann problems much more difficult to treat.

For general exterior perturbations of the domain and the characterization of the behavior of the spectra we refer to [2,3,25]. Also in [10] the case of Hölder perturbations of a domain is studied. In [15] they study perturbations of the domain under Robin boundary conditions and in [11] they consider fairly general elliptic operators with mixed boundary conditions, Robin and Dirichlet, although when dealing with results on perturbations of the boundary they perturb only the part of the boundary where the Dirichlet condition is imposed.

In many references some particular perturbation of the domain is considered. One of the most extensively studied examples is the so called dumbbell domain, which consists of two fixed domains joined by a thin channel. There are several results on the spectral behavior of the Laplace operators under this perturbation [2–4,9,22]. Also the nonlinear elliptic problem has been studied in [19,29] and the nonlinear parabolic problem in [23,24]. In [30] a functional framework to treat nonlinear elliptic problems when the domain is perturbed is developed. In this work, the family of domains is assumed to be nested, all of them contain the limiting domain and the sequence converges in measure to the limiting domain.

Another important examples of irregular perturbations of the domain are thin domains, where, roughly speaking, a domain in $\mathbb{R}^N$ is flattened in certain directions and converges to a lower dimensional set, see [18] for a pioneer work in this area and [28] for a monograph on this kind of perturbation. See also the work [27] for other type of thin domains.

Also, for interesting results on how the shape of the domain influences the structure and the number of solutions of nonlinear elliptic equations, we refer to [13,14].

In [1] the authors study the nonlinear dynamics of a reaction diffusion equation with Dirichlet boundary condition when the domain is perturbed, obtaining the continuity of the attractors. Also, in [26], the authors consider the case of smooth perturbations of the domain, that is, a family of domains Ω_ε which are dipheomorphic to a fixed one Ω_0 and the dipheomorphisms converge to the identity in C^2. With this dipheomorphisms and using the techniques developed in [21], the authors are able to prove the continuity of the attractors with Dirichlet boundary conditions in strong norms.

Most of the references above, specially those dealing with Neumann problems, either treat nice perturbations of the domain or, when the perturbation is not regular, the perturbed domains are geometrically defined in a concrete way (thin domains, dumbbell domains, etc.). In this respect our work is different. We do not impose any geometrical description on the perturbation but bring into light that in order to analyze the behavior of nonlinear dynamics it is just needed to understand correctly the spectral behavior of the linear operators.

Our analysis is based in an extensive and thorough study of the behavior of the linear part. Actually, in Section 2 we give necessary and sufficient conditions to obtain the spectral convergence of the linear operators under the class of domain perturbation satisfying (1.4), see Proposition 2.3 below. Next, we will see that by just requiring the spectral convergence of the linear operators, we obtain the convergence of the resolvent operators (Proposition 2.6) and a type of Trotter-Kato Approximation Theorem for linear semigroups (Proposition 2.7).

In Section 3 we prove, by using the variation of constants formula and the results on the linear semigroups of Section 2, that the family of nonlinear semigroups $\{T_\varepsilon(t),\ t\geqslant 0\}$ associated to (1.1) is continuous in ε at $\varepsilon=0$, uniformly in compact intervals of $(0,\infty)$. That is, if $u_0^\varepsilon\in H^1(\Omega_\varepsilon),\ 0\leqslant\varepsilon\leqslant\varepsilon_0$ with $||u^\varepsilon-u^0||_{H^1_\varepsilon}\to 0$ as $\varepsilon\to 0$, then for any $0<r<R<\infty$

$$\sup_{r\leqslant t\leqslant R} ||T_\varepsilon(t)(u^\varepsilon)-T_0(t)u^0||_{H^1_\varepsilon}\to 0 \quad \text{as } \varepsilon\to 0.$$

With this it is not difficult to show that the family of attractors $\mathscr{A}_\varepsilon$ and the set of equilibria $\mathscr{E}_\varepsilon$ are upper semicontinuous at $\varepsilon=0$.

In Section 4 we study the lower semicontinuity of the attractors. As a matter of fact we proceed as follows. Consider the problem

$$(\mathrm{P})_\varepsilon \quad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega_\varepsilon,\\ \dfrac{\partial u}{\partial n}=0 & \text{in } \partial\Omega_\varepsilon \end{cases}$$

for $0\leqslant\varepsilon\leqslant\varepsilon_0$. Assume that $(\mathrm{P})_0$ has exactly m distinct solutions, $u_1^0,\dots,u_m^0$ and that they are all hyperbolic, that is, zero is not an eigenvalue for the problems $\Delta+\partial_u f(\cdot,u_i(\cdot))I$ for $i=1,\dots,m$. In Section 4.1, we prove that, for small ε, P_ε has exactly m distinct solutions $u_1^\varepsilon,\dots,u_m^\varepsilon$ and $u_j^\varepsilon\to u_j^0$ in H^1_ε as $\varepsilon\to 0,\ 1\leqslant j\leqslant m$. This is done through a fixed point argument. Also, using the results of Section 2 on the convergence of the linear semigroups, we prove in Section 4.2 that the local unstable manifolds of the equilibrium points u_k^ε are continuous in H_ε^1 as $\varepsilon \to 0$.

It follows from the continuity of the local unstable manifolds that the attractors are lower semicontinuous at $\varepsilon = 0$. This can be proved in the following way. Since the system generated by (1.1) is gradient, then if $u_0 \in \mathscr{A}_0$ we have that u_0 belongs to the unstable manifold of u_k^0 for some $1 \leqslant k \leqslant n$. In particular, there will exist w_0 in the local unstable manifold of u_k^0 such that $u_0 = T_0(\tau)w_0$ for some $\tau > 0$. By the continuity of the local unstable manifolds, we can get w_ε in the local unstable manifold of u_k^ε such that $||w_\varepsilon - w_0||_{H_\varepsilon^1} \to 0$ as $\varepsilon \to 0$. Now, since the family of semigroups is continuous in H_ε^1 we have that $\mathscr{A}_\varepsilon \ni T_\varepsilon(\tau, w_\varepsilon) \to T_0(\tau, w_0) = u_0$ in H_ε^1 as $\varepsilon \to \infty$. This shows the lower semicontinuity of attractors. See Theorem 4.6 for more details.

Finally in Section 5 we give two examples of perturbation of the domain where the conditions of this paper apply. The first one is a C^0 perturbation of a fixed domain Ω_0, including the case where the boundary presents a high oscillatory behavior. The second one is a nonstandard dumbbell-type perturbation.

2. Linear theory

In this section we analyze the behavior of the linear parts of the operators and prove several results that will be used throughout the paper.

2.1. Spectral convergence characterization

It is very clear that the spectral behavior of the linear operators is extremely important when analyzing the continuity properties of nonlinear dynamics. We include in this section several results on the spectral behavior of operators of the type $-\Delta + V$, where V is a potential, with Neumann boundary conditions when the domain is perturbed. We are interested in obtaining necessary and sufficient conditions that guarantee that the eigenvalues and eigenfunctions behave continuously when the domain undergoes a perturbation satisfying (1.4).

The potentials may depend also on ε. We specify their behavior as $\varepsilon \to 0$ in the following definition.

Definition 2.1. A family $\{V_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ of potentials is said admissible if $V_\varepsilon \in L^\infty(\Omega_\varepsilon)$, $\sup_{0 \leqslant \varepsilon \leqslant \varepsilon_0} ||V_\varepsilon||_{L^\infty(\Omega_\varepsilon)} \leqslant C < \infty$ and $V_\varepsilon \to V_0$ weakly in $L^2(\mathbb{R}^N)$.

To fix the notations we consider the eigenvalue problems

$$\begin{cases} -\Delta u + V_\varepsilon u = \lambda u, & \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0, & \partial\Omega_\varepsilon, \end{cases}$$

where $\{V_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ is admissible. We denote by $\{\lambda_n^\varepsilon\}_{n=1}^\infty$, for $\varepsilon \in [0, \varepsilon_0]$, the set of eigenvalues, ordered and counting multiplicity, of the operator $-\Delta + V_\varepsilon$ with Neumann boundary conditions in Ω_ε and by $\{\phi_n^\varepsilon\}_{n=1}^\infty$ a corresponding complete family of orthonormalized eigenfunctions.

We will say that the spectra behaves continuously at $\varepsilon = 0$, if for fixed $n \in \mathbb{N}$ we have that $\lambda_n^\varepsilon \to \lambda_n^0$ as $\varepsilon \to 0$ and the spectral projections converge in H_ε^1, that is, if $a \notin \{\lambda_n^0\}_{n=0}^\infty$, and $\lambda_n^0 < a < \lambda_{n+1}^0$, then if we define the projections $P_a^\varepsilon : L^2(\mathbb{R}^N) \to H^1(\Omega_\varepsilon)$, $P_a^\varepsilon(\psi) = \sum_{i=1}^n (\phi_i^\varepsilon, \psi)_{L^2(\Omega_\varepsilon)} \phi_i^\varepsilon$ then

$$\sup\{\|P_a^\varepsilon(\psi) - P_a^0(\psi)\|_{H_\varepsilon^1},\ \psi \in L^2(\mathbb{R}^N), \|\psi\|_{L^2(\mathbb{R}^N)} = 1\} \to 0 \quad \text{as } \varepsilon \to 0.$$

The convergence of the spectral projections is equivalent to the following: for each sequence $\varepsilon_k \to 0$ there exists a subsequence, that we denote again by ε_k and a complete system of orthonormal eigenfunctions of the limiting problem $\{\phi_n^0\}_{n=1}^\infty$ such that $\|\phi_n^{\varepsilon_k} - \phi_n^0\|_{H_{\varepsilon_k}^1} \to 0$ as $k \to \infty$.

Notice that condition (1.4) implies that there exists a nonincreasing sequence ρ_ε with $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ such that if we define

$$K_\varepsilon = \{x \in \Omega_0 : \operatorname{dist}(x, \partial\Omega_0) > \rho_\varepsilon\} \tag{2.1}$$

then $K_\varepsilon \subset \Omega_\varepsilon$ for all $0 < \varepsilon \leqslant \varepsilon_0$. The family of open sets $\{K_\varepsilon\}_{0<\varepsilon\leqslant\varepsilon_0}$ can be regarded, as $\varepsilon \to 0$, as a smooth interior perturbation of the domain Ω_0. In particular, since the domain Ω_0 is Lipschitz, the family K_ε is uniformly Lipschitz in ε. This implies the existence of extension operators $E_\varepsilon : H^1(K_\varepsilon) \to H^1(\mathbb{R}^N)$, which are also extension operators from $L^2(K_\varepsilon) \to L^2(\mathbb{R}^N)$, and the norms $\|E_\varepsilon\|_{\mathscr{L}(H^1(K_\varepsilon), H^1(\mathbb{R}^N))}$ and $\|E_\varepsilon\|_{\mathscr{L}(L^2(K_\varepsilon), L^2(\mathbb{R}^N))}$ are uniformly bounded in ε for $0 < \varepsilon \leqslant \varepsilon_0$.

Remark 2.2. Notice that we do not exclude the possibility that $\rho_\varepsilon = 0$ and therefore $K_\varepsilon = \Omega_0$. This will be the case when Ω_ε is an exterior perturbation of the domain, that is, $\Omega_0 \subset \Omega_\varepsilon$.

In order to characterize when the spectra behaves continuously we define

$$\tau_\varepsilon = \inf_{\substack{\phi \in H^1(\Omega_\varepsilon) \\ \phi = 0, \text{ in } K_\varepsilon}} \frac{\int_{\Omega_\varepsilon} |\nabla \phi|^2}{\int_{\Omega_\varepsilon} |\phi|^2}. \tag{2.2}$$

Observe that, in case $\Omega_\varepsilon \backslash \bar{K}_\varepsilon$ is smooth, τ_ε is the first eigenvalue of the following problem:

$$\begin{cases} -\Delta u = \tau u, & \Omega_\varepsilon \backslash \bar{K}_\varepsilon, \\ u = 0, & \partial K_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0, & \partial \Omega_\varepsilon. \end{cases}$$

We have the following useful characterization

Proposition 2.3. *Assume the family of domains* $\{\Omega_\varepsilon\}_{0\leqslant\varepsilon\leqslant\varepsilon_0}$ *satisfies* (1.4). *Then, the following four statements are equivalent*:

(i) *The spectra of* $-\Delta+V_\varepsilon$ *behave continuously as* $\varepsilon\to 0$ *for any admissible family of potentials* $\{V_\varepsilon, 0\leqslant\varepsilon\leqslant\varepsilon_0\}$.

(ii) $\tau_\varepsilon\to\infty$ *as* $\varepsilon\to 0$.

(iii) *For any family of functions* ψ_ε *with* $\|\psi_\varepsilon\|_{H^1(\Omega_\varepsilon)}\leqslant C$ *then* $\|\psi_\varepsilon\|_{L^2(\Omega_\varepsilon\setminus\bar{K}_\varepsilon)}\to 0$ *as* $\varepsilon\to 0$.

(iv) *For any family of functions* ψ_ε *with* $\|\psi_\varepsilon\|_{H^1(\Omega_\varepsilon)}\leqslant C$, *there exists a sequence* ψ_{ε_k} *and a function* $\psi_0\in H^1(\Omega_0)$ *such that* $\psi_{\varepsilon_k}\to\psi_0$, *in* $L^2(\mathbb{R}^N)$ *and for any* $\chi\in H^1(\mathbb{R}^N)$ *we have that*

$$\int_{\Omega_{\varepsilon_k}}\nabla\psi_{\varepsilon_k}\nabla\chi\to\int_{\Omega_0}\nabla\psi_0\nabla\chi.$$

Moreover, if any of the four statements above is true then the following also holds

(v) $|\Omega_\varepsilon\setminus K_\varepsilon|\to 0$ *as* $\varepsilon\to 0$.

Remark 2.4. A somehow similar, although weaker, statement of this lemma can be found in the works [2,3].

Notice that statements (ii)–(iv) are independent of the potential V_ε. Hence (i) is equivalent to the fact that the spectra of $-\Delta$ behaves continuously as $\varepsilon\to 0$.

We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (v), (ii) + (v) $\Rightarrow$ (iii) $\Rightarrow$ (ii), (iii) + (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) and (iv) + (v) $\Rightarrow$ (i).

Proof. That (iii) implies (ii) is easy since if there exists a sequence $\varepsilon_k\to 0$ with τ_{ε_k} bounded, then, by the definition of τ_ε, we obtain a sequence of functions with $L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})$ norm equal one and the $H^1(\Omega_{\varepsilon_k})$ norm bounded.

That (ii) implies (v) is also easy. Notice that by the definition of K_{ε_k}, $|\Omega_0\setminus K_{\varepsilon_k}|\to 0$ as $\varepsilon_k\to 0$ and therefore we just need to show that $|\Omega_{\varepsilon_k}\setminus\Omega_0|\to 0$ as $\varepsilon_k\to 0$. If this were not true then we will have a positive $\eta>0$ and a sequence $\varepsilon_k\to 0$ such that $|\Omega_{\varepsilon_k}\setminus\Omega_0|\geqslant\eta$. Let $\rho=\rho(\eta)$ be a small number such that $|\{x\in\mathbb{R}^N\setminus\Omega_0, \operatorname{dist}(x,\Omega_0)\leqslant\rho\}|\leqslant\eta/2$. This implies that $|\{x\in\Omega_{\varepsilon_k}, \operatorname{dist}(x,\Omega_0)\geqslant\rho\}|\geqslant\eta/2$. Let us construct a smooth function $\gamma(x)$ with $\gamma(x)=0$ in Ω_0, $\gamma(x)=1$, $x\in\mathbb{R}^N\setminus\Omega_0$ with $\operatorname{dist}(x,\Omega_0)\geqslant\rho$. Then obviously $\gamma\in H^1(\Omega_{\varepsilon_k})$ with $\|\nabla\gamma\|_{L^2(\Omega_{\varepsilon_k})}\leqslant C$ and $\|\gamma\|_{L^2(\Omega_{\varepsilon_k})}\geqslant(\eta/2)^{\frac{1}{2}}$. This implies that τ_{ε_k} is bounded.

That (ii) implies (iii) is proved as follows. If it is not true then there will exist a sequence of functions ϕ_{ε_k} with $\|\phi_{\varepsilon_k}\|_{H^1(\Omega_{\varepsilon_k})}\leqslant C_1$ and $\|\phi_{\varepsilon_k}\|_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})}\geqslant C_2>0$, for some constants C_1 and C_2 independent of ε_k. If we consider the functions $\psi_{\varepsilon_k}=E_{\varepsilon_k}(\phi_{\varepsilon_k}|_{K_{\varepsilon_k}})$ then $\psi_{\varepsilon_k}\in H^1(\mathbb{R}^N)$ with $\|\psi_{\varepsilon_k}\|_{H^1(\mathbb{R}^N)}\leqslant C$ independent of ε_k. Moreover, by Hölder's inequality and Sobolev embeddings, we have

$$\|\psi_{\varepsilon_k}\|_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})}\leqslant\|\psi_{\varepsilon_k}\|_{L^{2N/(N-2)}(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})}|\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k}|^{\frac{1}{N}}\leqslant C\|\psi_{\varepsilon_k}\|_{H^1(\mathbb{R}^N)}|\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k}|^{\frac{1}{N}},\quad (N\geqslant 3),$$

$$||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} \leqslant ||\psi_{\varepsilon_k}||_{L^{2p}(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})}|\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k}|^{\frac{1}{2}-\frac{1}{p}} \leqslant C||\psi_{\varepsilon_k}||_{H^1(\mathbb{R}^N)}|\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k}|^{\frac{1}{2}-\frac{1}{2p}}, \quad (N=1,2),$$

where p can be choosen arbitrarily large in the last inequality. These two last inequalities imply that there exists a $\theta > 0$, such that

$$||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} \leqslant C|\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k}|^{\theta} \to 0 \quad \text{as } \varepsilon_k \to 0.$$

We consider now $\chi_{\varepsilon_k} = \phi_{\varepsilon_k} - \psi_{\varepsilon_k}$. By construction $\chi_{\varepsilon_k} = 0$ in K_{ε_k} and $||\chi_{\varepsilon_k}||_{H^1(\Omega_{\varepsilon_k})} \leqslant C$. Moreover, $||\chi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} \geqslant ||\phi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} - ||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} \geqslant C_2/2$ as long as ε_k is small enough. This contradicts (ii).

That (iii) implies (iv) is proved as follows. If ψ_{ε_k} is a sequence with $||\psi_{\varepsilon_k}||_{H^1(\Omega_{\varepsilon_k})} \leqslant C$, then we can extract a subsequence of ψ_{ε_k}, that we denote again by ψ_{ε_k} and we obtain a function $\psi_0 \in H^1(\Omega_0)$ such that $\psi_{\varepsilon_k} \to \psi_0$, w-$H^1(K)$, s-$L^2(K)$, for any $K \subset\subset \Omega_0$. Let us prove that we actually have $\psi_{\varepsilon_k} \to \psi_0$ in $L^2(\mathbb{R}^N)$. Notice first that with a similar argument as we have done in the proof that (ii) implies (iii) we have that there exists a $\rho > 0$ such that $||\psi_{\varepsilon_k}||_{L^2(K_{\varepsilon_k}\setminus K_\delta)} \leqslant C|K_{\varepsilon_k}\setminus K_\delta|^{\rho}$, for $0 < \varepsilon_k < \delta$, with a constant C independent of k and δ. Let $\eta > 0$ a small number. Choose δ small enough so that $||\psi_{\varepsilon_k}||_{L^2(K_{\varepsilon_k}\setminus K_\delta)} \leqslant \frac{\eta}{4}$, for any $\varepsilon_k < \delta$ and $||\psi_0||_{L^2(\Omega_0\setminus K_\delta)} \leqslant \frac{\eta}{4}$. Then,

$$||\psi_{\varepsilon_k} - \psi_0||^2_{L^2(\mathbb{R}^N)} = ||\psi_{\varepsilon_k} - \psi_0||^2_{L^2(K_\delta)} + ||\psi_{\varepsilon_k} - \psi_0||^2_{L^2(\mathbb{R}^N\setminus K_\delta)}.$$

But

$$\begin{aligned} ||\psi_{\varepsilon_k} - \psi_0||_{L^2(\mathbb{R}^N\setminus K_\delta)} &\leqslant ||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} + ||\psi_{\varepsilon_k}||_{L^2(K_{\varepsilon_k}\setminus K_\delta)} + ||\psi_0||_{L^2(\Omega_0\setminus K_\delta)} \\ &\leqslant \frac{\eta}{2} + ||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})}. \end{aligned}$$

Choosing $\varepsilon_k > 0$ small enough so that $||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k}\setminus K_{\varepsilon_k})} \leqslant \frac{\eta}{4}$ and $||\psi_{\varepsilon_k} - \psi_0||^2_{L^2(K_\delta)} \leqslant \frac{\eta}{4}$, we have that

$$||\psi_{\varepsilon_k} - \psi_0||_{L^2(\mathbb{R}^N)} \leqslant \eta.$$

This shows the convergence in $L^2(\mathbb{R}^N)$.

Now if $\chi \in H^1(\mathbb{R}^N)$ and if $\eta > 0$ is a small number, choose $\delta > 0$ small enough such that $||\chi||_{H^1((\Omega_{\varepsilon_k}\cup\Omega_0)\setminus\bar{K}_\delta)} \leqslant \eta$. Then, for $0 < \varepsilon_k < \delta$, we have

$$\begin{aligned} &\left| \int_{\Omega_{\varepsilon_k}} \nabla\psi_{\varepsilon_k}\nabla\chi - \int_{\Omega_0} \nabla\psi_0\nabla\chi \right| \\ &\quad \leqslant \left| \int_{K_\delta} (\nabla\psi_{\varepsilon_k} - \nabla\psi_0)\nabla\chi \right| + \int_{\Omega_{\varepsilon_k}\setminus K_\delta} |\nabla\psi_{\varepsilon_k}||\nabla\chi| + \int_{\Omega_0\setminus K_\delta} |\nabla\psi_0||\nabla\chi| \\ &\quad \leqslant \left| \int_{K_\delta} (\nabla\psi_{\varepsilon_k} - \nabla\psi_0)\nabla\chi \right| + 2C\eta \to 2C\eta \quad \text{as } \varepsilon_k \to 0. \end{aligned}$$

Since $\eta > 0$ is arbitrary, (iv) holds.

That (iv) implies (iii) is proved as follows. If ψ_ε is a family of functions with $||\psi_\varepsilon||_{H^1(\Omega_\varepsilon)} \leqslant C$ then, there is a sequence ψ_{ε_k} and a function $\psi_0 \in H^1(\Omega_0)$ such that $||\psi_{\varepsilon_k} - \psi_0||_{L^2(\mathbb{R}^N)} \to 0$ as $\varepsilon_k \to 0$. Hence $||\psi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k} \setminus \bar{K}_{\varepsilon_k})} \leqslant ||\psi_{\varepsilon_k} - \psi_0||_{L^2(\Omega_{\varepsilon_k} \setminus \bar{K}_{\varepsilon_k})} + ||\psi_0||_{L^2(\Omega_0 \setminus \bar{K}_{\varepsilon_k})} \to 0$ as $\varepsilon_k \to 0$.

Let us prove now that (i) implies (ii). If this is not the case then we will have again a sequence ε_k approaching zero and a positive number a with $\tau_{\varepsilon_k} < a$, for all k. From the definition of τ_{ε_k} we can get functions ϕ_{ε_k} with $\phi_{\varepsilon_k} = 0$ in Ω_{ε_k}, $||\phi_{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k})} = 1$ and $||\nabla \phi_{\varepsilon_k}||^2_{L^2(\Omega_{\varepsilon_k})} \leqslant a$.

Observe that

$$\int_{\Omega_{\varepsilon_k}} |\nabla \phi_{\varepsilon_k}|^2 + \int_{\Omega_{\varepsilon_k}} V_{\varepsilon_k} |\phi_{\varepsilon_k}|^2 \leqslant a + ||V_{\varepsilon_k}||_{L^\infty(\Omega_{\varepsilon_k})} \leqslant \tilde{a},$$

for some constant $\tilde{a}$ independent of ε_k. Choose $n \in \mathbb{N}$ with the property that $\tilde{a} < \lambda_n^0 < \lambda_{n+1}^0$, denote by $\phi_1^{\varepsilon_k}, \dots, \phi_n^{\varepsilon_k}$ the first n eigenfunctions and consider the linear subspace $[\phi_1^{\varepsilon_k}, \dots, \phi_n^{\varepsilon_k}, \phi_{\varepsilon_k}] \subset H^1(\Omega_{\varepsilon_k})$. By the spectral convergence we can get a subsequence, that we denote by ε_k again and eigenfunctions of the limiting problem $\phi_1^0, \dots, \phi_n^0$ such that $||\phi_i^{\varepsilon_k} - \phi_i^0||_{H^1_{\varepsilon_k}} \to 0$ as $\varepsilon_k \to 0$. This implies that $||\phi_i^{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k} \setminus K_{\varepsilon_k})} \to 0$ as $\varepsilon_k \to 0$. From here we get that

$$\int_{\Omega_{\varepsilon_k}} \phi_i^{\varepsilon_k} \phi_{\varepsilon_k} \to 0, \quad \text{as } \varepsilon_k \to 0, \ \text{for } i = 1, \dots, n,$$

which means that $[\phi_1^{\varepsilon_k}, \dots, \phi_n^{\varepsilon_k}, \phi_{\varepsilon_k}]$ is almost an orthonormal system in $L^2(\Omega_{\varepsilon_k})$. By the min–max characterization of the eigenvalues, we have that

$$\lambda_{n+1}^{\varepsilon_k} \leqslant \max_{\phi \in [\phi_1^{\varepsilon_k}, \dots, \phi_n^{\varepsilon_k}, \phi_{\varepsilon_k}]} \frac{\int_{\Omega_{\varepsilon_k}} |\nabla \phi|^2 + \int_{\Omega_{\varepsilon_k}} V_{\varepsilon_k} |\phi|^2}{\int_{\Omega_{\varepsilon_k}} |\phi|^2}.$$

But if $\phi \in [\phi_1^{\varepsilon_k}, \dots, \phi_n^{\varepsilon_k}, \phi_{\varepsilon_k}]$ we can write $\phi = \sum_{i=1}^n \alpha_i \phi_i^{\varepsilon_k} + \beta \phi_{\varepsilon_k}$. Using that $\phi_i^{\varepsilon_k}$ is an eigenfunction corresponding to the eigenvalue $\lambda_i^{\varepsilon_k}$ and that the family $\{\phi_1^{\varepsilon_k}, \dots, \phi_n^{\varepsilon_k}, \phi_{\varepsilon_k}\}$ is almost orthonormal, by direct calculation of the above quotient we get that

$$\lambda_{n+1}^{\varepsilon_k} \leqslant \frac{\sum_{i=1}^n \alpha_i^2 \lambda_i^{\varepsilon_k} + \tilde{a}\beta^2 + o(1)}{\sum_{i=1}^n \alpha_i^2 + \tilde{a} + o(1)} \leqslant \lambda_n^0 + o(1).$$

This contradicts the continuity of the eigenvalues given by (i).

The proof that (iv) implies (i) is as follows. Fix n with the property that $\lambda_n^0 < \lambda_{n+1}^0$ and consider the family of eigenfunctions $\{\phi_1^0, \dots, \phi_n^0\}$. If we denote by E a extension operator from $H^1(\Omega_0)$ to $H^1(\mathbb{R}^N)$, and by $\mathscr{T}_\varepsilon$ the restriction operator to Ω_ε, we construct the functions $\xi_i^\varepsilon = \mathscr{T}_\varepsilon E \phi_i^0$, $i = 1, \dots, n$. Since (iv) implies (v) we easily see that $||\xi_i^\varepsilon||_{H^1(\Omega_\varepsilon\setminus\Omega_0)}\to 0$ as $\varepsilon\to 0$ for $i=1,\dots,n$. By the min–max characterization of eigenvalues, we easily obtain that $\lambda_i^\varepsilon\leqslant\lambda_i^0+o(1)$ as $\varepsilon\to 0$.

We can choose a sequence $\varepsilon_k\to 0$ and numbers $\kappa_i\leqslant\lambda_i^0,\ i=1,\dots,n,$ such that $\lambda_i^{\varepsilon_k}\to\kappa_i$, for $i=1,\dots,n$. Since $\phi_i^{\varepsilon_k}$, for $i=1,\dots,n$ is a bounded sequence in $H^1(\Omega_{\varepsilon_k})$, then by (iv) we can extract another subsequence, that we still denote by $\phi_i^{\varepsilon_k}$, and get functions $\xi_i^0\in H^1(\Omega_0),\ i=1,\dots,n,$ such that $\phi_i^{\varepsilon_k}\to\xi_i^0$ in $L^2(\mathbb{R}^N)$ and

$$\int_{\Omega_{\varepsilon_k}}\nabla\phi_i^{\varepsilon_k}\nabla\chi\to\int_{\Omega_0}\nabla\xi_i^0\nabla\chi,\quad i=1,\dots,n$$

for any $\chi\in H^1(\mathbb{R}^N)$.

In particular $\int_{\Omega_0}\xi_i^0\xi_j^0=\delta_{ij}$ and passing to the weak limit in the equation, we get

$$\int_{\Omega_0}\nabla\xi_i^0\nabla\chi+\int_{\Omega_0}V_0\xi_i^0\chi=\kappa_i\int_{\Omega_0}\xi_i^0\chi,\quad i=1,\dots,n.$$

This implies that necessarily κ_i and ξ_i^0 are eigenvalues and eigenfunctions of the limiting problem. Since we already know that $\kappa_i\leqslant\lambda_i^0$ we necessarily have that $\kappa_i=\lambda_i^0$ for $i=1,\dots,n$ and $\{\xi_1^0,\dots,\xi_n^0\}$ is a system of orthonormal eigenfunctions associated to $\lambda_1^0,\dots,\lambda_n^0$.

In order to prove the convergence in $H^1_{\varepsilon_k}$ we notice that $\phi_i^0,\ i=1,\dots,n,$ satisfy

$$\int_{\Omega_{\varepsilon_k}}|\nabla\phi_i^{\varepsilon_k}|^2=\lambda_i^{\varepsilon_k}\int_{\Omega_{\varepsilon_k}}|\phi_i^{\varepsilon_k}|^2-\int_{\Omega_{\varepsilon_k}}V_{\varepsilon_k}|\phi_i^{\varepsilon_k}|^2\to\lambda_i^0\int_{\Omega_0}|\xi_i^0|^2-\int_{\Omega_0}V_0|\xi_i^0|^2=\int_{\Omega_0}|\nabla\xi_i^0|^2,$$

where we have used that $\phi_i^{\varepsilon_k}\to\xi_i^0$ in $L^2(\mathbb{R}^N)$, the weak convergence of V_{ε_k} to V_0 and the uniform bound of $||V_{\varepsilon_k}||_{L^\infty(\Omega_{\varepsilon_k})}$. Hence,

$$\int_{R^N}|\nabla\phi_i^{\varepsilon_k}-\nabla\xi_i^0|^2=\int_{\Omega_{\varepsilon_k}}|\nabla\phi_i^{\varepsilon_k}|^2+\int_{\Omega_0}|\nabla\xi_i^0|^2-2\int_{\Omega_{\varepsilon_k}}\nabla\phi_i^{\varepsilon_k}\nabla\xi_i^0.$$

But,

$$\int_{\Omega_{\varepsilon_k}}|\nabla\phi_i^{\varepsilon_k}|^2\to\int_{\Omega_0}|\nabla\xi_i^0|^2\quad\text{as }\varepsilon_k\to 0$$

and if we define $\tilde{\xi}_i^0\in H^1(\mathbb{R}^N)$ an extension of ξ_i^0 we get that

$$\int_{\Omega_{\varepsilon_k}}\nabla\phi_i^{\varepsilon_k}\nabla\xi_i^0=\int_{\Omega_{\varepsilon_k}}\nabla\phi_i^{\varepsilon_k}\nabla\tilde{\xi}_i^0+\int_{\Omega_{\varepsilon_k}}\nabla\phi_i^{\varepsilon_k}(\nabla\xi_i^0-\nabla\tilde{\xi}_i^0)\to\int_{\Omega_0}|\nabla\xi_i^0|^2\quad\text{as }\varepsilon_k\to 0$$

because

$$\left|\int_{\Omega_{\varepsilon_k}} \nabla\phi_i^{\varepsilon_k}(\nabla\xi_i^0 - \nabla\tilde{\xi}_i^0)\right| \leqslant ||\nabla\phi_i^{\varepsilon_k}||_{L^2(\Omega_{\varepsilon_k})}||\nabla\xi_i^0 - \nabla\tilde{\xi}_i^0||_{L^2(\Omega_{\varepsilon_k})} \to 0 \quad \text{as } \varepsilon_k \to 0.$$

This implies that

$$\int_{R^N} |\nabla\phi_i^{\varepsilon_k} - \nabla\xi_i^0|^2 \to 0 \quad \text{as } \varepsilon_k \to 0.$$

And the proposition is proved.

2.2. *Convergence of the resolvent operators*

We analyse in this section the behavior of the resolvent operators.

Definition 2.5. We say that a family $\{\Omega_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ is admissible if it satisfies (1.4) and one of conditions (i)–(iv) of Proposition 2.3.

We have the following result.

Proposition 2.6. *Assume that the family of potentials $\{V_\varepsilon, 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ and the family of domains $\{\Omega_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ are admissible. Assume also that $0 \notin \sigma(-\Delta + V_0)$. Then, for ε small enough $0 \notin \sigma(-\Delta + V_\varepsilon)$ and there exists a constant C independent of ε such that*

$$||(-\Delta + V_\varepsilon)^{-1} g_\varepsilon||_{H^1(\Omega_\varepsilon)} \leqslant C||g_\varepsilon||_{L^2(\Omega_\varepsilon)}, \quad g_\varepsilon \in L^2(\Omega_\varepsilon). \tag{2.3}$$

Moreover, if $g_\varepsilon \to g_0$ weakly in $L^2(\mathbb{R}^N)$, then

$$||(-\Delta + V_\varepsilon)^{-1} g_\varepsilon - (-\Delta + V_0)^{-1} g_0||_{H^1_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.4}$$

Proof. Let us show first (2.3). By the continuity of the spectra given by Proposition 2.3 we have that for ε small enough $0 \notin \sigma(-\Delta + V_\varepsilon)$. In particular, for $g_\varepsilon \in L^2(\Omega_\varepsilon)$ given we have a unique solution $w_\varepsilon \in H^1(\Omega_\varepsilon)$ of

$$\begin{cases} -\Delta w_\varepsilon + V_\varepsilon w_\varepsilon = g_\varepsilon, & \Omega_\varepsilon, \\ \dfrac{\partial w_\varepsilon}{\partial n} = 0, & \partial\Omega_\varepsilon. \end{cases} \tag{2.5}$$

We show first that if $||g_\varepsilon||_{L^2(\Omega_\varepsilon)} \leqslant C$, with C independent of ε, then $||w_\varepsilon||_{L^2(\Omega_\varepsilon)}$ is bounded. Suppose not, then there is a subsequence, which we again denote by $\{w_\varepsilon\}$, such that $||w_\varepsilon||_{L^2(\Omega_\varepsilon)} \to \infty$. Consider $\tilde{w}_\varepsilon = \frac{w_\varepsilon}{||w_\varepsilon||_{L^2(\Omega_\varepsilon)}}$, so that $||\tilde{w}_\varepsilon||_{L^2(\Omega_\varepsilon)} = 1$.

Then

$$\begin{cases} -\Delta \tilde{w}_\varepsilon + V_\varepsilon \tilde{w}_\varepsilon = \dfrac{g_\varepsilon}{||w_\varepsilon||_{L^2(\Omega_\varepsilon)}}, & \Omega_\varepsilon, \\ \dfrac{\partial \tilde{w}_\varepsilon}{\partial n} = 0, & \partial\Omega_\varepsilon. \end{cases} \tag{2.6}$$

Multiplying this equation by $\tilde{w}_\varepsilon$ and integrating by parts we obtain that

$$\int_{\Omega_\varepsilon} |\nabla \tilde{w}_\varepsilon|^2 + \int_{\Omega_\varepsilon} V_\varepsilon |\tilde{w}_\varepsilon|^2 = \int_{\Omega_\varepsilon} \frac{\tilde{g}_\varepsilon}{||w_\varepsilon||_{L^2(\Omega_\varepsilon)}} \tilde{w}_\varepsilon$$

from where it follows that

$$\int_{\Omega_\varepsilon} |\nabla \tilde{w}_\varepsilon|^2 \leqslant C,$$

with C independent of ε. Applying Proposition 2.3(iv) we can extract a sequence, denoted still by $\tilde{w}_\varepsilon$, so that $\tilde{w}_\varepsilon \to \tilde{w}_0$ in $L^2(\mathbb{R}^N)$ and for any $\chi \in H^1(\mathbb{R}^N)$ we have

$$\int_{\Omega_\varepsilon} \nabla \tilde{w}_\varepsilon \nabla \chi \to \int_{\Omega_0} \nabla \tilde{w}_0 \nabla \chi.$$

Notice in particular that $||\tilde{w}_0||_{L^2(\Omega_0)} = 1$.

Let $\xi \in H^1(\Omega_0)$ and consider $\tilde{\xi} \in H^1(\mathbb{R}^N)$ an extension of ξ to $\mathbb{R}^N$. If we multiply the equation (2.6) by $\tilde{\xi} \in H^1(\Omega_\varepsilon)$ and integrating by parts we have that

$$\int_{\Omega_\varepsilon} \nabla \tilde{w}_\varepsilon \nabla \tilde{\xi} + \int_{\Omega_\varepsilon} V_\varepsilon \tilde{w}_\varepsilon \tilde{\xi} = \int_{\Omega_\varepsilon} \frac{g_\varepsilon}{||w_\varepsilon||_{L^2(\Omega_\varepsilon)}} \tilde{\xi}.$$

Taking the limit, we get that

$$\int_{\Omega_0} \nabla \tilde{w}_0 \nabla \xi + \int_{\Omega_0} V_0 \tilde{w}_0 \xi = 0,$$

where we have used that $||V_\varepsilon||_{L^\infty(\Omega_\varepsilon)} \leqslant C$, $V_\varepsilon \to V_0$, w-$L^2(\mathbb{R}^N)$ and $\tilde{w}_\varepsilon \to \tilde{w}_0$ in $L^2(\mathbb{R}^N)$. Thus

$$\begin{cases} -\Delta \tilde{w}_0 + V_0 \tilde{w}_0 = 0, & \Omega_0, \\ \dfrac{\partial \tilde{w}_0}{\partial n} = 0, & \partial\Omega_0, \end{cases} \tag{2.7}$$

and since $0 \notin \sigma(-\Delta + V_0)$, we get $\tilde{w}_0 = 0$. This contradicts the fact that $||\tilde{w}_0||_{L^2(\Omega_0)} = 1$. Hence, we obtain that $||w_\varepsilon||_{L^2(\Omega_\varepsilon)}$ is uniformly bounded in ε.

To show that $||\nabla w_\varepsilon||_{L^2(\Omega_\varepsilon)}$ is uniformly bounded in ε we note that V_ε are uniformly bounded in $L^\infty(\Omega_\varepsilon)$ and that

$$\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 = -\int_{\Omega_\varepsilon} V_\varepsilon |w_\varepsilon|^2 + \int_{\Omega_\varepsilon} g_\varepsilon w_\varepsilon.$$

To show (2.4), notice that by the weak convergence of g_ε, we have that g_ε is uniformly bounded in $L^2(\mathbb{R}^N)$. Applying (2.3) we obtain that $\|(-\Delta + V_\varepsilon)^{-1} g_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ is uniformly bounded in ε. Using (iv) in Proposition 2.3 and taking the limit in the equation we obtain that if $u_\varepsilon = (-\Delta + V_\varepsilon)^{-1} g_\varepsilon$ and $u_0 = (-\Delta + V_0)^{-1} g_0$, then $u_\varepsilon \to u_0$ in $L^2(\mathbb{R}^N)$ and $\nabla u_\varepsilon \to \nabla u_0$ w-$L^2(\mathbb{R}^N)$. Now with a similar argument as in the proof that (iv) implies (i) in Proposition 2.3 we obtain that $u_\varepsilon \to u_0$ in H^1_ε. This concludes the proof of the lemma. $\square$

2.3. *Convergence of the linear semigroups*

With the continuity of the spectra of the operators $-\Delta + V_\varepsilon$ we can obtain estimates on the behavior of the linear semigroups that will be very useful for the analysis of the nonlinear dynamics.

We consider the operators $A_\varepsilon = \Delta - V_\varepsilon$ as unbounded operators in $L^2(\Omega_\varepsilon)$, for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. They generate analytic semigroups $e^{A_\varepsilon t}$ in $L^2(\Omega_\varepsilon)$, $H^1(\Omega_\varepsilon)$ and in general in the scale of fractional powers of the operator.

Notice that the semigroup $e^{A_\varepsilon t}$ acts on functions defined in Ω_ε. We will need to estimate expressions of the type $e^{A_\varepsilon t} u_0$ where, for instance $u_0 \in L^2(\Omega_0)$. As we said in the introduction, by this we mean that we extend the function u_0 by zero outside Ω_0 and restrict to Ω_ε. In this way we can also regard $u_0 \in L^2(\Omega_\varepsilon)$ and evaluate $e^{A_\varepsilon t} u_0$. Similarly we can give a meaning to $e^{A_0 t} u_\varepsilon$.

We have the following result

Proposition 2.7. *Assume that the family of domains $\{\Omega_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ and the family of potentials $\{V_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ are admissible. Let $a > 0$ be such that $\lambda_n^0 < a < \lambda_{n+1}^0$ and consider the spectral projection over the linear space generated by the first n eigenfunctions P_a^ε defined in Section* 2.1. *Denote also by b a number such that $b < \lambda_1^0$. Then, there exists a number $\frac{1}{2} < \gamma < 1$ and a function $\theta(\varepsilon)$ with $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that*

$$\|e^{A_\varepsilon t} u_\varepsilon - e^{A_0 t} u_\varepsilon\|_{H^1_\varepsilon} \leqslant M\theta(\varepsilon) t^{-\gamma} e^{-bt} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad u_\varepsilon \in L^2(\Omega_\varepsilon), \quad t > 0,$$

$$\|e^{A_\varepsilon t}(I - P_a^\varepsilon) u_\varepsilon - e^{A_0 t}(I - P_a^0) u_\varepsilon\|_{H^1_\varepsilon} \leqslant M\theta(\varepsilon) t^{-\gamma} e^{-at} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad u_\varepsilon \in L^2(\Omega_\varepsilon), \quad t > 0.$$

Proof. Let us prove the second inequality. So let us consider n and a given, satisfying the hypothesis of the proposition. Notice that we can choose a constant M independent of ε such that

$$\|e^{A_\varepsilon t}(I - P_a^\varepsilon) u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant M t^{-\frac{1}{2}} e^{-at} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad u_\varepsilon \in L^2(\Omega_\varepsilon), \quad t > 0, \quad \varepsilon \in [0, \varepsilon_0).$$

Now, we separate the estimate for t small and t large. Choose $\gamma \in (\frac{1}{2}, 1)$ fixed. Let $\delta > 0$ be a small parameter and let us consider two different cases according to $t \in (0, \delta]$ or $t > \delta$.

(i) If $t\in(0,\delta]$ we easily check that

$$\|e^{A_\varepsilon t}(I-P_\varepsilon^a)u_\varepsilon - e^{A_0 t}(I-P_0^a)u_\varepsilon\|_{H^1_\varepsilon} \leqslant 2Mt^{-\frac{1}{2}}e^{-at}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$
$$\leqslant 2M\delta^{\gamma-\frac{1}{2}}t^{-\gamma}e^{-at}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{2.8}$$

(ii) If $t>\delta$ we proceed as follows. Notice first that we can always choose a positive number $l=l(\delta)$ such that if $z\geqslant l$ then $ze^{-2zt}\leqslant \delta t^{-\gamma}e^{-at}$ for all $t\geqslant\delta$. Since we have $\lambda_k^\varepsilon \xrightarrow{\varepsilon\to 0}\lambda_k^0$ and $\lambda_k^0\xrightarrow{k\to\infty}+\infty$, there exists $N=N(\delta)>n$ such that $\lambda_k^\varepsilon\geqslant l(\delta)$, $\varepsilon\in[0,\varepsilon_0)$. Without loss of generality we can assume that we have $\lambda^0_{N(\delta)}<\lambda^0_{N(\delta)+1}$. Hence, from the spectral decomposition of the linear semigroups, we obtain

$$\begin{aligned}
&\|e^{A_\varepsilon t}(I-P_\varepsilon^a)u_\varepsilon - e^{A_0 t}(I-P_0^a)u_\varepsilon\|_{H^1_\varepsilon}\\
&\leqslant \left\|\sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^\varepsilon t}(u_\varepsilon,\phi_k^\varepsilon)\phi_k^\varepsilon - \sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^0 t}(u_\varepsilon,\phi_k^0)\phi_k^0\right\|_{H^1_\varepsilon}\\
&\quad + \left\|\sum_{N(\delta)+1}^{\infty} e^{-\lambda_k^\varepsilon t}(u_\varepsilon,\phi_k^\varepsilon)\phi_k^\varepsilon\right\|_{H^1(\Omega_\varepsilon)} + \left\|\sum_{N(\delta)+1}^{\infty} e^{-\lambda_k^0 t}(u_\varepsilon,\phi_k^0)\phi_k^0\right\|_{H^1(\Omega_0)}\\
&= I_1+I_2+I_3.
\end{aligned} \tag{2.9}$$

Analyzing I_2, I_3 and I_1, respectively, we get

$$I_2\leqslant \sum_{N(\delta)+1}^{\infty} \lambda_k^\varepsilon e^{-2\lambda_k^\varepsilon t}|(u_\varepsilon,\phi_n^\varepsilon)|^2\leqslant \delta t^{-\gamma}e^{-at}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)},$$
$$I_3\leqslant \sum_{N(\delta)+1}^{\infty} \lambda_k^0 e^{-2\lambda_k^0 t}|(u_0,\phi_n^0)|^2\leqslant \delta t^{-\gamma}e^{-at}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)},$$

$$\begin{aligned}
I_1 &= \left\|\sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^\varepsilon t}(u_\varepsilon,\phi_k^\varepsilon)\phi_k^\varepsilon - \sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^0 t}(u_\varepsilon,\phi_k^0)\phi_k^0\right\|_{H^1_\varepsilon}\\
&\leqslant \left\|\sum_{k=n+1}^{N(\delta)} (e^{-\lambda_k^\varepsilon t}-e^{-\lambda_k^0 t})(u_\varepsilon,\phi_k^\varepsilon)\phi_k^\varepsilon\right\|_{H^1_\varepsilon} + \left\|\sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^0 t}((u_\varepsilon,\phi_k^\varepsilon)\phi_k^\varepsilon-(u_\varepsilon,\phi_k^0)\phi_k^0)\right\|_{H^1_\varepsilon}\\
&\leqslant \sum_{k=n+1}^{N(\delta)} ((\lambda_k^\varepsilon)^{\frac{1}{2}}+1)|e^{-\lambda_k^\varepsilon t}-e^{-\lambda_k^0 t}|\,\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\
&\quad + \sum_{i=r}^{k(\delta)} e^{-\mu_i t}\left\|\sum_{k=n_i+1}^{n_{i+1}} ((u_\varepsilon,\phi_k^\varepsilon)\phi_k^\varepsilon-(u_\varepsilon,\phi_k^0)\phi_k^0)\right\|_{H^1_\varepsilon}.
\end{aligned}$$

Moreover, from the convergence of the eigenvalues and of the spectral projections, we can find $\varepsilon_1(\delta) \in (0, \varepsilon_0)$ so that

$$\sum_{n=1}^{N(\delta)} ((\lambda_k^\varepsilon)^{\frac{1}{2}} + 1)|e^{-\lambda_n^\varepsilon t} - e^{-\lambda_n^0 t}| \leqslant \delta t^{-\gamma} e^{-at}, \quad \varepsilon \in (0, \varepsilon_1(\delta)),$$

$$\begin{aligned}&\sum_{i=r}^{k(\delta)} e^{-\mu_i t} \left\| \sum_{k=n_i+1}^{n_{i+1}} ((u_\varepsilon, \phi_k^\varepsilon)\phi_k^\varepsilon - (u_\varepsilon, \phi_k^0)\phi_k^0) \right\|_{H^1(\Omega_0)} \\ &\quad \leqslant e^{-\lambda_n^0 t} \delta ||u_\varepsilon||_{L^2(\Omega_\varepsilon)} \leqslant C\delta t^{-\gamma} e^{-at} ||u_\varepsilon||_{L^2(\Omega_\varepsilon)} \quad \varepsilon \in (0, \varepsilon_1(\delta)).\end{aligned}$$

From the estimates for I_1, I_2 and I_3 we obtain

$$\begin{aligned}&||e^{A_\varepsilon t}(I - P_a^\varepsilon)u_\varepsilon - e^{A_0 t}(I - P_a^0)u_\varepsilon||_{H_\varepsilon^1} \\ &\quad \leqslant C\delta t^{-\gamma} e^{-at} ||u_\varepsilon||_{L^2(\Omega_\varepsilon)}, \quad t > \delta, \ \varepsilon \in (0, \varepsilon_1(\delta)).\end{aligned} \tag{2.10}$$

Finally, since δ is an arbitrary small number, inequalities (2.8) and (2.10) prove the result.

The proof of the first inequality of the proposition is very similar to the one provided for the second inequality. The role of a is played now by b and $P_a^\varepsilon = 0$, $P_a^0 = 0$.

This concludes the proof of the proposition. □

3. Upper semicontinuity of attractors and of the set of equilibria

In the previous section we have studied in detail the behavior of the linear parts of the operators under the perturbation we are considering and have proved a result on the continuity of the linear semigroups, Proposition 2.7. We will see in this section that the attractors and the stationary states, solutions of the nonlinear elliptic problem, are upper semicontinuous with respect to this perturbations.

To this end we will relate the continuity of the linear semigroups with the continuity of the nonlinear semigroups for dissipative parabolic equations by using the variation of constants formula. This in turn will imply the upper semicontinuity of the attractors and the stationary states. See also [5,7,27] for other examples that use a similar technique.

We will show the following result

Proposition 3.1. *Assume that the family of domains* $\{\Omega_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ *is admissible. Then, there exist* $0 \leqslant \gamma < 1$, *a function* $c(\varepsilon)$ *with* $c(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and a constant* M

such that

$$\|T_\varepsilon(t,u_\varepsilon)-T_0(t,u_\varepsilon)\|_{H^1_\varepsilon}\leqslant Mc(\varepsilon)t^{-\gamma},\quad t\in(0,\tau],\quad \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\leqslant R,\quad \varepsilon\in(0,\varepsilon_0),\tag{3.1}$$

where $M=M(\tau,R)$.

Moreover the attractors are upper semicontinuous at $\varepsilon=0$ *in* H^1_ε, *in the sense that*

$$\sup_{u_\varepsilon\in\mathscr{A}_\varepsilon}\left[\inf_{u_0\in\mathscr{A}_0}\{\|u_\varepsilon-u_0\|_{H^1_\varepsilon}\}\right]\to 0\quad \textit{as}\ \varepsilon\to 0.\tag{3.2}$$

Also, if we denote by $\mathscr{E}_\varepsilon,\ \varepsilon\in[0,\varepsilon_0]$ *the set of stationary states of* (1.1), *then*

$$\sup_{u_\varepsilon\in\mathscr{E}_\varepsilon}\left[\inf_{u_0\in\mathscr{E}_0}\{\|u_\varepsilon-u_0\|_{H^1_\varepsilon}\}\right]\to 0\quad \textit{as}\ \varepsilon\to 0.\tag{3.3}$$

Remark 3.2. For this section and for the rest of the paper we will denote by F and F' the Nemitsky operators of f, $\frac{\partial f}{\partial u}$, respectively. That is,

$$F(u)(x)=f(x,u(x)),\quad F'(u)(x)=\frac{\partial f}{\partial u}(x,u(x)).$$

Proof. Notice that the nonlinear semigroups $T_\varepsilon(t)$ are given by the variation of constants formula:

$$T_\varepsilon(t,u_\varepsilon)=e^{A_\varepsilon t}u_\varepsilon+\int_0^t e^{A_\varepsilon(t-s)}F(T_\varepsilon(s,u_\varepsilon))\,ds,\quad \varepsilon\in[0,\varepsilon_0).\tag{3.4}$$

Hence, calculating $T_\varepsilon(t,u_\varepsilon)-T_0(t,u_\varepsilon)$ and with some elementary computations we obtain

$$\begin{aligned}\|T_\varepsilon(t,u_\varepsilon)-T_0(t,u_\varepsilon)\|_{H^1_\varepsilon}\leqslant{}& \|e^{A_\varepsilon t}u_\varepsilon-e^{A_0 t}u_\varepsilon\|_{H^1_\varepsilon}\\ &+\int_0^t\|e^{A_\varepsilon t}F(T_\varepsilon(s,u_\varepsilon))-e^{A_0 t}F(T_\varepsilon(s,u_\varepsilon))\|_{H^1_\varepsilon}\,ds\\ &+\int_0^t\|e^{A_0 t}(F(T_\varepsilon(s,u_\varepsilon))-F(T_0(s,u_\varepsilon)))\|_{H^1_\varepsilon}\,ds,\quad \varepsilon\in[0,\varepsilon_0).\end{aligned}$$

Applying now Proposition 2.7 we get

$$\begin{aligned}\|T_\varepsilon(t,u_\varepsilon)-T_0(t,u_\varepsilon)\|_{H^1_\varepsilon}\leqslant{}& M\theta(\varepsilon)t^{-\gamma}e^{-bt}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\\ &+M\theta(\varepsilon)\int_0^t(t-s)^{-\gamma}e^{-b(t-s)}\|F(T_\varepsilon(s,u_\varepsilon))\|_{L^2(\Omega_\varepsilon)}\\ &+M\int_0^t(t-s)^{-1/2}e^{-b(t-s)}C\|T_\varepsilon(t,u_\varepsilon)-T_0(t,u_\varepsilon)\|_{H^1_\varepsilon}.\end{aligned}$$

But since $||u_\varepsilon||_{L^2(\Omega_\varepsilon)} \leqslant R$ and f is a bounded function, the first two terms in the last inequality can be bounded by $M\theta(\varepsilon)t^{-\gamma}$, with $M = M(\tau, R)$. Applying now Gronwall's lemma, see [20], we obtain statement (3.1).

Now, the upper semicontinuity of the attractors in H^1_ε, statement (3.2) follows directly from (3.1) and the fact that $\mathscr{A}_0$ attracts $\bigcup_{0<\varepsilon\leqslant\varepsilon_0} \mathscr{A}_\varepsilon$ in the topology of $H^1(\Omega_0)$, see for instance [16].

To show the upper semicontinuity in H^1_ε of the stationary states we will prove that for any sequence of $\varepsilon \to 0$ and for any $u_\varepsilon \in \mathscr{E}_\varepsilon$ we can extract a subsequence, that we still denote by ε, and obtain a $u_0 \in \mathscr{E}_0$ such that $||u_\varepsilon - u_0||_{H^1_\varepsilon} \to 0$ as $\varepsilon \to 0$. From the upper semicontinuity of the attractors given by (3.2), we obtain the existence of a $u_0 \in \mathscr{A}_0$ such that $||u_\varepsilon - u_0||_{H^1_\varepsilon} \to 0$ as $\varepsilon \to 0$. To show that $u_0 \in \mathscr{E}_0$ we first observe that for any $t>0$, $||u_\varepsilon - T_0(t,u_0)||_{H^1_\varepsilon} \to ||u_0 - T_0(t,u_0)||_{H^1(\Omega_0)}$. Moreover, for a fixed $\tau > 0$ and for any $t \in (0,\tau)$ we have that,

$$||u_\varepsilon - T_0(t,u_0)||_{H^1_\varepsilon} = ||T_\varepsilon(t,u_\varepsilon) - T_0(t,u_0)||_{H^1_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0,$$

where we have used that u_ε is a stationary state and (3.1). In particular, we have that for each $t>0$, $u_0 = T_0(t,u_0)$, which implies that u_0 is a stationary state. This concludes the proof of the Proposition.

4. Continuity of equilibria, unstable manifolds and attractors

In order to obtain lower semicontinuity of attractors in H^1_ε we must ensure that the set of equilibria $\mathscr{E}_\varepsilon$ behaves lower-semicontinuously. In this section we prove that, for the sort of domain perturbations considered here and assuming that the equilibria of the limiting problem are all hyperbolic, $\mathscr{E}_\varepsilon$ is a finite set with constant cardinality; that is, $\mathscr{E}_\varepsilon = \{u_1^\varepsilon, \dots, u_n^\varepsilon\}$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$. This set behaves continuously with respect to ε in H^1_ε, that is,

$$\max_{1\leqslant k\leqslant n} \{||u_k^\varepsilon - u_k^0||_{H^1_\varepsilon}\} \xrightarrow{\varepsilon\to 0} 0.$$

We also prove, in this section, that the local unstable manifolds of equilibrium solutions are continuous as $\varepsilon \to 0$. For that we use the convergence of equilibria to obtain the continuity of the spectrum of the linearization around such equilibria and consequently the continuity of the local unstable manifolds.

4.1. Continuity of the set of equilibria

Consider the following family of elliptic problems:

$$(\mathrm{P})_\varepsilon \quad \begin{cases} \Delta u + f(x,u) = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\varepsilon \end{cases}$$

for each $0 \leqslant \varepsilon \leqslant \varepsilon_0$ $(\varepsilon_0 > 0)$. We can show the following

Proposition 4.1. *Assume that the family of domains* $\{\Omega_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ *is admissible. Assume also that problem* $(\mathrm{P})_0$ *has a solution* u^0 *and that zero is not in the spectrum of the operator* $\Delta + \frac{\partial f}{\partial u}(\cdot, u^0(\cdot))I : H^2_n(\Omega_0) \subset L^2(\Omega_0) \to L^2(\Omega_0)$. *Consider the extension operator* $E : H^1(\Omega_0) \to H^1(\mathbb{R}^N)$ *and let* $u^{0,\varepsilon} = E(u^0)_{|_{\Omega_\varepsilon}} \in H^1(\Omega_\varepsilon)$. *Then, there exists* $\varepsilon_0 > 0$ *and* $\delta > 0$ *so that problem* $(\mathrm{P})_\varepsilon$ *has exactly one solution,* u^ε, *in* $\{w_\varepsilon, ||w_\varepsilon - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} \leqslant \delta\}$ *for* $0 < \varepsilon \leqslant \varepsilon_0$. *Furthermore,*

$$||u^\varepsilon - u^0||_{H^1_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Define the operators

$$\begin{aligned} &\Theta_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon), \\ &\Theta_\varepsilon(z_\varepsilon) = (-\Delta + F'(u^{0,\varepsilon})I)^{-1}(F(z_\varepsilon) + F'(u^{0,\varepsilon})z_\varepsilon) \end{aligned} \tag{4.1}$$

(see Remark 3.2 for the meaning of F and F'). The operators Θ_ε are well defined by applying Proposition 2.6, since $F'(u^{0,\varepsilon}) \to F'(u^0)$ in $L^2(\mathbb{R}^N)$ and $0 \notin \sigma(\Delta + F'(u^0)I)$. Notice also that v_ε is a fixed point of Θ_ε if and only if v_ε is a solution of $(\mathrm{P})_\varepsilon$.

We will show that there exists $\delta > 0$ and $\varepsilon_0 > 0$, such that the operator Θ_ε, for $0 < \varepsilon < \varepsilon_0$, is a strict contraction from $B_\delta(u^{0,\varepsilon}) = \{v_\varepsilon \in H^1(\Omega_\varepsilon) : ||v_\varepsilon - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} \leqslant \delta\}$ into itself.

To prove this, let us start by showing that $\Theta_\varepsilon : B_\delta(u^{0,\varepsilon}) \to H^1(\Omega_\varepsilon)$ is a strict contraction, that is, there exists a $\rho < 1$ such that $||\Theta_\varepsilon v_\varepsilon - \Theta_\varepsilon w_\varepsilon||_{H^1(\Omega_\varepsilon)} \leqslant \rho ||v_\varepsilon - w_\varepsilon||_{H^1(\Omega_\varepsilon)}$ for any $v_\varepsilon,\ w_\varepsilon \in B_\delta(u^{0,\varepsilon})$. We have,

$$\begin{aligned} &||\Theta_\varepsilon(v_\varepsilon) - \Theta_\varepsilon(w_\varepsilon)||_{H^1(\Omega_\varepsilon)} \\ &\quad \leqslant ||(-\Delta + F'(u^{0,\varepsilon})I)^{-1}||_{\mathscr{L}(L^2(\Omega_\varepsilon), H^1(\Omega_\varepsilon))} ||F(v_\varepsilon) - F(w_\varepsilon) - F'(u^{0,\varepsilon})(v_\varepsilon - w_\varepsilon)||_{L^2(\Omega_\varepsilon)} \\ &\quad \leqslant C||F(v_\varepsilon) - F(w_\varepsilon) - F'(u^{0,\varepsilon})(v_\varepsilon - w_\varepsilon)||_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{4.2}$$

Where we have used Lemma 2.6 to obtain that $||(-\Delta + F'(u^{0,\varepsilon})I)^{-1}||_{\mathscr{L}(L^2(\Omega_\varepsilon), H^1(\Omega_\varepsilon)} \leqslant C$ for some constant C independent of ε. Next we study $||F(v_\varepsilon) - F(w_\varepsilon) - F'(u^{0,\varepsilon}) (v_\varepsilon - w_\varepsilon)||_{L^2(\Omega_\varepsilon)}$. We prove

Lemma 4.2. *There exists a constant* C *such that for all* $z_\varepsilon \in H^1(\Omega_\varepsilon)$, *all* $\delta > 0$ *and all* $v_\varepsilon, w_\varepsilon$ *with* $||v_\varepsilon - z_\varepsilon||_{H^1(\Omega_\varepsilon)} < \delta,\ ||v_\varepsilon - z_\varepsilon||_{H^1(\Omega_\varepsilon)} < \delta$, *we have*

$$||F(v_\varepsilon) - F(w_\varepsilon) - F'(z_\varepsilon)(v_\varepsilon - w_\varepsilon)||_{L^2(\Omega_\varepsilon)} \leqslant C\left(\frac{1}{\tau_\varepsilon} + \delta^{2/N}\right)||v_\varepsilon - w_\varepsilon||_{H^1(\Omega_\varepsilon)},$$

where τ_ε *is given by* (2.2).

If we assume the lemma proved, then we have

$$||\Theta_\varepsilon(v_\varepsilon) - \Theta_\varepsilon(w_\varepsilon)||_{H^1(\Omega_\varepsilon)} \leqslant C\left(\frac{1}{\tau_\varepsilon} + \delta^{2/N}\right)||v_\varepsilon - w_\varepsilon||_{H^1(\Omega_\varepsilon)}.$$

Now, given $\rho<1$ choose ε small enough such that $C\frac{1}{\tau_\varepsilon}\leqslant\frac{\rho}{2}$ and δ small enough so that $C\delta^{2/N}<\frac{\rho}{2}$. This shows Θ_ε is a strict contraction from $B_\delta(u^{0,\varepsilon})$ into $H^1(\Omega_\varepsilon)$.

In order to prove that Θ_ε maps $B_\delta(u^{0,\varepsilon})$ into itself we show first that $||\Theta_\varepsilon u^{0,\varepsilon} - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} \to 0$ as $\varepsilon\to 0$, for all $k = 1, \dots, m$. Notice that

$$\begin{aligned} ||\Theta_\varepsilon u^{0,\varepsilon} - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} &\leqslant ||\Theta_\varepsilon u^{0,\varepsilon} - u^0||_{H^1_\varepsilon} + ||u^{0,\varepsilon} - u^0||_{H^1_\varepsilon} \\ &= ||\Theta_\varepsilon u^{0,\varepsilon} - u^0||_{H^1_\varepsilon} + ||u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon\setminus\Omega_0)} \end{aligned}$$

But $||u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon\setminus\Omega_0)} \to 0$ as $\varepsilon\to 0$. Hence we just need to show that $||\Theta_\varepsilon u^{0,\varepsilon} - u^0||_{H^1_\varepsilon} \to 0$ as $\varepsilon\to 0$. If we denote by $v_\varepsilon = \Theta_\varepsilon u^{0,\varepsilon}$, then $v_\varepsilon \in H^1(\Omega_\varepsilon)$ is the solution of

$$\begin{cases} -\Delta v_\varepsilon + F'(u^{0,\varepsilon})v_\varepsilon = F(u^{0,\varepsilon}) + F'(u^{0,\varepsilon})u^{0,\varepsilon}, & \Omega_\varepsilon, \\ \dfrac{\partial v_\varepsilon}{\partial n} = 0, & \partial\Omega_\varepsilon \end{cases}$$

and u^0 is the solution of

$$\begin{cases} -\Delta u^0 + F'(u^0)u^0 = F(u^0) + F'(u^0)u^0, & \Omega_0, \\ \dfrac{\partial u^0}{\partial n} = 0, & \partial\Omega_0. \end{cases}$$

But by the resolvent convergence estimates (2.4) we get that $||v_\varepsilon - u^0||_{H^1_\varepsilon} \to 0$ as $\varepsilon\to 0$.

To show that Θ_ε maps $B_\delta(u^{0,\varepsilon})$ into itself we just observe that if $v_\varepsilon \in B_\delta(u^{0,\varepsilon})$

$$\begin{aligned} ||\Theta_\varepsilon v_\varepsilon - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} &\leqslant ||\Theta_\varepsilon v_\varepsilon - \Theta_\varepsilon u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} + ||\Theta_\varepsilon u^{0,\varepsilon} - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} \\ &\leqslant \rho\delta + ||\Theta_\varepsilon u^{0,\varepsilon} - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)}. \end{aligned}$$

Choosing ε small enough again we can guarantee that $||\Theta_\varepsilon u^{0,\varepsilon} - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} < (1-\rho)\delta$ and therefore $||\Theta_\varepsilon v_\varepsilon - u^{0,\varepsilon}||_{H^1(\Omega_\varepsilon)} < \delta$. This concludes the proof of the Proposition.

Proof of Lemma 4.2. Note that

$$|F(v_\varepsilon(x)) - F(w_\varepsilon(x)) - F'(z_\varepsilon(x))(v_\varepsilon(x) - w_\varepsilon(x))| \leqslant \bar{C}\gamma_{\varepsilon,\delta}(x)|v_\varepsilon(x) - w_\varepsilon|,$$

where

$$\gamma_{\varepsilon,\delta}(x) = \min\{1, |v_\varepsilon(x) - z_\varepsilon(x)| + |w_\varepsilon(x) - z_\varepsilon(x)|\}.$$

It follows, from the definition of $\gamma_{\varepsilon,\delta}$, that $\|\gamma_{\varepsilon,\delta}\|_{L^\infty(\Omega_\varepsilon)} \leqslant 1$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Moreover $\|\gamma_{\varepsilon,\delta}\|_{L^2(\Omega_\varepsilon)} \leqslant \|v_\varepsilon - z_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|w_\varepsilon - z_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant 2\delta$, for all v_ε, $w_\varepsilon \in B_\delta(u^{0,\varepsilon})$. Using Hölder's inequality, we get

$$\|\gamma_{\varepsilon,\delta}\|_{L^p(\Omega_\varepsilon)} \leqslant (2\delta)^{2/p} \leqslant 2(\delta)^{2/p}, \quad 2 \leqslant p < \infty, \text{ for all } v_\varepsilon, w_\varepsilon \in B_\delta(u^{0,\varepsilon}).$$

Now if $\varphi_\varepsilon = v_\varepsilon - w_\varepsilon$ we denote by $\tilde{\varphi}_\varepsilon = E_\varepsilon(\varphi_{\varepsilon_{|K_\varepsilon}})_{|\Omega_\varepsilon}$. Then

$$\begin{aligned}\|\tilde{\varphi}_\varepsilon - \varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} &= \|\tilde{\varphi}_\varepsilon - \varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} \leqslant \frac{1}{\tau_\varepsilon}\|\nabla\tilde{\varphi}_\varepsilon - \nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus K_\varepsilon)} \\ &\leqslant C\frac{1}{\tau_\varepsilon}(\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \|\tilde{\varphi}_\varepsilon\|_{H^1(\mathbb{R}^N)}) \leqslant C\frac{1}{\tau_\varepsilon}(\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \|\varphi_\varepsilon\|_{H^1(K_\varepsilon)}) \\ &\leqslant C\frac{2}{\tau_\varepsilon}\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)},\end{aligned}$$

where we have used that $E_\varepsilon : H^1(K_\varepsilon) \to H^1(\mathbb{R}^N)$ is bounded and τ_ε is the first eigenvalue of $-\Delta$ in $\Omega_\varepsilon \setminus K_\varepsilon$ with Dirichlet boundary condition in ∂K_ε and Neumann boundary condition in $\partial\Omega_\varepsilon$. Now

$$\begin{aligned}\|\gamma_{\varepsilon,\delta}\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} &\leqslant \|\gamma_{\varepsilon,\delta}(\varphi_\varepsilon - \tilde{\varphi}_\varepsilon)\|_{L^2(\Omega_\varepsilon)} + \|\gamma_{\varepsilon,\delta}\tilde{\varphi}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ &\leqslant \|\gamma_{\varepsilon,\delta}\|_{L^\infty(\Omega_\varepsilon)}\|\varphi_\varepsilon - \tilde{\varphi}_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\gamma_{\varepsilon,\delta}\|_{L^N(\Omega_\varepsilon)}\|\tilde{\varphi}_\varepsilon\|_{L^{2N/(N-2)}(\mathbb{R}^N)} \\ &\leqslant \left(C\frac{2}{\tau_\varepsilon} + C\delta^{2/N}\right)\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)}.\end{aligned}$$

This proves the lemma. □

As an immediate consequence of this proposition, we have

Corollary 4.3. *Assume the conditions of Proposition* 4.1 *hold. Assume moreover that problem* $(\mathrm{P})_0$ *has exactly m solutions* $u_1^0, \dots, u_m^0$ *and that all of them are hyperbolic in the sense that* 0 *is not in the spectrum of* $\Delta + F'(u_k^0)I : H_n^2(\Omega_0) \subset L^2(\Omega_0) \to L^2(\Omega_0)$ *for* $k = 1, \dots, m$. *Then there exists a small* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *problem* $(\mathrm{P})_\varepsilon$ *has exactly m solutions* $u_1^\varepsilon, \dots, u_m^\varepsilon$. *Moreover, we have*

$$\|u_k^\varepsilon - u_k^0\|_{H_\varepsilon^1} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. By Proposition 3.1 we have that for any solution u^ε of $(\mathrm{P})_\varepsilon$ for ε small enough lies in a neighborhood of the set of equilibria $(\mathrm{P})_0$. But by Proposition 4.1, in a neighborhood of u_k^0 there is only one solution of $(\mathrm{P})_\varepsilon$ which converges to u_k^0 in H_ε^1. This proves the result.

4.2. Continuity of the unstable manifolds

In this section we show that the local unstable manifolds of u^ε, for $k = 1, \dots, m$ fixed, are continuous in H^1_ε as $\varepsilon \to 0$. The existence of this manifold follows from standard invariant manifold theory, see [20], although its proof is adapted to encompass the possibility that the space changes according to a parameter and to keep track of the dependence of the invariant manifold upon the parameter. After this, we show that the unstable manifolds are close for small ε. For this we will use the convergence results on the linear part obtained in Section 2.

We have the following

Proposition 4.4. *Assume that the family of domains $\{\Omega_\varepsilon : 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ is admissible. Assume also that u^0 is a solution of problem $(\mathrm{P})_0$ and that zero is not in the spectrum of the operator $\Delta + \frac{\partial f}{\partial u}(\cdot, u^0(\cdot))I : H^2_n(\Omega_0) \subset L^2(\Omega_0) \to L^2(\Omega_0)$. By Proposition 4.1, $(\mathrm{P})_\varepsilon$ has a unique solution, u^ε, near u_0. Then, there exist δ, $\varepsilon_0 > 0$ such that u^ε has a local unstable manifold $W^u_{\mathrm{loc}}(u^\varepsilon) \subset H^1(\Omega_\varepsilon)$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and if we denote by*

$$W^u_\delta(u^\varepsilon) = \{w \in W^u_{loc}(u^\varepsilon), ||w - u^\varepsilon||_{H^1(\Omega_\varepsilon)} < \delta\}, \quad 0 \leqslant \varepsilon \leqslant \varepsilon_0,$$

then $W^u_\delta(u^\varepsilon)$ converges in H^1_ε to $W^u_\delta(u^0)$ as $\varepsilon \to 0$, that is

$$\sup_{w_\varepsilon \in W^u_\delta(u^\varepsilon)} \inf_{w_0 \in W^u_\delta(u^0)} ||w_\varepsilon - w_0||_{H^1_\varepsilon} + \sup_{w_0 \in W^u_\delta(u^0)} \inf_{w_\varepsilon \in W^u_\delta(u^\varepsilon)} ||w_\varepsilon - w_0||_{H^1_\varepsilon} \to 0 \quad as\ \varepsilon \to 0.$$

Proof. Notice that by Proposition 4.1, we have that $||u^\varepsilon - u^0||_{H^1_\varepsilon} \to 0$ as $\varepsilon \to 0$. This implies by Proposition 2.3 that the spectra of $-\Delta + F'(u_\varepsilon)$ behave continuously as $\varepsilon \to 0$, see Remark 3.2 for the meaning of F and F'.

Rewriting (1.1) for $w = u - u^\varepsilon$ to deal with the neighborhood of u^ε we arrive at

$$\begin{cases} w_t = \Delta w + F'(u^\varepsilon)w + F(w + u^\varepsilon) - F(u^\varepsilon) - F'(u^\varepsilon)w, & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial w}{\partial n} = 0, & \text{in } \partial\Omega_\varepsilon. \end{cases} \tag{4.3}$$

Denote as usual by $\{\lambda^\varepsilon_i\}_{i=1}^\infty$ the eigenvalues of $(\Delta + F'(u^\varepsilon)I)$ and by $\{\phi^\varepsilon_i\}_{i=1}^\infty$ a corresponding orthonormal system of eigenfunctions. If $\lambda^0_1, \dots, \lambda^0_n$ are positive and $\lambda^0_{n+1}, \lambda^0_{n+2}, \dots$ are negative, let $\beta > 0$ and $\varepsilon_0 > 0$ such that $\lambda^\varepsilon_1 \geqslant \dots \geqslant \lambda^\varepsilon_n \geqslant \beta > 0 > -\beta \geqslant \lambda^\varepsilon_{n+1} \geqslant \lambda^\varepsilon_{n+2} \dots$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Denote by $W_\varepsilon = [\phi^\varepsilon_1, \dots, \phi^\varepsilon_n]$ and $W^\perp_\varepsilon = \{\psi \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \psi\phi = 0,\ \forall \phi \in W\}$. As we have done previously, denote by $P^\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon)$ the orthogonal projections on W_ε

$$P^\varepsilon\psi = \sum_{i=1}^n \left(\int_{\Omega_\varepsilon} \psi\phi^\varepsilon_i\right)\phi^\varepsilon_i$$

and $Q^\varepsilon = I - P^\varepsilon$.

If $\psi \in W_\varepsilon$ then $\psi = \sum_{i=1}^n (\int_{\Omega_\varepsilon} \psi \phi_i^\varepsilon)\phi_i^\varepsilon$ and

$$\|\psi\|_{H^1(\Omega_\varepsilon)} = \left(\sum_{i=1}^n (1+\lambda_i^\varepsilon)\left(\int_{\Omega_\varepsilon} \psi\phi_i^\varepsilon\right)^2\right)^{\frac{1}{2}}$$

and since $\lambda_i^\varepsilon \to \lambda_i^0$, $1 \leqslant i < \infty$, we have that W_ε is isomorphic to $\mathbb{R}^n$ through the isomorphism

$$W_\varepsilon \ni \psi \xrightarrow{T_\varepsilon} \left(\int_{\Omega_\varepsilon} \psi\phi_1^\varepsilon, \dots, \int_{\Omega_\varepsilon} \psi\phi_n^\varepsilon\right) \in \mathbb{R}^n.$$

T_ε is bounded with bounded inverse T_ε^{-1} and the norms of T_ε and T_ε^{-1} are uniformly bounded $0 \leqslant \varepsilon \leqslant \varepsilon_0$.

Now we decompose Eq. (4.3) in the following way. If w is a solution to (4.3) we write

$$w = \sum_{i=1}^n v_i\phi_i^\varepsilon + z,$$

where $v_i = \int_{\Omega_\varepsilon} w\phi_i^\varepsilon$. Hence

$$\dot{v}_i = \lambda_i^\varepsilon v_i + \int_{\Omega_\varepsilon} [F(w+u^\varepsilon) - F(u^\varepsilon) - F'(u^\varepsilon)w]\phi_i^\varepsilon$$

and

$$\begin{cases} z_t = \Delta z + F'(u^\varepsilon)z + F(w+u^\varepsilon) - F(u^\varepsilon) - F'(u^\varepsilon)w \\ \qquad - \sum\limits_{i=1}^n (\int_{\Omega_\varepsilon} [F(w+u^\varepsilon) - F(u^\varepsilon) - F'(u^\varepsilon)w]\phi_i^\varepsilon)\phi_i^\varepsilon, \\ \dfrac{\partial z}{\partial n} = 0. \end{cases}$$

We write $v = (v_1, \dots, v_n)^\top$ and $H_\varepsilon(v,z) = (H_1(v,z), \dots, H_n(v,z))^\top$ where

$$H_j^\varepsilon(v,z) = \int_{\Omega_\varepsilon} \left[F\left(\sum_{i=1}^n v_i\phi_i^\varepsilon + z + u^\varepsilon\right) - F(u^\varepsilon) - F'(u^\varepsilon)\left(\sum_{i=1}^n v_i\phi_i^\varepsilon + z\right)\right]\phi_j^\varepsilon,$$

and

$$G_\varepsilon(v,z) = F\left(\sum_{i=1}^n v_i\phi_i^\varepsilon + z + u^\varepsilon\right) - F(u^\varepsilon) - F'(u^\varepsilon)\left(\sum_{i=1}^n v_i\phi_i^\varepsilon + z\right) - \sum_{i=1}^n H_i(v,z)\phi_i^\varepsilon.$$

Hence, we have that, $H_\varepsilon(0,0) = 0, G_\varepsilon(0,0) = 0$. From Lemma 4.2 we obtain that given $\rho > 0$ there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that if $\|v\|_{\mathbb{R}^n} + \|z\|_{H^1(\Omega_\varepsilon)} < \delta$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0$

we have

$$\|H_\varepsilon(v,z)\|_{\mathbb{R}^n} < \rho,$$

$$\|G_\varepsilon(v,z)\|_{L^2(\Omega_\varepsilon)} < \rho,$$

$$\|H_\varepsilon(v,z) - H_\varepsilon(\tilde{v},\tilde{z})\|_{\mathbb{R}^n} < \rho(\|v-\tilde{v}\|_{\mathbb{R}^n} + \|z-\tilde{z}\|_{H^1(\Omega_\varepsilon)}),$$

$$\|G_\varepsilon(v,z) - G_\varepsilon(\tilde{v},\tilde{z})\|_{L^2(\Omega_\varepsilon)} < \rho(\|v-\tilde{v}\|_{\mathbb{R}^n} + \|z-\tilde{z}\|_{H^1(\Omega_\varepsilon)}). \tag{4.4}$$

The fact that we can choose ρ and δ uniformly for $0 \leqslant \varepsilon \leqslant \varepsilon_0$ satisfying the inequalities above is the key point to obtain that the local unstable manifolds are defined in a small neighborhood of the equilibrium point u_ε uniformly for $0 \leqslant \varepsilon \leqslant \varepsilon_0$.

We can extend H_ε, G_ε outside $B_\delta(u^\varepsilon)$ in such a way that bounds (4.4) hold for all $v \in \mathbb{R}^N$, $z \in H^1(\Omega_\varepsilon)$.

Denote by $A_\varepsilon = (\Delta + F'(u^\varepsilon)I)_{|_{W_\varepsilon^\perp}}$, $B_\varepsilon = \mathrm{diag}(\lambda_1^\varepsilon, \dots, \lambda_n^\varepsilon)$. Then, Eq. (4.3) can be rewritten in the following form:

$$\dot{v} = B_\varepsilon v + H_\varepsilon(v,z),$$

$$\dot{z} = A_\varepsilon z + G_\varepsilon(v,z), \tag{4.5}$$

$v \in \mathbb{R}^n$, $z \in W_\varepsilon^\perp$, where H_ε, G_ε satisfy (4.4) for all $v \in \mathbb{R}^n$, $z \in W_\varepsilon^\perp$.

Also, for some positive M, β, independent of ε, $0 \leqslant \varepsilon \leqslant \varepsilon_0$

$$\|e^{A_\varepsilon t} z\|_{H^1(\Omega_\varepsilon)} \leqslant M e^{-\beta t} \|z\|_{H^1(\Omega_\varepsilon)}, \quad t \geqslant 0,$$

$$\|e^{A_\varepsilon t} z\|_{H^1(\Omega_\varepsilon)} \leqslant M t^{-\frac{1}{2}} e^{-\beta t} \|z\|_{L^2(\Omega_\varepsilon)}, \quad t \geqslant 0,$$

$$\|e^{B_\varepsilon t} v\|_{\mathbb{R}^n} \leqslant M e^{\beta t} \|v\|_{\mathbb{R}^n}, \quad t \leqslant 0.$$

Now we will show that for a suitably small $\rho > 0$, there is an unstable manifold for u^ε

$$S^\varepsilon = \{(v,z) : z = \sigma_\varepsilon^*(v), \ v \in \mathbb{R}^n\},$$

where $\sigma_\varepsilon^* : \mathbb{R}^n \to W_\varepsilon^\perp$ is bounded and Lipschitz continuous. Furthermore,

$$\sup_{v \in \mathbb{R}^n} \|\sigma_\varepsilon^*(v) - \sigma_0^*(v)\|_{H_\varepsilon^1} \xrightarrow{\varepsilon \to 0} 0.$$

In order to show this, we will first prove the existence of the invariant manifold. For $D > 0$, $\Delta > 0$, $0 < \theta < 1$, given, if $\rho > 0$ is such that

$$\rho M \beta^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}) \leqslant D,$$

$$\rho M^2 (1 + \Delta) \beta^{-\frac{1}{2}} \leqslant \Delta,$$

$$\beta - \rho M(1+\Delta) \geqslant \frac{\beta}{2},$$
$$\rho M \Gamma\left(\frac{1}{2}\right)\left[\frac{1}{\beta^{\frac{1}{2}}} + \frac{1+\Delta}{\beta - \rho M(1+\Delta)}\right] \leqslant \theta < 1.$$

Let $\sigma_\varepsilon : \mathbb{R}^n \to W_\varepsilon^\perp$ satisfying

$$|||\sigma_\varepsilon||| \coloneqq \sup_{v \in \mathbb{R}^n} ||\sigma_\varepsilon(v)||_{H^1(\Omega_\varepsilon)} \leqslant D, \quad ||\sigma_\varepsilon(v) - \sigma_\varepsilon(\tilde{v})||_{H^1(\Omega_\varepsilon)} \leqslant \Delta ||v - \tilde{v}||_{\mathbb{R}^n}. \tag{4.6}$$

Let $v_\varepsilon(t) = \psi(t, \tau, \eta, \sigma_\varepsilon)$ be the solution of

$$\frac{dv_\varepsilon}{dt} = B_\varepsilon v_\varepsilon + H_\varepsilon(v_\varepsilon, \sigma_\varepsilon(v_\varepsilon)), \quad \text{for } t < \tau, \ v_\varepsilon(\tau) = \eta, \tag{4.7}$$

and define

$$\Phi(\sigma_\varepsilon)(\eta) = \int_{-\infty}^{\tau} e^{A_\varepsilon(\tau - s)} G_\varepsilon(v_\varepsilon(s), \sigma_\varepsilon(v_\varepsilon(s)))\, ds. \tag{4.8}$$

Note that

$$||\Phi(\sigma_\varepsilon)(\cdot)||_{H^1(\Omega_\varepsilon)} \leqslant \int_{-\infty}^{\tau} \rho M (\tau - s)^{-\frac{1}{2}} e^{-\beta(\tau - s)}\, ds = \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right). \tag{4.9}$$

From the choice of ρ we have that, $||\Phi(\sigma_\varepsilon)(\cdot)||_{H^1(\Omega_\varepsilon)} \leqslant D$. Next, suppose that σ_ε and $\tilde{\sigma}_\varepsilon$ are functions satisfying (4.6), $\eta, \tilde{\eta} \in \mathbb{R}^n$ and denote $v_\varepsilon(t) = \psi(t, \tau, \eta, \sigma_\varepsilon)$, $\tilde{v}_\varepsilon(t) = \psi(t, \tau, \tilde{\eta}, \tilde{\sigma}_\varepsilon)$. Then,

$$v_\varepsilon(t) - \tilde{v}_\varepsilon(t) = e^{B_\varepsilon(t - \tau)}(\eta - \tilde{\eta}) + \int_\tau^t e^{B_\varepsilon(t-s)}[H_\varepsilon(v_\varepsilon, \sigma_\varepsilon(v_\varepsilon)) - H_\varepsilon(\tilde{v}_\varepsilon, \tilde{\sigma}_\varepsilon(\widetilde{v_\varepsilon}))]\, ds.$$

With some simple and standard computations we obtain

$$\begin{aligned} ||v_\varepsilon(t) - \widetilde{v_\varepsilon}(t)||_{\mathbb{R}^n} \leqslant{} & M e^{\beta(t-\tau)} ||\eta - \tilde{\eta}||_{\mathbb{R}^n} + \rho M(1+\Delta) \int_t^\tau e^{\beta(t-s)} ||v_\varepsilon - \tilde{v}_\varepsilon||_{\mathbb{R}^n}\, ds \\ & + \rho M |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon|||_{H^1(\Omega_\varepsilon)} \int_t^\tau e^{\beta(t-s)}\, ds. \end{aligned}$$

Let $\phi(t) = e^{-\beta(t-\tau)} ||v_\varepsilon(t) - \widetilde{v_\varepsilon}(t)||_{\mathbb{R}^n}$. Then,

$$\phi(t) \leqslant M||\eta - \tilde{\eta}||_{\mathbb{R}^n} + \rho M \int_t^\tau e^{\beta(\tau - s)}\, ds |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon|||_{H^1(\Omega_\varepsilon)} + M\rho(1+\Delta) \int_t^\tau \phi(s)\, ds.$$

By Gronwall's inequality

$$
\begin{aligned}
||v_\varepsilon(t) - \widetilde{v_\varepsilon}(t)||_{\mathbb{R}^n} \leqslant & \Bigg[M||\eta - \tilde{\eta}||_{\mathbb{R}^n} e^{\beta\,(t-\tau)} \\
& + \rho M \int_t^\tau e^{\beta(t-s)}\, ds |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon|||_{H^1(\Omega_\varepsilon)} \Bigg] e^{-\rho M(1+\Delta)(t-\tau)} \\
& \leqslant [M||\eta - \tilde{\eta}||_{\mathbb{R}^n} + \rho M \beta^{-1} |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon|||_{H^1(\Omega_\varepsilon)}] e^{-\rho M(1+\Delta)(t-\tau)}\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&||\Phi(\sigma_\varepsilon)(\eta) - \Phi(\tilde{\sigma}_\varepsilon)(\tilde{\eta})||_{H^1(\Omega_\varepsilon)} \\
&\leqslant M \int_{-\infty}^\tau (\tau - s)^{-\frac{1}{2}} e^{-\beta(\tau-s)} ||G_\varepsilon(v_\varepsilon, \sigma_\varepsilon(v_\varepsilon)) - G_\varepsilon(\tilde{v}_\varepsilon, \tilde{\sigma}_\varepsilon(\widetilde{v_\varepsilon}))||_{L^2(\Omega_\varepsilon)}\, ds \\
&\leqslant \rho M \int_{-\infty}^\tau (\tau - s)^{-\frac{1}{2}} e^{-\beta(\tau-s)} (||\sigma_\varepsilon(v_\varepsilon) - \tilde{\sigma}_\varepsilon(\widetilde{v_\varepsilon})||_{H^1(\Omega_\varepsilon)} + ||v_\varepsilon - \widetilde{v_\varepsilon}||_{\mathbb{R}^n})\, ds \\
&\leqslant \rho M \int_{-\infty}^\tau (\tau - s)^{-\frac{1}{2}} e^{-\beta(\tau-s)} [(1+\Delta)||v_\varepsilon - \widetilde{v_\varepsilon}||_{\mathbb{R}^n} + |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon|||]\, ds.
\end{aligned}
$$

Using the estimates for $||v_\varepsilon - \tilde{v}_\varepsilon||_{\mathbb{R}^n}$ we obtain

$$
\begin{aligned}
||\Phi(\sigma_\varepsilon)(\eta) - \Phi(\tilde{\sigma}_\varepsilon)(\tilde{\eta})|| \leqslant & \rho M \Gamma\left(\frac{1}{2}\right) \left[\beta^{-\frac{1}{2}} + \frac{1+\Delta}{\beta - \rho M(1+\Delta)}\right] |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon||| \\
& + \rho M^2 (1+\Delta) \beta^{-\frac{1}{2}} ||\eta - \tilde{\eta}||_{\mathbb{R}^n}.
\end{aligned}
$$

Let

$$
I_\sigma(\varepsilon) = \rho M \Gamma\left(\frac{1}{2}\right) \left[\beta^{-\frac{1}{2}} + \frac{1+\Delta}{\beta - \rho M(1+\Delta)}\right]
$$

and

$$
I_\eta(\varepsilon) = \rho M^2 (1+\Delta) \beta^{-\frac{1}{2}}.
$$

It is easy to see that, given $\theta < 1$, there exists a ρ_0 such that, for $\rho \leqslant \rho_0$, $I_\sigma(\varepsilon) \leqslant \theta$ and $I_\eta(\varepsilon) \leqslant \Delta$ and

$$
||\Phi(\sigma_\varepsilon)(\eta) - \Phi(\tilde{\sigma}_\varepsilon)(\tilde{\eta})||_{H^1(\Omega_\varepsilon)} \leqslant \Delta ||\eta - \eta'||_{\mathbb{R}^n} + \theta |||\sigma_\varepsilon - \tilde{\sigma}_\varepsilon|||. \tag{4.10}
$$

Inequalities (4.9) and (4.10) imply that G is a contraction map from the class of functions that satisfy (4.6) into itself. Therefore, it has a unique fixed point $\sigma_n^* = \Phi(\sigma_n^*)$ in this class.

It remains to prove that $S = \{(v, \sigma_\varepsilon^*(v)) : v \in \mathbb{R}^n\}$ is an invariant manifold for (4.5). Let $(v_0, z_0) \in S$, $z_0 = \sigma_\varepsilon^*(v_0)$. Denote by $v_\varepsilon^*(t)$ the solution of the following initial value problem

$$\frac{dv}{dt} = B_\varepsilon v + H_\varepsilon(v, \sigma_\varepsilon^*(v)), \quad v(0) = v_0.$$

This defines a curve $(v_\varepsilon^*(t), \sigma_\varepsilon^*(v_\varepsilon^*(t))) \in S,\ t \in \mathbb{R}$. But the only solution of

$$\dot{z} = A_\varepsilon z + G_\varepsilon(v_\varepsilon^*(t), \sigma_\varepsilon^*(v_\varepsilon^*(t))),$$

which remains bounded as $t \to -\infty$ is

$$z^*(t) = \int_{-\infty}^{t} e^{A_\varepsilon(t-s)} G_\varepsilon(v_\varepsilon^*(s), \sigma_\varepsilon^*(v_\varepsilon^*(s)))\, ds = \sigma_\varepsilon^*(v_\varepsilon^*(t)).$$

Therefore, $(v_\varepsilon^*(t), \sigma_\varepsilon^*(v_\varepsilon^*(t)))$ is a solution of (4.5) through (v_0, z_0) and the invariance is proved.

Next, we show that the fixed points σ_ε^* depend continuously upon ε at $\varepsilon = 0$. This is accomplished in the following manner. If $0 \leqslant \varepsilon \leqslant \varepsilon_0$ is such that the unstable manifold is given by the graph of $\sigma_\varepsilon^*,\ 0 \leqslant \varepsilon \leqslant \varepsilon_0$, we want to show that

$$\sup_{\eta \in \mathbb{R}^n} ||\sigma_\varepsilon^*(\eta) - \sigma_0^*(\eta)||_{H_\varepsilon^1} = |||\sigma_\varepsilon^* - \sigma_0^*||| \to 0 \quad \text{as } \varepsilon \to 0.$$

It follows from Proposition 2.7 that

$$\begin{aligned}
||\sigma_\varepsilon^*(\eta) - \sigma_0^*(\eta)||_{H_\varepsilon^1} &\leqslant \int_{-\infty}^{\tau} ||e^{A_\varepsilon(\tau-s)} G_\varepsilon(v_\varepsilon, \sigma_\varepsilon^*(v_\varepsilon)) - e^{A_0(\tau-s)} G_0(v_0, \sigma_0^*(v_0))||_{H_\varepsilon^1}\, ds \\
&\leqslant M\theta(\varepsilon) \int_{-\infty}^{\tau} e^{\beta(\tau-s)} (\tau-s)^{-\gamma} ||G_\varepsilon(v_\varepsilon, \sigma_\varepsilon^*(v_\varepsilon))||_{L^2(\Omega_\varepsilon)}\, ds + M \\
&\quad \times \int_{-\infty}^{\tau} e^{\beta(\tau-s)} (\tau-s)^{-\frac{1}{2}} ||G_\varepsilon(v_\varepsilon, \sigma_\varepsilon^*(v_\varepsilon)) - G_0(v_0, \sigma_0^*(v_0))||_{L^2(\mathbb{R}^N)}\, ds \\
&\leqslant o(1) + \rho M \beta^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) |||\sigma_\varepsilon^* - \sigma_0^*||| \\
&\quad + \rho M(1+\Delta) \int_{-\infty}^{\tau} e^{-\beta(\tau-s)} (\tau-s)^{-\frac{1}{2}} ||v_\varepsilon - v_0||_{\mathbb{R}^n}\, ds.
\end{aligned}$$

Thus, it is enough to estimate $||v_\varepsilon - v_0||_{\mathbb{R}^n}$. Note that

$$\begin{aligned}
||v_\varepsilon - v_0||_{\mathbb{R}^n} &\leqslant \int_t^{\tau} ||e^{B_\varepsilon(t-s)} - e^{B_0(t-s)}||\, ||H_\varepsilon(v_\varepsilon, \sigma_\varepsilon^*(v_\varepsilon))||_{\mathbb{R}^n}\, ds \\
&\quad + \int_t^{\tau} ||e^{B_0(t-s)}||\, ||H_\varepsilon(v_\varepsilon, \sigma_\varepsilon^*(v_\varepsilon)) - H_0(v_0, \sigma_0^*(v_0))||_{\mathbb{R}^n}\, ds \\
&\leqslant \rho M \beta^{-1} [o(1) + |||\sigma_\varepsilon^* - \sigma_0^*|||] + \rho M(1+\Delta) \int_t^{\tau} e^{\beta(t-s)} ||v_\varepsilon - v_0||_{\mathbb{R}^n}\, ds.
\end{aligned}$$

Therefore

$$\|v_\varepsilon - v_0\|_{\mathbb{R}^n} \leqslant \rho M \beta^{-1}[o(1) + |||\sigma^*_\varepsilon - \sigma^*_0|||]e^{-\rho M(1+\Delta)(\tau - t)}$$

which shows that

$$\sup_{\eta \in \mathbb{R}^n} \|\sigma^*_\varepsilon(\eta) - \sigma^*_0(\eta)\|_{H^1_\varepsilon} \xrightarrow{\varepsilon \to 0} 0.$$

This proves the result. □

As an immediate consequence of this proposition, we have

Corollary 4.5. *Assume the conditions of Proposition* 4.4 *hold, that problem* $(\mathrm{P})_0$ *has exactly* m *solutions* $u^0_1, \dots, u^0_m$ *and that all of them are hyperbolic. Then there exist* $\varepsilon_0, \delta > 0$ *small enough such that problem* $(\mathrm{P})_\varepsilon$ *has exactly* m *solutions and their local unstable manifolds* $W^u_\delta(u^\varepsilon_k)$, $k = 1, \dots, m$ *behave continuously in* H^1_ε *as* $\varepsilon \to 0$.

4.3. Continuity of attractors

We are now in position to prove the central result of our work.

Theorem 4.6. *Assume that the family of domains* $\{\Omega_\varepsilon, 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$ *is admissible and that every equilibrium of the unperturbed problem* $(\mathrm{P})_0$ *is hyperbolic. Then the attractors* $\mathscr{A}_\varepsilon$ *behave continuously in* H^1_ε *as* $\varepsilon \to 0$, *that is*

$$\sup_{u_\varepsilon \in \mathscr{A}_\varepsilon} \inf_{u_0 \in \mathscr{A}_0} \|u_\varepsilon - u_0\|_{H^1_\varepsilon} + \sup_{u_0 \in \mathscr{A}_0} \inf_{u_\varepsilon \in \mathscr{A}_\varepsilon} \|u_\varepsilon - u_0\|_{H^1_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Since we have already shown in Proposition 3.1 the upper semicontinuity of attractors, we just need to show the lower semicontinuity. This will follow from the continuity of the local unstable manifolds. To see this, we argue in the following way. If $u_0 \in \mathscr{A}_0$ then u_0 belongs to the unstable manifold of u^0_k for some $1 \leqslant k \leqslant m$. Let $\delta > 0$ be the one obtained in Proposition 4.4. If τ is such that $w_0 = T_0(-\tau, u_0) \in W^u_\delta(u^0_k)$, from the continuity of the unstable manifolds there is a sequence $w_\varepsilon \in W^u_\delta(u^\varepsilon_k)$ which converges to w_0 in H^1_ε as $\varepsilon \to 0$. Now, since the family of semigroups is continuous in H^1_ε we have that $\mathscr{A}_\varepsilon \ni T_\varepsilon(\tau, w_\varepsilon) \to T_0(\tau, w_0) = u_0$ in H^1_ε as $\varepsilon \to 0$. Showing the lower semicontinuity of attractors. This proves the theorem. □

Remark 4.7. Notice that, if moreover we assume the transversality of the stable and unstable manifolds in $\mathscr{A}_0$ then for ε small enough the flow in the attractor $\mathscr{A}_\varepsilon$ is C^0-conjugate with the flow in $\mathscr{A}_0$, in the sense that there exist homeomorphisms h_ε : $\mathscr{A}_\varepsilon \to \mathscr{A}_0$ such that for all $u_\varepsilon \in \mathscr{A}_\varepsilon$, we have $T_\varepsilon(t)(u_\varepsilon) = h^{-1}_\varepsilon T_0(t) h_\varepsilon(u_\varepsilon)$, for all $t \geqslant 0$, see [17].

Remark 4.8. The dynamics of (1.1) have been compared in the space H^1_ε. This means that, for instance, in the case of exterior perturbations of the domain the restriction to Ω_0 of equilibria, unstable manifolds and attractors of (1.1) in Ω_ε converges in $H^1(\Omega_0)$ to the equilibria, unstable manifolds and attractor of the same problem in Ω_0.

We may explore now the possibility of obtaining convergence in stronger norms. For this what we need is to have uniform bounds of the attractors in stronger norms. In order to accomplish this we first note that we may easily obtain uniform $L^\infty(\Omega_\varepsilon)$ bounds of u_t in the attractors, that is, there exists a constant C independent of ε such that

$$\sup\{||u_t(t,\phi_\varepsilon)||_{L^\infty(\Omega_\varepsilon)}, \phi_\varepsilon \in \mathscr{A}_\varepsilon, t\in\mathbb{R}, 0\leqslant\varepsilon\leqslant\varepsilon_0\}\leqslant C.$$

To obtain this we follow the steps given in Proposition 5.1 of [7].

Hence, we can view Eq. (1.1) for fixed t as an elliptic equation $-\Delta u_\varepsilon + u_\varepsilon = F(u_\varepsilon) + u_\varepsilon - u_{\varepsilon,t}$ and notice that the right-hand side is uniformly bounded in $L^\infty(\Omega_\varepsilon)$ when $u_\varepsilon(t)$ is an orbit in the attractor $\mathscr{A}_\varepsilon$. Therefore, the problem of obtaining uniform bounds in stronger norms is reduced to obtaining uniform bounds for the solution of the elliptic problem

$$\begin{cases} -\Delta u + u = g, & \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0, & \partial\Omega_\varepsilon \end{cases} \tag{4.11}$$

when $g\in L^\infty(\Omega_\varepsilon)$, $||g||_{L^\infty(\Omega_\varepsilon)}\leqslant C$, with C independent of ε.

Hence if, for instance, the family of domains Ω_ε is uniformly Hölder then there exists a $\alpha>0$ and a constant C such that if u is the solution of (4.11) then $||u||_{C^\alpha(\Omega_\varepsilon)}\leqslant C$ (see [10]). This allows to obtain convergence in C^β for any $0<\beta<\alpha$.

5. Examples

Let us consider in this section two examples where Proposition 2.3 applies and, therefore, all the results of this paper apply. The first example is a C^0 perturbation of the domain and the second one is a nonstandard dumbbell type domain.

5.1. A C^0 perturbation of the domain

Let $\Omega_0\subset\mathbb{R}^N$ be a $C^{0,1}$ domain and assume that for any point $\xi\in\partial\Omega_0$, up to a rigid motion we have that

$$\begin{aligned}&\Omega_0\cap\{x\in\mathbb{R}^N : |x_i-\xi_i|<\delta\}\\ &\quad=\{x=(x',x_N) : x_N=\xi_N+f_0(x'), |x_i-\xi_i|<\delta, i=1,\dots,N-1\}\end{aligned}$$

for certain Lipschitz function f_0 and where, as it is done customarily, we denote by $x' = (x_1, \dots, x_{N-1})$ so that $x = (x', x_N)$.

In order to simplify the notation assume that $\xi = 0$. Hence

$$\Omega_0 \cap \{x \in \mathbb{R}^N : |x_i| < \delta\} = \{x = (x', x_N) : x_N < f_0(x'), |x_i| < \delta, i = 1, \dots, N-1\}.$$

Assume that

$$\Omega_\varepsilon \cap \{x \in \mathbb{R}^N : |x_i| < \delta\} = \{x = (x', x_N) : x_N < f_\varepsilon(x'), |x_i| < \delta, i = 1, \dots, N-1\},$$

where $f_\varepsilon \to f_0$ uniformly in $\{x' : |x_i'| < \delta\}$.

Notice also that by definition

$$\partial K_\varepsilon \cap \{x \in \mathbb{R}^N : |x_i| < \delta\} = \{x = (x', x_N) : x_N = g_\varepsilon(x'), |x_i| < \delta, i = 1, \dots, N-1\}$$

for certain function g_ε with $g_\varepsilon < f_0$, $g_\varepsilon < f_\varepsilon$ and $g_\varepsilon \to f_0$ uniformly in $\{x' : |x_i'| < \delta\}$.

If we denote by

$$R_{\varepsilon,\delta} = (\Omega_\varepsilon \backslash K_\varepsilon) \cap \{x; |x_i| < \delta\} = \{x = (x', x_N) : |x_i| < \delta, g_\varepsilon(x') < x_N < f_\varepsilon(x')\},$$

we have

$$||\nabla u_\varepsilon||^2_{L^2(R_{\varepsilon,\delta})} = \int_{-\delta}^{\delta} \dots \int_{-\delta}^{\delta} \int_{g_\varepsilon(x')}^{f_\varepsilon(x')} \left|\frac{\partial u}{\partial x_n}\right|^2 dx_N \, dx'.$$

But for x' fixed, applying Poincaré inequality in one dimension, we have

$$\int_{g_\varepsilon(x')}^{f_\varepsilon(x')} \left|\frac{\partial(u_\varepsilon \circ \chi^{-1})}{\partial x_n}\right|^2 dx_N \geqslant \frac{\pi^2}{4|f_\varepsilon(x') - g_\varepsilon(x')|^2} \int_{g_\varepsilon(x')}^{f_\varepsilon(x')} |u_\varepsilon|^2 \, dx_N$$

which implies that

$$||\nabla u_\varepsilon||^2_{L^2(R_{\varepsilon,\delta})} \geqslant \frac{\pi^2}{4||f_\varepsilon - g_\varepsilon||^2_{L^\infty}} ||u_\varepsilon||^2_{L^2(R_{\varepsilon,\delta})}$$

and since $f_\varepsilon, g_\varepsilon \to f_0$ uniformly in $\{x' : |x_i'| < \delta\}$ then there exists $\kappa_\varepsilon \to \infty$ as $\varepsilon \to 0$, such that

$$||\nabla u_\varepsilon||^2_{L^2(R_{\varepsilon,\delta})} \geqslant \kappa_\varepsilon ||u_\varepsilon||^2_{L^2(R_{\varepsilon,\delta})}.$$

Since this argument can be done for a finite covering of $\partial\Omega_0$ we obtain that

$$||\nabla u_\varepsilon||^2_{L^2(\Omega_\varepsilon \backslash K_\varepsilon)} \geqslant C\kappa_\varepsilon ||u_\varepsilon||^2_{L^2(\Omega_\varepsilon \backslash K_\varepsilon)}$$

for certain constant C independent of ε. This shows that (ii) holds.

Notice that the only requirement on f_ε is the uniform convergence to f_0. In particular, we may consider perturbations with a highly oscillating behavior.

For instance

$$f_\varepsilon(x') = f_0(x') + \varepsilon F\left(\frac{x_1}{\varepsilon^{\alpha_1}}, \ldots, \frac{x_{N-1}}{\varepsilon^{\alpha_{N-1}}}\right),$$

where $F : \mathbb{R}^{N-1} \to \mathbb{R}$ is a smooth bounded function.

5.2. *A nonstandard dumbbell-type perturbation*

A typical dumbbell domain consists of a pair of disjoints domains Ω_L and Ω_R which are joined by a thin channel R_ε. Usually, the shape of the channel is given by (for instance in two dimensions)

$$R_\varepsilon = \{(x, y) : x \in (0, L), 0 < y < \varepsilon g_\varepsilon(x)\},$$

where $g_\varepsilon \to g_0$ uniformly in $[0, L]$ and g_0 is some smooth strictly positive function.

The unperturbed domain is given by $\Omega_0 = \Omega_L \cup \Omega_R$. The dumbbell domain is given by $\Omega_\varepsilon = \Omega_L \cup R_\varepsilon \cup \Omega_R$. It represents a prototype of nonconvex perturbation and it has been extensively studied from many points of view. Notice that we have $\Omega_0 \subset \Omega_\varepsilon$ and therefore the sets K_ε in (2.1) can be taken $K_\varepsilon = \Omega_0$, (see also Remark 2.2). In terms of the spectral behavior of the Laplace operator, the results in [2,3,22] say that there is a net contribution of the spectra of the Laplace operator coming from the thin channel. That is, the eigenvalues and eigenfunctions of the dumbbell domain converge as $\varepsilon \to 0$ to the eigenvalues and eigenfunctions of the unperturbed domain $\Omega_0 = \Omega_L \cup \Omega_R$ and to the eigenvalues and eigenfunctions of a problem coming from the channel:

$$\begin{cases} -\dfrac{1}{g_0}(g_0 u_x)_x = \mu u, & x \in (0, L), \\ u(0) = 0, & u(1) = 0. \end{cases} \tag{5.1}$$

Moreover, it is known that the eigenvalues of

$$\begin{cases} -\Delta u = \tau u, & x \in R_\varepsilon, \\ u = 0, & \partial R_\varepsilon \cap \partial(\Omega_L \cup \Omega_R), \\ \dfrac{\partial u}{\partial n} = 0, & \partial R_\varepsilon \backslash \partial(\Omega_L \cup \Omega_R) \end{cases} \tag{5.2}$$

converge to the eigenvalues of (5.1), see [2,3,18].

In particular, (ii) of Proposition 2.3 does not hold and we cannot apply the results in this paper.

Here, we are going to construct a dumbbell domain $\Omega_\varepsilon \subset \mathbb{R}^N$, $N \geqslant 2$, with a thin channel R_ε such that property (ii) of Proposition 2.3 holds, that is, the first eigenvalue of (5.2) diverges to infinity as the parameter $\varepsilon \to 0$. For this dumbbell domain we obtain the convergence of the spectra given by Proposition 2.3, that is, the eigenvalues and eigenfunctions in Ω_ε converge to the eigenvalues and eigenfunctions of Ω_0, so that no contribution from the channel occurs. Hence, all the results of this paper will apply to this example.

The channel R_ε will be constructed as follows:

$$R_\varepsilon = \{(x, x'); x \in (0, L), x' \in \mathbb{R}^{N-1}, |x'| < g_\varepsilon(x)\},$$

where

$$g_\varepsilon(x) = \begin{cases} \left(\dfrac{1}{2} - \dfrac{x}{2L}\right)^{\frac{1}{\varepsilon}}, & 0 < x < L/2, \\ \left(\dfrac{x}{2L}\right)^{\frac{1}{\varepsilon}}, & L/2 < x < L. \end{cases}$$

Hence, consider the eigenvalue problem (5.2) in R_ε and denote by τ_ε the first eigenvalue. Since $g_\varepsilon(L - x) = g_\varepsilon(x)$, by symmetry we will have that the first eigenfunction will satisfy the same symmetry condition and therefore if we define

$$\tilde{R}_\varepsilon = R_\varepsilon \cap \{L/2 < x < L\},$$

the first eigenvalue of (5.2) coincides with the first eigenvalue of

$$\begin{cases} -\Delta u = \tau u, & x \in \tilde{R}_\varepsilon, \\ u = 0, & \partial\tilde{R}_\varepsilon \cap \{x = L\}, \\ \dfrac{\partial u}{\partial n} = 0, & \partial\tilde{R}_\varepsilon \cap \{L/2 \leqslant x < L\}. \end{cases} \tag{5.3}$$

Denote by κ_ε the first eigenvalue of the problem

$$\begin{cases} -\dfrac{1}{x^{(N-1)/\varepsilon}}(x^{(N-1)/\varepsilon}u_x)_x = \kappa u, & L/2 < x < L, \\ u = 0, & x = L, \\ \dfrac{\partial u}{\partial x} = 0, & x = L/2. \end{cases} \tag{5.4}$$

Let us show that there exists a positive number a such that $\tau_\varepsilon \geqslant a\kappa_\varepsilon$. To see this, denote by $\gamma_\varepsilon(x, x')$ the eigenfunction corresponding to the eigenvalue τ_ε of problem (5.3). Assume it is normalized so that $||\gamma_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} = 1$ and hence, $||\nabla\gamma_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} = \tau_\varepsilon$. Denote also by $\bar{\gamma}_\varepsilon(x)$ the averaged function in the x' direction of γ_ε, that is, if $\Gamma_\varepsilon(x) = \{x' \in \mathbb{R}^{N-1},\ (x, x') \in \tilde{R}_\varepsilon\}$ then

$$\bar{\gamma}_\varepsilon(x) = \frac{1}{|\Gamma_\varepsilon(x)|}\int_{\Gamma_\varepsilon(x)} \gamma_\varepsilon(x, x')\, dx', \quad L/2 < x < L.$$

Notice that $|\Gamma_\varepsilon(x)| = (\frac{x}{2L})^{(N-1)/\varepsilon}|B(0, 1)|$, where $B(0, 1)$ is the $(N-1)$-dimensional unit ball.

Then we obviously have

$$\kappa_\varepsilon \leqslant \frac{\int_{L/2}^{L} x^{(N-1)/\varepsilon} |\bar{\gamma}_{\varepsilon,x}|^2 \, dx}{\int_{L/2}^{L} x^{(N-1)/\varepsilon} |\bar{\gamma}_\varepsilon|^2 \, dx}.$$

But, changing variables in the integral above,

$$\bar{\gamma}_\varepsilon(x) = \frac{1}{|B(0,1)|} \int_{B(0,1)} \gamma_\varepsilon(x, g_\varepsilon(x)x') \, dx'$$

which implies that for $L/2 < x < L$

$$\bar{\gamma}_{\varepsilon,x}(x) = \frac{1}{|B(0,1)|} \int_{B(0,1)} (\gamma_{\varepsilon,x}(x, g_\varepsilon(x)x') + g'_\varepsilon(x)x' \cdot \nabla_{x'} \gamma_\varepsilon(x, g_\varepsilon(x)x')) \, dx'.$$

Hence,

$$|\bar{\gamma}_{\varepsilon,x}(x)|^2 \leqslant C \int_{B(0,1)} (|\gamma_{\varepsilon,x}(x, g_\varepsilon(x)x')|^2 + |g'_\varepsilon(x)|^2 |\nabla_{x'} \gamma_\varepsilon(x, g_\varepsilon(x)x')|^2) \, dx'.$$

Since $|g'_\varepsilon(x)| \to 0$ as $\varepsilon \to 0$, uniformly in $x \in [L/2, L]$, we obtain

$$|\bar{\gamma}_{\varepsilon,x}(x)|^2 \leqslant C \int_{B(0,1)} |\nabla \gamma_\varepsilon(x, g_\varepsilon(x)x')|^2 \, dx' = \frac{C}{g_\varepsilon^{N-1}(x)} \int_{\Gamma_\varepsilon(x)} |\nabla \gamma_\varepsilon(x, x')|^2 \, dx'$$

which implies

$$\begin{aligned} \int_{L/2}^{L} x^{\frac{(N-1)}{\varepsilon}} |\bar{\gamma}_{\varepsilon,x}|^2 \, dx &\leqslant C \int_{L/2}^{L} (2L)^{\frac{N-1}{\varepsilon}} \int_{\Gamma_\varepsilon(x)} |\nabla \gamma_\varepsilon(x)|^2 \, dx' \, dx \\ &= C(2L)^{\frac{N-1}{\varepsilon}} \int_{\tilde{R}_\varepsilon} |\nabla \gamma_\varepsilon|^2 \, dx' \, dx. \end{aligned}$$

Moreover,

$$\int_{L/2}^{L} x^{\frac{N-1}{\varepsilon}} |\bar{\gamma}_\varepsilon(x)|^2 \, dx = \int_{L/2}^{L} (2L)^{\frac{N-1}{\varepsilon}} \int_{\Gamma_\varepsilon(x)} |\bar{\gamma}_\varepsilon(x)|^2 \, dx' \, dx = (2L)^{\frac{N-1}{\varepsilon}} \int_{\tilde{R}_\varepsilon} |\bar{\gamma}_\varepsilon|^2 \, dx' \, dx$$

which implies

$$\kappa_\varepsilon \leqslant C \frac{\int_{\tilde{R}_\varepsilon} |\nabla \gamma_\varepsilon|^2}{\int_{\tilde{R}_\varepsilon} |\bar{\gamma}_\varepsilon|^2} = C \frac{\tau_\varepsilon}{\int_{\tilde{R}_\varepsilon} |\bar{\gamma}_\varepsilon|^2}. \tag{5.5}$$

But, by the definition of $\bar{\gamma}_\varepsilon$ we have

$$||\bar{\gamma}_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} = ||\gamma_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} - ||\gamma_\varepsilon - \bar{\gamma}_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} = 1 - ||\gamma_\varepsilon - \bar{\gamma}_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} \tag{5.6}$$

and applying, for fixed x, the second Poincaré inequality in $\Gamma(x)$, we have

$$\int_{\Gamma_\varepsilon(x)} |\gamma_\varepsilon(x,x') - \gamma_\varepsilon(x)|^2 \leqslant C|g_\varepsilon(x)|^2 \int_{\Gamma_\varepsilon(x)} |\nabla_{x'}\gamma_\varepsilon(x,x')|^2 \leqslant C4^{\frac{-1}{\varepsilon}} \int_{\Gamma_\varepsilon(x)} |\nabla_{x'}\gamma_\varepsilon(x,x')|^2\,dx',$$

where we have used that $|g_\varepsilon(x)| \leqslant 2^{-1/\varepsilon}$, $x \in (L/2, L)$. Integrating in x, we get

$$||\gamma_\varepsilon - \bar{\gamma}_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} \leqslant C4^{-1/\varepsilon}||\nabla\gamma_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} = C4^{-1/\varepsilon}\tau_\varepsilon. \tag{5.7}$$

But it is not difficult to see that $\tau_\varepsilon \leqslant \frac{C}{\varepsilon^2}$ for ε small enough. To show this, just consider the function $\chi_\varepsilon(x,x') = L - x$ as a test function in the Rayleigh quotient for τ_ε. This implies that

$$||\gamma_\varepsilon - \bar{\gamma}_\varepsilon||^2_{L^2(\tilde{R}_\varepsilon)} \leqslant C2^{(1-N)/\varepsilon}\varepsilon^{-2} \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence, from this last statement, (5.5)–(5.7) we show that $\kappa_\varepsilon \leqslant C\tau_\varepsilon$ for some constant C independent of ε.

Let us see now that $\kappa_\varepsilon \to \infty$ as $\varepsilon \to 0$. Denote by ϕ_ε the positive eigenfunction associated to κ_ε. Assume also that we normalize the eigenfunction so that $||\phi_\varepsilon||_{L^\infty(L/2,L)} = 1$. By the maximum principle applied to (5.4), we have $\phi_\varepsilon(x) \leqslant \chi_\varepsilon(x)$ where

$$\begin{cases} -\dfrac{1}{x^{(N-1)/\varepsilon}}\,(x^{(N-1)/\varepsilon}\chi_x)_x = \kappa_\varepsilon, & L/2 < x < L, \\ \chi = 0, & x = L, \\ \dfrac{\partial\chi}{\partial x} = 0, & x = L/2. \end{cases} \tag{5.8}$$

By direct computation, the solution of the problem above is given by

$$\chi_\varepsilon(x) = -\frac{\kappa_\varepsilon L^2}{2(\frac{N-1}{\varepsilon}+1)}\left(\left(\frac{x}{L}\right)^2 - 1 + \frac{1}{2^{\frac{N-1}{\varepsilon}}(\frac{N-1}{\varepsilon}-1)}\left(\left(\frac{x}{L}\right)^{1-\frac{N-1}{\varepsilon}} - 1\right)\right)$$

which satisfies

$$||\chi_\varepsilon||_{L^\infty(L/2,L)} = \chi_\varepsilon(L/2) \leqslant C\varepsilon\kappa_\varepsilon.$$

This implies that

$$1 = ||\phi_\varepsilon||_{L^\infty(L/2,L)} \leqslant ||\chi_\varepsilon||_{L^\infty(L/2,L)} \leqslant C\varepsilon\kappa_\varepsilon$$

so

$$\kappa_\varepsilon \geqslant \tilde{C}/\varepsilon \to \infty \quad \text{as } \varepsilon \to 0.$$

Since $\tau_\varepsilon \geqslant a\kappa_\varepsilon$, we also obtain that $\tau_\varepsilon \to \infty$ as $\varepsilon \to 0$. Hence, Proposition 2.3(ii) holds and all the results of this paper apply to this perturbation. $\square$

Remark 5.1. For this kind of dumbbell domain the formation of nonconstant stable equilibrium solutions is a direct consequence of Proposition 3.1. If for instance we consider the nonlinearity $f(u) = u - u^3$, we have that for any domain the equilibria $u = 1$ and -1 are asymptotically stable. Hence if we consider u_0 an equilibria in $\Omega_0 = \Omega_l \cup \Omega_R$ given by $u_0 = 1$ in Ω_L and $u_0 = -1$ in Ω_R, we know that this equilibrium is asymptotically stable. By Proposition 3.1 there exists an equilibrium $u_\varepsilon \in H^1(\Omega_\varepsilon)$ which is near u_0 in H^1_ε and that the linearization around u_ε converges to the linearization of the limit problem around u_0. In particular u_ε is an asymptotically stable equilibrium (with the same index as u_0) and u_ε is obviously nonconstant.

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