Dissipative parabolic equations in locally uniform spaces

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Abstract

Cauchy problem for a semilinear second order parabolic equation
\[ u_t = \Delta u + f(x, u, \nabla u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \]
is considered within the semigroup approach in locally uniform spaces \( \dot{W}^{s,p}_{U} (\mathbb{R}^N) \). Global solvability, dissipativeness and the existence of an attractor are established under the same assumptions as for problems in bounded domains. In particular, condition \( sf(s, 0) < 0, \ |s| > s_0 > 0 \), together with gradient’s ‘subquadratic’ growth restriction, are shown to guarantee the existence of an attractor for the above mentioned equation. This result cannot be located in the previous references devoted to reaction-diffusion equations in the whole of \( \mathbb{R}^N \).

1 Introduction and main results

In this paper we study the Cauchy problem for an autonomous second order semilinear parabolic equation of the form
\[
\begin{cases}
  u_t = \Delta u + f(x, u, \nabla u), & t > 0, \ x \in \mathbb{R}^N, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\] (1.1)

Our goal is to give structure conditions on the nonlinearity guaranteeing global well posedness of (1.1), dissipativeness and the existence of an attractor.

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Note that several different versions of (1.1), including the case of systems, have been considered by many authors, e.g. [1, 9, 23, 18, 19, 27, 32, 33, 34, 6, 13, 12], where attractors have been constructed under suitable functional settings and suitable growth sign and structure conditions on the nonlinear term $f(x, u, \nabla u)$.

Here, we give a functional setting in very large spaces and prove the existence of a suitable notion of attractor that attract in suitable strong norms.

To illustrate the sort of difficulties that one faces when considering (1.1), consider the bistable reaction term $f = f(u) = u - u^3$. Notice that in dimension one this equation has traveling waves of all velocities connecting the constant equilibria 0 and ±1; [26, page 136]. Also a continuum of bounded space periodic equilibria is easily shown to exists. Moreover, for any dimension, one can show that for any reasonable initial data the sup norm of the solution will be asymptotically bounded by 1. On the other hand for any nonnegative smooth compactly supported initial data, the solution converges to 1 in $L^{\infty}_{loc}(\mathbb{R}^N)$; see e.g. [6]. Hence to account for all this dynamics one must choose a large space, containing bounded functions, but whose norm is not too strong. For example, travelling waves do not connect 0 and ±1 in the sup norm.

In the case $f$ depends on the gradient, then some subquadratic growth restriction is needed, otherwise solutions may blow-up in a finite time in $C^1$ norm, see [40].

Hence, in the cited references different approaches for the functional setting and for the structure conditions in the nonlinearity have been proposed. For example, weighted Sobolev spaces $W^{k,p}(\mathbb{R}^N)$ have been used in [1, 9, 18, 21, 22]. The space of bounded and uniformly continuous functions $BUC(\mathbb{R}^N)$, was used in [32]. More recently Bessel potentials spaces $H^s_p(\mathbb{R}^N)$ have been used in [6]. The so called locally uniform spaces have been used in [33, 34, 12, 13] and a systematic study of linear equations in such spaces was developed in [7]. Hence this paper is a natural nonlinear continuation of that one.

In each one of these settings, the authors impose some structure conditions on the nonlinear terms. For example in [9, 14] the assumption on the nonlinear term reads

$$f(x, u) = -\lambda_0 u + f_0(u) + g(x), \quad \text{with} \quad \lambda_0 > 0, \quad f_0(u)u \leq 0, \quad (1.2)$$

with $g$ a suitable given function, while in [18] reads (for vector valued solutions)

$$f(x, u, \nabla u) = -\lambda_0 u + f_0(u, \nabla u) + g(x), \quad \text{with} \quad \lambda_0 > 0, \quad f_0(u, \nabla u)u \leq 0. \quad (1.3)$$

Note that these sort of conditions imply strong restrictions on the sign of both the linear and the nonlinear parts of the equation. Slightly weaker assumptions on the linear part are used in [32] and even weaker ones have been considered in [6]. On the other hand assuming monotonicity conditions on $f_0$, conditions on the linear term are dropped in [13, 22], which implies that the bistable equation can be handled within these frameworks. As the two latter references work in large weighted spaces, weak attractivity properties are obtained. The same happens in [9]. Also, note that [9, 21, 22] do not allow the term $f_0$ to depend on $x$.

As mentioned before, our goal here is to give a functional setting in a very large space and prove the existence of a suitable notion of attractor that attracts in suitable strong norms.
In particular, our results show that if we consider the particular case of (1.1)
\[ u_t = \Delta u + f(u, \nabla u), \quad t > 0, \ x \in \mathbb{R}^N, \quad u(0, x) = u_0(x), \ x \in \mathbb{R}^N, \] (1.4)
where the nonlinear term does not depend on \( x \) and grows subquadratically with respect to \( \nabla u \), then this problem has an attractor if the associated ordinary differential equation for spatially homogeneous solutions
\[ U_t = f(U, 0), \quad t > 0, \ U(0) = U_0 \in \mathbb{R}, \] (1.5)
has an attractor. This, in turn, reduces to the condition
\[ sf(s, 0) < 0, \quad |s| > s_0 > 0. \]

In this paper we will use locally uniform spaces as phase spaces for (1.1). These spaces have been introduced as the Banach spaces consisting of all (equivalence classes of) measurable functions \( \phi: \mathbb{R}^N \to \mathbb{R} \) satisfying
\[ \| \phi \|_{L^p(\mathbb{R}^N)} := \sup_{y \in \mathbb{R}^N} \| \phi \|_{L^p(B(y, 1))} < \infty \]
for \( 1 \leq p \leq \infty \). Following [7] we denote these spaces by \( L^p_U(\mathbb{R}^N) \) and consider their closed subspaces \( \dot{L}^p_U(\mathbb{R}^N) \), of functions which are translation continuous with respect to \( \| \cdot \|_{L^p_U(\mathbb{R}^N)} \), that is
\[ \| \tau_y \phi - \phi \|_{\dot{L}^p_U(\mathbb{R}^N)} \to 0 \quad \text{as} \quad |y| \to 0 \]
where \( \{ \tau_y, y \in \mathbb{R}^N \} \) denotes the group of translations. Note in particular that \( L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N) \) and \( \dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N) \). Similarly, we denote by \( W^{k,p}_U(\mathbb{R}^N) \) and \( \dot{W}^{k,p}_U(\mathbb{R}^N) \), \( k \in \mathbb{N} \), Sobolev type spaces constructed above \( L^p_U(\mathbb{R}^N) \) and \( \dot{L}^p_U(\mathbb{R}^N) \) respectively. Thus, \( W^{k,p}_U(\mathbb{R}^N) \) is the Banach space consisting of elements \( \phi \in W^{k,p}_{loc}(\mathbb{R}^N) \) satisfying \( \| \phi \|_{W^{k,p}_U(\mathbb{R}^N)} = \sum_{|\sigma| \leq k} \| D^\sigma \phi \|_{L^p(\mathbb{R}^N)} < \infty \) and \( \dot{W}^{k,p}_U(\mathbb{R}^N) \) consists of all those elements of \( W^{k,p}_U(\mathbb{R}^N) \), which are translation continuous with respect to \( \| \cdot \|_{W^{k,p}_U(\mathbb{R}^N)} \). The case of noninteger \( k \), \( \dot{W}^{k,p}_U(\mathbb{R}^N) \), is defined by interpolation, see [7] and the Appendix for details. We just note here that translations are continuous in these spaces.

It is also shown in [7] that the differential operator \( A = \Delta \) considered in \( \dot{L}^p_U(\mathbb{R}^N) \), \( 1 < p < \infty \), with the domain \( \dot{W}^{2,p}_U(\mathbb{R}^N) \), generates a \( C^0 \) analytic semigroup \( e^{At} \in \mathcal{L}(L^p_U(\mathbb{R}^N)) \), \( t \geq 0 \). Moreover, the corresponding fractional power spaces are the complex interpolation spaces \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) := [\dot{L}^p_U(\mathbb{R}^N), \dot{W}^{2,p}_U(\mathbb{R}^N)]_\alpha \) for \( 0 \leq \alpha \leq 1 \).

Therefore, using the techniques in [26], we can solve
\[ \begin{cases} u_t - \Delta u = f(\cdot, u, \nabla u), \quad t > 0, \\ u(0) = u_0 \in \dot{W}^{2\alpha,p}_U(\mathbb{R}^N), \end{cases} \] (1.6)
constructing the solution as a continuous function from \([0, \tau_{u_0})\) into \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \) satisfying the integral equation
\[ u(t, u_0) = e^{At} u_0 + \int_0^t e^{A(t-s)} f(\cdot, u(s, u_0), \nabla u(s, u_0)) \, ds, \quad t \in [0, \tau_{u_0}) \] (1.7)
provided $f$ defines a locally Lipschitz map $\dot{W}_U^{2\alpha,p}(\mathbb{R}^N) \to \dot{L}_p^p(\mathbb{R}^N)$.

Once local existence is established, the main idea that drives the arguments is to obtain suitable $L^\infty(\mathbb{R}^N)$ bounds on the solutions. Then the variation of constants formula above allows us to transfer this estimate to gradient estimates. Here is where the subquadratic growth assumption in the gradient is required for this technique to work. Finally note that when $f$ depends on the gradient, $L^\infty(\mathbb{R}^N)$ bounds are obtained by comparison with a suitable ODE (see e.g. (1.5)). On the other hand, if $f$ depends only on $x$ and $u$ then comparison is carried out with a suitable linear parabolic PDE.

Hence we first present the following local well posedness result for (1.6).

**Proposition 1.1**

i) Suppose that $f : \mathbb{R}^{N+1} \to \mathbb{R}$, $f = f(x,s)$, is locally Lipschitz continuous with respect to $s$ uniformly for $x \in \mathbb{R}^N$, and Hölder continuous with respect to $x$ uniformly for $s$ varying in bounded subsets of $\mathbb{R}^N$.

Then, for $p > \frac{N}{2}$ and $\alpha \in [0,1)$, such that $2\alpha - \frac{N}{p} > 0$ and for each $u_0 \in \dot{W}_U^{2\alpha,p}(\mathbb{R}^N)$, there exists a unique solution $u = u(\cdot,u_0)$ to (1.6) defined on a maximal interval of existence $[0,\tau_{u_0})$.

ii) Suppose that $f : \mathbb{R}^{2N+1} \to \mathbb{R}$, $f = f(x,s,p)$, is a continuous function locally Lipschitz with respect to $(s,p)$, uniformly for $x \in \mathbb{R}^N$, and Hölder continuous with respect to $x$ uniformly for $(s,p)$ varying in bounded subsets of $\mathbb{R}^{N+1}$.

Then, for $p > N$ and $\alpha \in [0,1)$, such that $2\alpha - \frac{N}{p} > 1$ and for each $u_0 \in \dot{W}_U^{2\alpha,p}(\mathbb{R}^N)$, there exists a unique solution $u = u(\cdot,u_0)$ to (1.6) defined on a maximal interval of existence $[0,\tau_{u_0})$.

In either case, the solutions are given by (1.7) and satisfy

$$u \in C([0,\tau_{u_0}), \dot{W}_U^{2\alpha,p}(\mathbb{R}^N)) \cap C((0,\tau_{u_0}), \dot{W}_U^{2\alpha,p}(\mathbb{R}^N)) \cap C^1((0,\tau_{u_0}), \dot{W}_U^{2-\alpha,p}(\mathbb{R}^N))$$

(1.8)

where $2^-$ denotes any positive number smaller than 2.

The above local well posedness result follows easily from the consideration of [26] and the fact that, in the first case of the theorem, $\dot{W}_U^{2\alpha,p}(\mathbb{R}^N)$ is embedded in $BUC(\mathbb{R}^N)$ for $2\alpha - \frac{N}{p} > 0$. For the second case the argument is similar just noting that $\dot{W}_U^{2\alpha,p}(\mathbb{R}^N)$ is embedded in $C^1_{bd}(\mathbb{R}^N)$ for $2\alpha - \frac{N}{p} > 1$. In either case the nonlinear term $f$ defines a Nemytskii map from $\dot{W}_U^{2\alpha,p}(\mathbb{R}^N)$ to $\dot{L}_p^p(\mathbb{R}^N)$ which is Lipschitz continuous on bounded sets.

**Remark 1.2** Note that the assumptions on the nonlinearity in the Proposition can be somewhat relaxed if one assumes some particular structure. For example, if we assume

$$f(x,s,p) = g(x) + m(x)s + f_0(x,s,p),$$

which will be of practical interests for the applications, then it would suffice to assume $g,m \in \dot{L}_p^p(\mathbb{R}^N)$. Then the Proposition will hold provided that $f_0$ satisfies the regularity stated in the Proposition.
To prove that local solutions in Proposition 1.1 are actually global in time and have bounded orbits, we will require \( f \) to satisfy
\[
 sf(x, s, 0) \leq 0, \quad |s| \geq s_0 > 0, \quad x \in \mathbb{R}^N, \tag{1.9}
\]
and
\[
 |f(x, s, p)| \leq c(s)(1 + |p|^{\gamma_0}), \quad (x, s, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \tag{1.10}
\]
where \( \gamma_0 \in [1, 2) \) and \( c : [0, \infty) \rightarrow [0, \infty) \) is a continuous function. These conditions are analogous as in the case of semilinear equations in bounded domains (see [25, 36]). Note that once solutions are global then a \( C^0 \)-semigroup \( \{T(t)\} \) corresponding to (1.1) in \( \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \) can be defined by \( T(t)u_0 = u(t, u_0), \ t \geq 0, \ u_0 \in \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \).

The main results of this article concerning solutions of (1.1) are formulated in Theorems 1.3, 1.5, and Corollary 1.4. Namely,

**Theorem 1.3** Assume \( p > N \) and \( p \geq 2 \) and \( f(x, s, p) \) as in Proposition 1.1. Assume also that (1.10) holds with \( \gamma_0 \in [1, 2) \) and (1.9) is satisfied.

Then we have the following results

i) Problem (1.1) is globally well posed in \( \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \). Orbits of bounded sets of the semigroup \( \{T(t)\} \) of global solutions to (1.1) in \( \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \) are bounded and for each \( M > s_0 \),
\[
\mathcal{R} = \{ \phi \in \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) : \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq M \}, \tag{1.11}
\]
is positively invariant set under \( \{T(t)\} \).

ii) If, in addition, (1.9) is strengthened to the condition
\[
 sf(x, s, 0) \leq sh(s) < 0, \quad |s| \geq s_0 > 0, \quad x \in \mathbb{R}^N, \tag{1.12}
\]
for some continuous function \( h : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \), then there exists a closed bounded invariant set \( A \subset \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \), which “attracts the asymptotic dynamics of (1.1)” in the sense that
\[
\text{dist}_W(T(t)B, A) \rightarrow 0, \quad \text{as } t \rightarrow \infty,
\]
for any \( B \) bounded in \( \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \). Here \( W \) denotes either \( C_{\text{loc}}^{1+\mu}(\mathbb{R}^N) \), with \( \mu \in (0, 1 - \frac{N}{p}) \), or the weighted space \( W_\rho^{s,p}(\mathbb{R}^N) \), \( 0 \leq s < 2 \), where \( \rho \) is any translation of \( \rho_0(x) = (1 + |x|^2)^{-\nu} \) and \( \nu > \frac{N}{2} \).

If \( f \) does not depend on \( x \), that is, \( f = f(s, p) \), then \( A \) is moreover invariant with respect to the group of translations in \( \mathbb{R}^N \).

Note that attraction in \( W = C_{\text{loc}}^{1+\mu}(\mathbb{R}^N) \) means that for each \( R > 0 \) the attraction takes place in \( C^{1+\mu}(\overline{B}(0, R)) \). Also, note that the set \( A \) described in the Theorem 1.3 does not attract in the norm of the phase space \( \mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \) but the attraction takes place in a suitable weaker norm determined by the choices for the space \( W \). Then we say that \( A \) is an \( (\mathcal{W}_{U}^{2\alpha,p}(\mathbb{R}^N) - W) \) attractor. Indeed, it is easy to see from the bistable example above and the traveling wave
solutions that since \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N) \) the set \( A \) can not in fact attract in the norm of the phase space \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \).

Also note that in particular when \( f \) does not depend on \( x \) (i.e. \( f = f(u, \nabla u) \)) we have, from (1.12),

**Corollary 1.4** Assume \( f : \mathbb{R}^{N+1} \to \mathbb{R}, f = f(s,p), \) is a locally Lipschitz function satisfying (1.10). Then, for \( p > N \), there exists a bounded \( (\dot{W}^{1,p}_U(\mathbb{R}^N) - L^p_{\rho_0}(\mathbb{R}^N)) \) attractor for (1.4), with \( \rho_0(x) = (1 + |x|^2)^{-\nu} \) and \( \nu > \frac{N}{2} \), if and only if the ordinary differential equation (1.5) has an attractor.

Evidently, in a non-gradient case \( f = f(u) \), assumption (1.10) in Corollary 1.4 above can be omitted.

When the nonlinear term in (1.1) depends on \( x \) but does not depend on \( \nabla u \) an even less restrictive structure condition can be imposed on \( f \). In this case, we will assume that

\[
\begin{align*}
s f(x,s) & \leq C(x)|s|^2 + D(x)|s|, \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^N, \quad \text{where} \\
C, D & \in \dot{L}^p(\mathbb{R}^N) \quad \text{for certain} \quad p > \frac{N}{2}, \quad p \geq 2 \quad \text{and} \quad D \geq 0 \quad \text{a.e. in} \quad \mathbb{R}^N
\end{align*}
\]

(1.13) see [6] for similar conditions in an \( L^p(\mathbb{R}^N) \) setting.

We then have

**Theorem 1.5** Suppose that \( f : \mathbb{R}^{N+1} \to \mathbb{R}, f = f(x,s), \) is locally Lipschitz continuous with respect to \( s \) uniformly for \( x \in \mathbb{R}^N \), and Hölder continuous with respect to \( x \) uniformly for \( s \) varying in bounded subsets of \( \mathbb{R}^N \). Suppose also that (1.13) holds and the linear semigroup generated by \( \Delta + CI \) decays exponentially, that is, \( \Re \sigma(-\Delta - C(\cdot)I) > 0 \).

Then for \( p > \frac{N}{2} \) and \( \alpha \in [0, 1) \) such that \( 2\alpha - \frac{N}{p} > 0 \) there exists a closed bounded invariant set \( A \subset \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \), which is a \( (\dot{W}^{2\alpha,p}_U(\mathbb{R}^N) - W) \) attractor for \( \{T(t)\} \), where \( W \) denotes either \( C^\mu_{\text{loc}}(\mathbb{R}^N) \), with \( 2 - \frac{N}{p} > \mu > 0 \), or the weighted space \( W^{\alpha,p}_{\rho}(\mathbb{R}^N), 0 \leq s < 2 \), with \( \rho \) being any translation of \( \rho_0(x) = (1 + |x|^2)^{-\nu} \), \( \nu > \frac{N}{2} \).

If \( f \) does not depend on \( x \), that is, \( f = f(s,p) \), then \( A \) is moreover invariant with respect to the group of translations in \( \mathbb{R}^N \).

Note that the results in [7] allow us to consider the operator \( A = \Delta \) (and in fact much more general elliptic operators) in the larger space \( L^p_U(\mathbb{R}^N) \) or even in the much larger space \( L^p(\mathbb{R}^N) \). However in the former case the operator has non dense domain \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \) and therefore the linear semigroup is not continuous at \( t = 0 \). The nonlinear equation could then be handled using the techniques for non densely defined generators as in [31]. Note that this type of difficulties is the same as one finds in solving the heat equation in a bounded domain with Dirichlet boundary conditions or in \( \mathbb{R}^N \), with \( L^\infty \) as a phase space. This is also the reason why the \( BUC \) setting is somewhat easier for local existence than the \( L^\infty \) setting, see e.g. [32].
On the other hand, for the case of weighted spaces, although the linear semigroup is continuous, [7], the difficulties for the nonlinear equations come from the lack of Sobolev–like embeddings which make unavailable the standard techniques for handling nonlinear terms. In fact it was shown in [8] that generally, without specific structure assumption on nonlinear term, semilinear problems are ill posed in weighted spaces when the weight goes to zero at infinity. However with monotonicity assumptions of the form
\[ \frac{\partial f}{\partial s}(x, s) \leq \text{const}, \quad x \in \mathbb{R}^N, \ s \in \mathbb{R}, \]
the nonlinear semigroups can be constructed, see e.g. [13]. See also [35] for even less restrictive monotonicity conditions or suitable growth and structure conditions in weighted spaces.

Finally, note that we could have considered (1.1) in less regular spaces, e.g. without assuming \( p > N \) or \( p > N/2 \) in Proposition 1.1, but assuming some growth conditions on the nonlinearities, as in [6] for the \( L^p(\mathbb{R}^N) \) setting. However, for simplicity we will not pursue along this direction here.

The main results above will be proved in Section 2. In Section 3 we will discuss several examples and compare the structure conditions appearing in the references with the ones in this paper. Finally in the Appendix we will give suitable characterization of the spaces \( \dot{W}^{s,p}_{U}(\mathbb{R}^N) \) which are used in previous sections. Such a result will supplement earlier consideration of [7].

2 Proofs of the main results

Before proving the main results of this paper we recall several useful properties of the spaces \( \dot{W}^{s,p}_{U}(\mathbb{R}^N) \) and \( W^{s,p}_{\rho}(\mathbb{R}^N) \) that have been proved in [7]; see also the Appendix. Among them we mention that \( C^{\infty}_{bd}(\mathbb{R}^N) \) is dense in \( \dot{W}^{s,p}_{U}(\mathbb{R}^N) \) and if \( \rho \) is any translation of \( \rho_0(x) = (1 + |x|^2)^{-\nu} \) with \( \nu > \frac{N}{2} \), then we have the continuous embeddings
\[ \dot{W}^{s,p}_{U}(\mathbb{R}^N) \hookrightarrow W^{s,p}_{\rho}(\mathbb{R}^N), \] (2.1)
and
\[ \dot{W}^{s,p}_{U}(\mathbb{R}^N) \hookrightarrow C^{l+\mu}(\mathbb{R}^N), \quad l + 1 > s - \frac{N}{p} > l + \mu > l \] (2.2)
while the embeddings
\[ \dot{W}^{2,p}_{U}(\mathbb{R}^N) \hookrightarrow W^{1,p}_{\rho}(\mathbb{R}^N) \] (2.3)
and
\[ \dot{W}^{s,p}_{U}(\mathbb{R}^N) \hookrightarrow C^{l+\mu}(Q), \quad l + 1 > s - \frac{N}{p} > l + \mu > l, \] (2.4)
are compact, where \( Q \) is any compact set in \( \mathbb{R}^N \). See [7, Lemma 4.1] and [12].

Now, we note that the conclusions of Theorems 1.3 and 1.5 will be consequence of the following general results.
Proposition 2.1 Assume \( \{S(t)\} \) is a \( C^0 \)-semigroup on a metric space \( V \) for which there exists a bounded set \( B_0 \subset V \) absorbing bounded sets and \( W \) is a metric space containing \( B_0 \) such that any sequence \( \{S(t_n)v_n\} \) where \( t_n \to \infty \) and \( \{v_n\} \subset B_0 \) possesses a subsequence convergent in the metric of \( W \).

Then there exists a nonempty set \( A \subset W \) which is compact in \( W \) and attracts bounded subsets of \( V \) with respect to the Hausdorff semidistance \( d_W \).

Proof. The assertion follows from (2.5), defining \( A \) as
\[
A := \{ w \in W : S(t_n)v_n \xrightarrow{w} w \text{ for certain } \{v_n\} \subset B_0 \text{ and } t_n \to \infty \},
\]
and the property that \( B_0 \) is an absorbing set.

Note that the set \( A \) defined in the Proposition above is a minimal element in the class of compact subsets of \( W \) attracting bounded subsets of \( V \) in the metric of \( W \) and, if \( W = V \), it is a global attractor in the sense of [24].

The set \( A \) resulting from Proposition 2.1 will be called a \((V - W)\) attractor for \( \{S(t)\} \) if additionally \( A \) is also a closed subset of \( V \), invariant under \( \{S(t)\} \).

For example (2.5) holds whenever \( V \) is compactly embedded in \( W \) and in this case \( A \subset W \). However, in general, one has to impose conditions additional to (2.5) to conclude that \( A \) is contained in \( V \) and is an invariant set. For example, if \( V \) is a reflexive Banach space (or more generally the dual of a Banach space), then (2.5) together with weak (or weak-\( * \)) convergence, gives that \( A \subset V \).

Motivated by this we introduce the following condition, generalizing the both notions of asymptotic compactness [29] and asymptotic smoothness [24], that we denote \((V - W)\) asymptotic compactness.

For any \( t_n \to \infty \) and \( \{v_n\} \subset B_0 \), \( \{S(t_n)v_n\} \) contains a subsequence \( \{S(t_{n_k})v_{n_k}\} \) convergent in the metric of \( W \) to certain \( v \in V \cap W \) and,

\[
\text{if in addition, } S(t_{n_k} + t)v_{n_k} \xrightarrow{w} \tilde{v} \text{ for certain } t > 0, \text{ then } S(t)v = \tilde{v}.
\]

(2.6)

Note that the idea of using \( \omega \)-limit sets, in weaker topologies, of absorbing sets to construct a suitable notion of attractor has been used before, see e.g. [9], where the weak topology is considered in an \( L^2(\mathbb{R}^N) \) setting.

Corollary 2.2 Suppose that assumptions of Proposition 2.1 hold and, in addition, (2.6) is satisfied. Then \( \{S(t)\} \) possesses a \((V - W)\) attractor \( A \).

We remark that in applications we present in this paper, specific properties of equations of the form (1.1) will allow us to prove that \((V - W)\) attractor \( A \) is in fact bounded in \( V \), as stated in Theorems 1.3 and 1.5.

Now we derive necessary estimates and prove the results reported in Section 1. The advantage of the functional analytic setting in uniformly local spaces is that most estimates will follow as for equations in bounded domains.
2.1 Proof of Theorem 1.3

In this subsection we consider (1.1) with \( f : \mathbb{R}^{2N+1} \to \mathbb{R} \), locally Lipschitz with respect to \((s,p)\) uniformly for \( x \in \mathbb{R}^N \), and Hölder continuous with respect to \( x \) uniformly for \((s,p)\) varying in bounded subsets of \( \mathbb{R}^{N+1} \). As follows from Proposition 1.1, if \( p > N \), \( \alpha \in [0,1) \) and \( 2\alpha - \frac{N}{p} > 1 \), then (1.1) is locally well posed in \( \dot{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \). Under further assumptions of Theorem 1.3 we prove below that the solutions to (1.1) exist globally in time and the corresponding semigroup in \( \dot{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \) is bounded dissipative. We then show that the generalized asymptotic smoothness condition (2.6) is satisfied and construct a \((\dot{W}_{U}^{2\alpha,p}(\mathbb{R}^N),W)\) attractor, as in Corollary 2.2.

This will be done in a sequence of lemmas below.

Lemma 2.3 Suppose that \( f : \mathbb{R}^{2N+1} \to \mathbb{R} \), \( f = f(x,s,p) \), \( p = (p_1, \ldots, p_N) \) is locally Lipschitz continuous with respect to \( s \) and \( p \) uniformly for \( x \in \mathbb{R}^N \) as well as it is \( \mu \)-Hölder continuous with respect to \( x \), uniformly for \((s,p)\) varying in bounded subsets of \( \mathbb{R} \times \mathbb{R}^N \). Suppose also that (1.10) and (1.9) hold.

i) If \( u_0, v_0 \in C^{2+\mu}(\mathbb{R}^N) \), \( u = u(t,x,u_0), v = v(t,x,v_0) \in C^{1+\frac{\mu}{2},2+\mu}([0,T] \times \mathbb{R}^N) \), \( u(0,x,u_0) = u_0(x) \), \( v(0,x,v_0) = v_0(x) \), \( x \in \mathbb{R}^N \), functions \( u, v \) satisfy the differential inequality

\[
\dot{u} - \Delta u - f(x,u,\nabla u) \leq \dot{v} - \Delta v - f(x,v,\nabla v), \quad t \in [0,T], \quad x \in \mathbb{R}^N, \tag{2.7}
\]

and simultaneously

- either \( u \) fulfils (1.1) together with the relation
  \[
u_0(x) \leq v = v(t,x,v_0), \quad t \in [0,T], \quad x \in \mathbb{R}^N,\]
  or

- \( v \) fulfils both (1.1) and the inequality
  \[
u = u(t,x,u_0) \leq v_0(x), \quad t \in [0,T], \quad x \in \mathbb{R}^N,\]

then

\[
u \leq v \quad \text{in} \quad [0,T] \times \mathbb{R}^N.
\]

ii) If \( u_0, v_0 \in C^{2+\mu}(\mathbb{R}^N) \), \( u = u(t,x,u_0), v = v(t,x,v_0) \in C^{1+\frac{\mu}{2},2+\mu}([0,T] \times \mathbb{R}^N) \) and functions \( u, v \) solve (1.1), then \( u \leq v_0 \) in \( \mathbb{R}^N \) implies that \( u \leq v \) in \([0,T] \times \mathbb{R}^N\).

iii) If \( 2\alpha - \frac{N}{p} > 1 \), \( p \in (1,\infty) \), \( \alpha \in [0,1) \), \( u_0, v_0 \in \dot{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \), \( u_{0n} \to u_0, v_{0n} \to v_0 \) in \( \dot{W}_{U}^{2\alpha,p}(\mathbb{R}^N) \) as \( n \to \infty \), where \( u_{0n}, v_{0n} \in C^{\infty}_{bd}(\mathbb{R}^N) \) and in addition \( u_{0n} \leq v_{0n}, n \in \mathbb{N} \), then local solutions \( u(\cdot,u_0), v(\cdot,v_0) \) resulting from Proposition 1.1 satisfy in \( \mathbb{R}^N \) the relation

\[
u(t,u_0) \leq v(t,v_0),
\]

as long as both \( u \) and \( v \) exist.
Suppose also that both continuous with respect to are locally Lipschitz continuous with respect to. It is then an additional property of the solutions to (1.1) that $u(\cdot, u_0) \in C^{1+\frac{\mu}{2}+\rho}([0, T] \times \mathbb{R}^N)$ can be approximated by a sequence $\{w^n\}$ of classical solutions of the Dirichlet initial boundary value problems (2.8) in an increasing sequence of cylinders $(0, T) \times B^n$, where $B^n = \{x \in \mathbb{R}^N : |x| < n\}$.

Consider thus $u_0, v_0 \in C^{2+\mu}(\mathbb{R}^N)$ and the classical solutions\footnote{As follows from consideration of [30, Chapter V, §8] these solutions are continuous in $[0, T] \times \text{cl}B^n$ and they are Hölder $C^{1+\frac{\mu}{2}+\rho}$-functions on any compact sub-cylinder of $(0, T) \times B^n$.} $u_n = u_n(t, x, u_0)$ to the initial boundary value problem

$$\begin{align*}
u_{nt} &= \Delta u_n + f(x, u_n, \nabla u_n), \quad t \in (0, T), \quad x \in B^n, \\
u_n(t, x) &= u_0(x), \quad x \in \partial B^n, \quad t \in (0, T) \\
u_n(0, x) &= u_0(x), \quad x \in B^n. \\
\end{align*}$$

As shown in [30, Chapter V, §8], the sequence $\{u_n\}$ is point-wise convergent in $[0, T] \times \mathbb{R}^N$ to the solution $u = u(t, x, u_0) \in C^{1+\frac{\mu}{2}+\rho}([0, T] \times \mathbb{R}^N)$ of (1.1).

Assuming that $u_0(x) \leq v(t, x, v_0)$ for $t \in [0, T], x \in \mathbb{R}^N$, the relation (2.7) between the ‘defects’ (see [44, Chapter IV, §24]) ensures that function $v$ will be a supersolution to each of the problems (2.8). This gives the bounds $u_n \leq v$ in $(0, T] \times B^n$ for each $n \in \mathbb{N}$ and thus

$$u \leq v \text{ in } [0, T] \times \mathbb{R}^N.$$  

Since a similar reasoning applies in the situation when $v$ satisfies (1.1) together with

$$u = u(t, x, u_0) \leq v_0(x), \quad t \in [0, T], \quad x \in \mathbb{R}^N,$$

the proof of item i) is complete.

The proof of ii) is an obvious modification of the above procedure. Now the assumption for initial conditions is merely $u_0 \leq v_0$ in $\mathbb{R}^N$ because both $u$ and $v$ are the solutions to (1.1) and hence both $u$ and $v$ may be approximated by solutions of the Dirichlet boundary value problems in $(0, T) \times B^n$.

Since by our assumptions on parameters $W^{2a,p}_U(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$, assertion iii) is a consequence of ii) and continuity with respect to initial data of the solutions resulting from Proposition 1.1. □

Lemma 2.4 Suppose that $f, h : \mathbb{R}^{2N+1} \rightarrow \mathbb{R}, f = f(x, s, p), h = h(x, s, p), p = (p_1, \ldots, p_N)$ are locally Lipschitz continuous with respect to $s$ and $p$ uniformly for $x \in \mathbb{R}^N$ and $\mu$-Hölder continuous with respect to $x$, uniformly for $(s, p)$ varying in bounded subsets of $\mathbb{R} \times \mathbb{R}^N$. Suppose also that both $f$ and $h$ fulfill growth restriction (1.10) and structure condition (1.9).

i) If $u_0, v_0 \in C^{2+\mu}(\mathbb{R}^N)$, functions $u = u(t, x, u_0), v(t, x, v_0) \in C^{1+\frac{\mu}{2}+\rho}([0, T] \times \mathbb{R}^N)$ solve

$$\begin{align*}
u_{tt} - \Delta u &= f(x, u, \nabla u), \quad t \in [0, T], \quad x \in \mathbb{R}^N, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}^N, \\
u_{tt} - \Delta v &= h(x, v, \nabla v), \quad t \in [0, T], \quad x \in \mathbb{R}^N, \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}^N. \\
\end{align*}$$

Proof. Following [30, Chapter V, §8], if $u_0 \in C^{2+\mu}(\mathbb{R}^N)$ then (1.1) possesses a $C^{1+\frac{\mu}{2}+\rho}([0, T] \times \mathbb{R}^N)$-solution which, because of its regularity, coincides with $u(\cdot, u_0)$.
and moreover
\[ u_0 \leq v_0 \quad \text{in} \quad \mathbb{R}^N, \quad f(x, v, \nabla v) \leq h(x, v, \nabla v) \quad \text{in} \quad [0, T] \times \mathbb{R}^N, \]
(respectively \( u_0 \geq v_0 \) in \( \mathbb{R}^N \), \( f(x, v, \nabla v) \geq h(x, v, \nabla v) \) in \( [0, T] \times \mathbb{R}^N \))

then
\[ u \leq v \quad \text{in} \quad [0, T] \times \mathbb{R}^N, \]
(respectively \( u \geq v \) in \( [0, T] \times \mathbb{R}^N \)).

ii) If \( 2\alpha - \frac{N}{p} > 1 \), \( p \in (1, \infty) \), \( \alpha \in [0, 1) \), \( u_0 \in \dot{W}^{2\alpha, p}_U(\mathbb{R}^N) \), \( v_0 \in C^{2+\mu}(\mathbb{R}^N) \) and
\[ \sup_{x \in \mathbb{R}^N} u_0(x) < \inf_{x \in \mathbb{R}^N} v_0(x) \]
(respectively \( \inf_{x \in \mathbb{R}^N} u_0(x) > \sup_{x \in \mathbb{R}^N} v_0(x) \))

then local solutions \( u(\cdot, u_0), v(\cdot, v_0) \) resulting from Proposition 1.1 satisfy in \( \mathbb{R}^N \) the relation
\[ u(t, u_0) \leq v(t, v_0) \]
(respectively \( u(t, u_0) \geq v(t, v_0) \))
as long as both \( u \) and \( v \) exist.

Proof. If \( u_0 \leq v_0 \) in \( \mathbb{R}^n \) and \( \ddot{v} \in C^{1+\frac{\mu}{2}+\mu}_b([0, T] \times \mathbb{R}^N) \) solves
\[ \ddot{v} - \Delta \ddot{v} = h(x, \ddot{v}, \nabla \ddot{v}), \quad t \in [0, T], \quad x \in \mathbb{R}^N, \quad \ddot{v}(0, x) = u_0(x), \quad x \in \mathbb{R}^N, \]
then by Lemma 2.3 ii) we have \( \ddot{v} \leq v \) in \( [0, T] \times \mathbb{R}^N \) and hence
\[ u_0 \leq v(t, x, v_0), \quad t \in [0, T], \quad x \in \mathbb{R}^N. \]

Since our assumptions ensure validity of (2.7) we obtain from Lemma 2.3 ii) that
\[ u \leq v \quad \text{in} \quad [0, T] \times \mathbb{R}^N, \]
as required in part i).

For the proof of ii) recall that the set \( C_b^\infty(\mathbb{R}^N) \) is dense in \( \dot{W}^{2\alpha, p}_U(\mathbb{R}^N) \), see the Appendix. The result then follows from i) and continuity with respect to initial data of the solutions resulting from Proposition 1.1. \( \square \)

Lemma 2.5 If (1.10) holds with \( \gamma_0 \in [1, 2) \) and (1.9) is satisfied then the solutions to (1.1) resulting from Proposition 1.1 are bounded in \( L^\infty(\mathbb{R}^N) \) and in \( \dot{W}^{2\alpha, p}_U(\mathbb{R}^N) \) uniformly for \( t \in [0, \infty) \) and \( u_0 \) varying in bounded subsets of \( \dot{W}^{2\alpha, p}_U(\mathbb{R}^N) \).
Proof. Since the constants $\pm M$ where $M > \max\{\|u_0\|_{L^\infty(\mathbb{R}^N)}, s_0\}$ are respectively supersolution and subsolution to (1.1), Lemma 2.3 i) ensures that local solutions to (1.1) through $u_0 \in C_{bd}^\infty(\mathbb{R}^N)$ are bounded a priori in $L^\infty(\mathbb{R}^N)$ uniformly for $t \in [0, \infty)$ and for $u_0$ varying in bounded subsets of $W^{2, p}_U(\mathbb{R}^N)$. The same property holds for the solutions through $u_0 \in W^{2, p}_U(\mathbb{R}^N)$ because $C_{bd}^\infty(\mathbb{R}^N)$ is dense in $W^{2, p}_U(\mathbb{R}^N)$, thus

$$
\sup_{t \in [0, \infty]} \sup_{\|u_0\|_{W^{2, p}_U(\mathbb{R}^N)} \leq R} \|u(t, u_0)\|_{L^\infty(\mathbb{R}^N)} \leq c(R). \tag{2.11}
$$

We next observe that (1.10) with $\gamma_0 \in [1, 2)$ leads to the estimate

$$
\|f(x, u, \nabla u)\|_{L^p_c(\mathbb{R}^N)} \leq g(\|u\|_{L^\infty(\mathbb{R}^N)})(1 + \|u\|^\theta_{W^{2, p}_U(\mathbb{R}^N)}), \tag{2.12}
$$

where $\max\{\alpha, \frac{20}{7}\} < \gamma < 1, \theta \in (\frac{1}{2\gamma}, \frac{1}{7})$, and $g : [0, \infty) \to [0, \infty)$ is a continuous function. There exists also $\tau_R > 0$ such that $\|u(t, u_0)\|_{W^{2, p}_U(\mathbb{R}^N)}$ is uniformly bounded with respect to $t \in [0, \tau_R]$, $\|u_0\|_{W^{2, p}_U(\mathbb{R}^N)} \leq R$; i.e.

$$
\mathcal{V} = \{v : \exists \|u_0\|_{W^{2, p}_U(\mathbb{R}^N)} \leq R, v = u(\tau_R, u_0)\} \text{ is a bounded subset of } \hat{W}^{2, p}_U(\mathbb{R}^N) \tag{2.13}
$$

(see [14, Proposition 2.3.1, 3.2.1]).

Using (2.12), (2.13) and a modification of the variation of constants formula (1.7) we obtain

$$
\begin{align*}
\|u(t, v)\|_{\hat{W}^{2, p}_U(\mathbb{R}^N)} &\leq \|e^{(\Delta - \omega I)t}\|_{L^p_c(\mathbb{R}^N), L^p_c(\mathbb{R}^N)} \|v\|_{W^{2, p}_U(\mathbb{R}^N)} \\
+ \int_0^t \|e^{(\Delta - \omega I)(t-s)}\|_{L^p_c(\mathbb{R}^N), W^{2, p}_U(\mathbb{R}^N)} g(\|u(s, v)\|_{L^\infty(\mathbb{R}^N)}) \left(1 + \|u(s, v)\|^\theta_{W^{2, p}_U(\mathbb{R}^N)}\right) ds \tag{2.14} \\
+ \int_0^t \|e^{(\Delta - \omega I)(t-s)}\|_{L^p_c(\mathbb{R}^N), W^{2, p}_U(\mathbb{R}^N)} \|u(s, v)\|_{L^p_c(\mathbb{R}^N)} ds.
\end{align*}
$$

We chose $\omega \in \mathbb{IR}$ in (2.14) in such a way that $\text{Re } \sigma (-\Delta + \omega I) > a > 0$ and then

$$
\|e^{(\Delta - \omega I)(t-s)}\|_{L^p_c(\mathbb{R}^N), W^{2, p}_U(\mathbb{R}^N)} \leq c_\beta \frac{e^{-at}}{\beta^3}, \quad t > 0, \quad \beta \in [0, 1]. \tag{2.15}
$$

Also $\|u(s, v)\|_{L^p_c(\mathbb{R}^N)}$ does not exceed a multiple of $\|u(s, v)\|_{L^\infty(\mathbb{R}^N)}$, which has already been estimated in (2.11). This allows us to get from (2.14) the inequality

$$
\begin{align*}
\sup_{s \in [0, t]} \sup_{v \in \mathcal{V}} \|u(s, v)\|_{W^{2, p}_U(\mathbb{R}^N)} &\leq c_0 \sup_{v \in \mathcal{V}} \|v\|_{W^{2, p}_U(\mathbb{R}^N)} \\
+ \sup_{s \in (0, C(R)]} \left(g(s) + \omega s\right) \left(1 + \sup_{s \in [0, t]} \sup_{v \in \mathcal{V}} \|u(s, v)\|^\theta_{W^{2, p}_U(\mathbb{R}^N)}\right) \int_0^\infty c_\gamma \frac{e^{-ay}}{y^\gamma} dy,
\end{align*}
$$

and observe that $z := \sup_{s \in [0, t]} \sup_{v \in \mathcal{V}} \|u(t, v)\|_{W^{2, p}_U(\mathbb{R}^N)}$ satisfies the relation

$$
0 \leq z \leq \text{const}(1 + z^\theta). \tag{2.16}
$$
Note that the value of \( const \) essentially depends on \( R \) and any number satisfying (2.16) is bounded above by the positive root \( z_0 = \tilde{c}(R) \) of the equation \( z = const(1 + z^\theta) \). The latter ensures the estimate

\[
\sup_{t \in [0, \infty)} \sup_{v \in V} \|u(t, v)\|_{W^{2\alpha,p}(\mathbb{R}^N)} \leq \tilde{c}(R). \tag{2.17}
\]

Having (2.17) and the uniform boundedness of \( \|u(t, u_0)\|_{W^{2\alpha,p}(\mathbb{R}^N)} \) with respect to \( t \in [0, \tau_R] \), \( \|u_0\|_{W^{2\alpha,p}(\mathbb{R}^N)} \leq R \) it is next routine to get

\[
\sup_{t \in [0, \infty)} \sup_{\|v\|_{\mathbb{R}^N} \leq R} \|u(t, u_0)\|_{W^{2\alpha,p}(\mathbb{R}^N)} \leq const(R), \tag{2.18}
\]

which is the desired bound on the solutions. \( \square \)

**Remark 2.6** As a consequence of Lemma 2.5 the solutions to (1.1) in \( \dot{W}^{2\alpha,p}(\mathbb{R}^N) \) exist globally in time and orbits of bounded sets of the semigroup \( \{T(t)\} \) corresponding to (1.1) in \( \dot{W}^{2\alpha,p}(\mathbb{R}^N) \) are bounded. Recalling that constant functions \( \pm M \) satisfying \( M > \max\{\|u_0\|_{L^\infty(\mathbb{R}^N)}, s_0\} \) are respectively supersolution and subsolution to (1.1) it is clear that the set \( \mathcal{R} \) defined in (1.11) is positively invariant. Part i) of Theorem 1.3 is thus proved.

**Lemma 2.7** If, in addition to assumptions of Lemma 2.5, condition (1.12) holds, then the \( L^\infty(\mathbb{R}^N) \) and \( \dot{W}^{2\alpha,p}(\mathbb{R}^N) \), \( 2\alpha - \frac{N}{p} > 1 \), estimates (2.11), (2.17) are asymptotically independent of \( u_0 \) varying in bounded subsets of \( \dot{W}^{2\alpha,p}(\mathbb{R}^N) \).

**Proof.** Note that, from (1.12), considering the solutions of the ODE

\[
\dot{U} = h(U), \quad t > 0, \tag{2.19}
\]

with initial condition \( U(0) = s \in \mathbb{R} \), if \( u_0 \) belongs to a ball in \( \dot{W}^{2\alpha,p}(\mathbb{R}^N) \) of radius \( R > 0 \) and \( M > s_0 \) is chosen sufficiently large then, by Lemma 2.4, \( u(t, u_0) \) will remain between the solutions \( U(t, -M) \) and \( U(t, M) \) to (2.19). Evidently \( U(t, \pm M) \) approaches appropriate zeros of \( h(s) \) as \( t \to \infty \), which translates into the estimate

\[
\limsup_{t \to \infty} \sup_{\|u_0\|_{W^{2\alpha,p}(\mathbb{R}^N)} \leq R} \|u(t, u_0)\|_{L^\infty(\mathbb{R}^N)} \leq s_0. \tag{2.20}
\]

The above relation ensures that if \( R > 0 \) is arbitrarily fixed, there exists \( t_R > 0 \) such that any solution with \( \|u_0\|_{W^{2\alpha,p}(\mathbb{R}^N)} \leq R \) remains bounded by \( s_0 \) uniformly for \( t > t_R \) and \( x \in \mathbb{R}^N \). From (2.14), (2.11), (2.17), (2.20) we then have

\[
\|u(t, v)\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)} \leq \|e^{\lambda t}L(u_0)\|_{L(p, L_p(\mathbb{R}^N), 2\alpha)} \|v\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)}
\]

\[
+ \int_0^\tau \|e^{\lambda (t-s)}L(u_0)\|_{L(p, \dot{W}^{2\alpha,p}(\mathbb{R}^N), 2\alpha)} \sup_{s \in [0, C(R)]} (g(s) + \omega s) \left(1 + [\tilde{c}(R)]^\theta\right) ds
\]

\[
+ \int_\tau^T \|e^{\lambda (t-s)}L(u_0)\|_{L(p, \dot{W}^{2\alpha,p}(\mathbb{R}^N), 2\alpha)} \sup_{s \in [0, s_0]} (g(s) + \omega s) \left(1 + \|u(s, v)\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)}\right) ds, \quad \tau > t_R,
\]

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and consequently, using (2.15),
\[
\limsup_{t \to \infty} \sup_{v \in V} \|u(t, v)\|_{\dot{W}^{2, \alpha,p}(\mathbb{R}^N)} \\
\leq \sup_{s \in [0, s_0]} (g(s) + \omega s) \left( 1 + \sup_{s \in [r, \tau]} \sup_{v \in V} \|u(s, v)\|_{\dot{W}^{2, \alpha,p}(\mathbb{R}^N)}^\vartheta \right) \int_0^\infty c_\gamma e^{-\alpha y} \, dy, \quad \tau > t_R.
\]

The above inequality ensures that \( \limsup_{t \to \infty} \sup_{v \in V} \|u(t, v)\|_{\dot{W}^{2, \alpha,p}(\mathbb{R}^N)} \) satisfies the relation like (2.16) but now with constant independent of \( R > 0 \). We thus obtain
\[
\limsup_{t \to \infty} \sup_{v \in V} \|u(t, v)\|_{\dot{W}^{2, \alpha,p}(\mathbb{R}^N)} \leq z_0,
\]
where \( z_0 \) is the positive root of the equation
\[
z = \left[ \sup_{s \in [0, s_0]} (g(s) + \omega s) \int_0^\infty c_\gamma e^{-\alpha y} \, dy \right] (1 + z_0^\vartheta)
\]
and thus \( z_0 \) does not depend on \( R > 0 \).

After establishing asymptotic bounds on the solutions the next result shows that condition (2.5) is satisfied. The more difficult part is, of course, proving that the limit points are in fact in \( \dot{W}^{2, \alpha,p}(\mathbb{R}^N) \).

**Lemma 2.8** Under the assumptions of Lemma 2.7 the semigroup \( \{T(t)\} \) of global solutions to (1.1) in \( \dot{W}^{2\alpha,\alpha,p}(\mathbb{R}^N) \) possesses an absorbing set \( B_0 \), bounded in \( \dot{W}^{2,p}(\mathbb{R}^N) \) and positively invariant under \( \{T(t)\} \).

Furthermore, for any sequence of the form \( \{T(t_n)u_n\} \), where \( t_n \to \infty \) and \( \{u_n\} \subset B_0 \), there exists a subsequence that converges in \( W \) to certain \( w \in \dot{W}^{2,\alpha,p}(\mathbb{R}^N) \), where \( W \) denotes either \( C_{loc}^{1+\mu}(\mathbb{R}^N) \), with \( 1 - \frac{N}{p} > \mu > 0 \), or the weighted space \( W^{s,p}_\rho(\mathbb{R}^N), 0 \leq s < 2 \), where \( \rho \) is any translation of \( \rho_0(x) = (1 + |x|^2)^{-\nu} \) and \( \nu > \frac{N}{2} \).

**Proof.** It has already been mentioned in Remark 2.6 that (1.1) generates on \( \dot{W}^{2,\alpha,p}(\mathbb{R}^N) \) a \( C^0 \) semigroup \( \{T(t)\} \) with bounded orbits of bounded sets. Estimate (2.21) ensures now that \( \{T(t)\} \) has a bounded absorbing set \( B_0 \subset \dot{W}^{2,\alpha,p}(\mathbb{R}^N) \). The positive orbit \( \gamma^+(B_0) \) of the absorbing set \( B_0 \) is also bounded in \( \dot{W}^{2,\alpha,p}(\mathbb{R}^N) \). Moreover, as a result of [14, Lemma 3.2.1], the set \( T(1)\gamma^+(B_0) \) is in fact bounded in \( \dot{W}^{2,p}(\mathbb{R}^N) \) so that the first assertion of the lemma holds with \( B_0 := T(1)\gamma^+(B_0) \).

For the proof of the second assertion consider a sequence \( \{T(t_n)u_n\} \), where \( t_n \to \infty \) and \( \{u_n\} \subset B_0 \). Using the compact embeddings (2.3) and (2.4) we choose a subsequence \( \{T(t_{n_k})u_{n_k}\} \), which is convergent in \( W^{1,p}_\rho(\mathbb{R}^N) \) and in \( C_{loc}^{1+\mu}(\mathbb{R}^N) \), to a function \( w \) in these spaces. Hence, it remains to prove that \( w \in \dot{W}^{2,\alpha,p}(\mathbb{R}^N) \).

Since \( \{T(t_{n_k})u_{n_k}\} \subset B_0 \) is bounded in \( \dot{W}^{2,p}(\mathbb{R}^N) \), then in particular we have that \( \|T(t_{n_k})u_{n_k}\|_{W^{2,p}(B(y,1))} \leq C \) for every \( k \) and \( y \in \mathbb{R}^N \). Then by weak convergence in \( W^{2,p}(B(y,1)) \) we get that \( \|w\|_{W^{2,p}(B(y,1))} \leq C \) for every \( y \in \mathbb{R}^N \) and then \( w \in \dot{W}^{2,\alpha,p}(\mathbb{R}^N) \), as will be shown in the Appendix. \( \square \)
Remark 2.9: From the proof of Lemma 2.8, we obtain that under the assumptions of Lemma 2.7 there exists $C_{B_0} > 0$ such that if $w$ is a limit in $W^s_{\rho}(\mathbb{R}^N)$ with certain $s \in [0,2)$ of a sequence $\{T(t_n)u_n\}$, where $t_n \to \infty$ and $\{u_n\} \subset B_0$, then

$$\|w\|_{W^s_{\rho}(\mathbb{R}^N)} \leq C_{B_0}.$$  

Hence the set $A$ in Proposition 2.1 is actually bounded in $W^2_{U}(\mathbb{R}^N) \hookrightarrow \dot{W}^{2\alpha}_{U}(\mathbb{R}^N)$. Therefore, as a consequence of [14, Lemma 3.2.1], $A$ is in fact bounded in $W^{2\alpha}_{U}(\mathbb{R}^N)$.

Now, in the next two lemmas, we show that the generalized asymptotic smoothness condition (2.6) is satisfied.

Lemma 2.10: Under the assumptions of Lemma 2.7, if $\gamma \in [0,1)$, $\{u_n\} \subset B_0$, $u_0 \in W^{2\alpha}_{U}(\mathbb{R}^N) \cap W^{2\gamma}_{U}(\mathbb{R}^N)$ and $\{u_n\}$ tends to $u_0$ in $W^{2\gamma}_{\rho}(\mathbb{R}^N)$, then $\{u(t, u_n)\}$ converges to $u(t, u_0)$ in $W^{2\gamma}_{\rho}(\mathbb{R}^N)$ for each $t > 0$.

Proof. As shown in [7], operator $A$ considered in $L^{p}_\rho(\mathbb{R}^N)$ with the domain $W^{2\alpha}_{\rho}(\mathbb{R}^N)$ is sectorial in $L^{p}_\rho(\mathbb{R}^N)$, $1 < p < \infty$. Furthermore, fractional power spaces associated to $A$ in $L^{p}_\rho(\mathbb{R}^N)$ coincide with the complex interpolation spaces, (see the Appendix)

$$W^{2\gamma}_{\rho}(\mathbb{R}^N) = [L^{p}_\rho(\mathbb{R}^N), W^{2\alpha}_{\rho}(\mathbb{R}^N)]_{\beta}.$$  

(2.22)

If $\mathcal{F}_f(\phi)(x) := f(x, \phi(x), \nabla \phi(x))$, $\phi \in \dot{W}^{2\gamma}_{U}(\mathbb{R}^N)$ and $B_0$ is an invariant absorbing set from Lemma 2.8 then

$$\|\mathcal{F}_f(v) - \mathcal{F}_f(w)\|_{L^{\rho}_{\beta}(\mathbb{R}^N)} \leq L\|v - w\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)}, \quad v, w \in B_0, \quad \beta \in \left[\frac{1}{2}, 1\right),$$  

(2.23)

because $f$ is locally Lipschitz and $\sup_{\phi \in B_0} \max_{|\sigma| \leq 1} |D^\sigma \phi(x)|$ is bounded by a multiple of $\sup_{\phi \in B_0} \|\phi\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)}$. Also, if $\{u_n\} \subset B_0$ converges to $u_0$ in $W^{2\gamma}_{\rho}(\mathbb{R}^N)$ for certain $\gamma \in [0,1]$, then Lemma 2.8 ensures that in fact $u_0$ belongs to $W^{2\gamma}_{\rho}(\mathbb{R}^N)$ with any $\gamma \in [0,1]$ and $\{u_n\}$ is convergent to $u_0$ in the latter spaces as well. It is thus clear that it suffices to prove the assertion for $\gamma$ close to 1.

Thanks to (2.23), for the solutions $u(t, u_n)$, $u(t, u_0)$ to (1.1) and each $\gamma \in [\frac{1}{2}, 1)$, $0 < t < T < \infty$ we have

$$\|u(t, u_n) - u(t, u_0)\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)} \leq \|e^{\Delta t}\|_{L^{\rho}_{\beta}(\mathbb{R}^N), L^{\rho}_{\alpha}(\mathbb{R}^N)}\|u_n - u_0\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)}$$

$$+ \int_0^t \|e^{\Delta(t-s)}\|_{L^{\rho}_{\beta}(\mathbb{R}^N), W^{2\gamma}_{\rho}(\mathbb{R}^N)}\|\mathcal{F}_f(u(s, u_n)) - \mathcal{F}_f(u(s, u_0))\|_{L^{\rho}_{\beta}(\mathbb{R}^N)}ds$$

$$\leq c(\tau)\|u_n - u_0\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)} + L\tau^\gamma(\gamma)\int_0^t (t-s)^{-\gamma}\|u(s, u_n) - u(s, u_0)\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)}ds.$$  

Application of the singular Gronwall lemma (see [26]) allows us to conclude that $\|u(t, u_n) - u(t, u_0)\|_{W^{2\gamma}_{\rho}(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for arbitrarily chosen $t > 0$.  


Lemma 2.11 Suppose that the assumptions of Lemma 2.7 hold.

If \( t_n \to \infty \) and \( \{u_n\} \subset B_0 \), then \( \{T(t_n)u_n\} \) contains a subsequence \( \{T(t_{n_k})u_{n_k}\} \) convergent in \( C^{1+\mu}_{loc}(IR^N) \), \( \mu \in (0, 1 - \frac{N}{p}) \), to certain \( v \in W^{2s,p}_{U}(IR^N) \) and, in addition,

\[
T(t_{n_k} + t)u_{n_k} \xrightarrow{C^{1+\mu}_{loc}(IR^N)} T(t)v \quad \text{for each } t > 0. 
\]

Proof. Let \( t_n \to \infty \) and \( \{u_n\} \subset B_0 \). From Lemma 2.8 there exists \( \{T(t_{n_k})u_{n_k}\} \) and \( v \in W^{2s,p}_{U}(IR^N) \) such that \( \{T(t_{n_k})u_{n_k}\} \) converges to \( v \) both in \( C^{1+\mu}_{loc}(IR^N) \), \( \mu \in (0, 1 - \frac{N}{p}) \), and in \( W^{s,p}_{\rho}(IR^N) \) \( s \in [0, 2) \). Hence from Lemma 2.10, we get

\[
T(t_{n_k} + t)u_{n_k} \xrightarrow{W^{s,p}_{\rho}(IR^N)} T(t)v \quad \text{for each } t > 0. 
\]

Since we can apply Lemma 2.8 to arbitrarily chosen subsequence of \( \{T(t_{n_k} + t)u_{n_k}\} \), we get from (2.24) that

\[
T(t_{n_k} + t)u_{n_k} \xrightarrow{C^{1+\mu}_{loc}(IR^N)} T(t)v \quad \text{for each } t > 0, \mu \in (0, 1 - \frac{N}{p}). \quad \Box
\]

Hence (2.6) is satisfied and Corollary 2.2 ensures the existence of a \( (\hat{W}^{2s,p}_{U}(IR^N) - W) \) attractor for (1.1), where \( W \) is either a Hölder space \( C^{1+\mu}_{loc}(IR^N) \), \( \mu \in (0, 1 - \frac{N}{p}) \), or a weighted space \( W^{s,p}_{\rho}(IR^N) \), \( s \in [0, 2) \). These attractors are independent of the choice of \( W \) and can be described as

\[
A := \{w \in \hat{W}^{2s,p}_{U}(IR^N) : T(t)u_n \xrightarrow{W} w \ \text{for certain } \{u_n\} \subset B_0 \ \text{and} \ t_n \to \infty \},
\]

see Remark 2.9.

To complete the proof of Theorem 1.3 we prove that when \( f \) does not depend on \( x \), then \( A \) is invariant with respect to the group of translations in \( IR^N \).

Lemma 2.12 Suppose that the assumptions of Lemma 2.7 hold and assume moreover that \( f \) does not depend on \( x \), that is, \( f = f(s, p) \). Then

\[
w(\cdot) \in A \quad \text{if and only if} \quad w(\cdot - z) \in A \quad \text{for each } z \in IR^N.
\]

Proof. Note that if \( \phi \in \hat{W}^{2s,p}_{U}(IR^N) \) then \( \phi(\cdot - z) \in \hat{W}^{2s,p}_{U}(IR^N) \) for each \( z \in IR^N \) and recall that \( B_0 \) is positively invariant absorbing set for \( \{T(t)\} \), which is also bounded in \( \hat{W}^{2s,p}_{U}(IR^N) \). Fix \( z \in IR^N \), \( w \in A \) and \( t_n \to \infty \), \( \{u_n\} \subset B_0 \) such that \( T(t_n)u_n \to w \) in \( W \) being either a Hölder space \( C^{1+\mu}_{loc}(IR^N) \), \( \mu \in (0, 1 - \frac{N}{p}) \), or a weighted space \( W^{s,p}_{\rho}(IR^N) \), \( s \in [0, 2) \). In particular \( T(t_n)u_n \to w \) point-wise in \( IR^N \). Then \( w(\cdot - z) \in \hat{W}^{2s,p}_{U}(IR^N) \), the sequence \( \{u_n(\cdot - z)\} \) is bounded in \( \hat{W}^{2s,p}_{U}(IR^N) \) and consequently \( T(t)\{u_n(\cdot - z)\} \subset B_0 \) for \( t \geq t_0 \).

Define further

\[
\tilde{t}_n = t_{n_0 + n} - t_0, \quad \tilde{u}_n = T(t_0)u_{n_0 + n}(\cdot - z), \ n \in \mathbb{N}.
\]
Note that \( \tilde{t}_n \to \infty, \{\tilde{u}_n\} \subset B_0 \) and consider \( T(\tilde{t}_n)\{\tilde{u}_n\} \).

By Lemma 2.8 there is a subsequence \( \{T(\tilde{t}_{n_k})\tilde{u}_{n_k}\} \) convergent to a certain \( \tilde{w} \in A \) in \( W \). In particular the convergence is point-wise in \( \mathbb{R}^N \). Since \( T(\tilde{t}_{n_k})\tilde{u}_{n_k}(\cdot) = T(t_{n_0+n_k})u_{n_0+n_k}(\cdot-z) \) and \( T(t_{n_0+n_k})u_{n_0+n_k}(x) \to w(x) \) for \( x \in \mathbb{R}^N \) we obtain that \( T(\tilde{t}_{n_k})\tilde{u}_{n_k}(x) \to w(x-z) \) for \( x \in \mathbb{R}^N \). This ensures that \( \tilde{w}(\cdot) = w(\cdot-z) \) and thus \( w(\cdot-z) \in A \). \( \square \)

**Remark 2.13** Note that in the situation above translations in \( \mathbb{R}^N \) are symmetries of the equation, in the sense that for all \( t \geq 0 \) and \( y \in \mathbb{R}^N \) we have

\[
\tau_y T(t) = T(t) \tau_y.
\]

In this context the result in Lemma 2.12 can be generalized in the following terms: if \( g \) is a bounded linear transformation in \( \dot{W}^{2,\alpha,p}_U(\mathbb{R}^N) \) such that

\[
g \circ T(t) = T(t) \circ g
\]

then \( g(A) = A \); see Propositions 1.2 and 1.3 in [37].

**2.2 Proof of Corollary 1.4**

If (1.5) has an attractor in \( \mathbb{R} \), then \( sf(s,0) < 0 \) for large \( |s| \). Then the ‘if’ part follows from Theorem 1.3 with \( h(s) = f(s,0) \).

On the other hand, assume that (1.4) has a bounded \((\dot{W}^{1,\alpha,p}_U(\mathbb{R}^N) - L^p_0(\mathbb{R}^N))\)-attractor \( A \), where \( \rho_0(x) = (1 + |x|^2)^{-\nu} \) with \( \nu > N/2 \). In particular, if \( B(r_A) \) is a ball in \( L^p_0(\mathbb{R}^N) \) centered at zero and containing \( A \), then the orbit of any bounded set \( B \subset \dot{W}^{1,\alpha,p}_U(\mathbb{R}^N) \) must enter \( B(r_A) \) in certain time \( t_B \geq 0 \) and remain inside this ball for all \( t \geq t_B \). Note that this happens as well with the solutions \( U(\cdot, U_0) \) of (1.5) because they also solve (1.4).

Solutions \( U(\cdot, U_0) \) are thus defined globally in time and, moreover,

\[
\lim_{t \to +\infty} \|U(t, U_0)\|_{L^p_0(\mathbb{R}^N)} = \lim_{t \to +\infty} \sup \|U(t, U_0)\| \left( \int_{\mathbb{R}^N} \rho_0 dx \right)^{\frac{1}{p}} \leq r_A.
\]

Hence (1.5) has an attractor.

**2.3 Proof of Theorem 1.5**

In this subsection we discuss the Cauchy problem for equation

\[
\dot{u} = \Delta u + f(x, u), \ x \in \mathbb{R}^N, \ t > 0,
\]

(2.25)

where the function \( f: \mathbb{R}^{N+1} \to \mathbb{R} \) is locally Lipschitz continuous with respect to \( u \) uniformly for \( x \in \mathbb{R}^N \), and Hölder continuous with respect to \( x \) uniformly for \( u \) varying in bounded subsets of \( \mathbb{R}^N \). We consider \( u_0 \in \dot{W}^{2,\alpha,p}_U(\mathbb{R}^N) \) with \( \alpha \in [0, 1), p > \frac{N}{2}, p \geq 2, 2\alpha - \frac{N}{p} > 0 \) and use comparison techniques (see e.g. [5, Appendix A]) to derive suitable estimates of the solutions.
Lemma 2.14 Under the above assumptions and (1.13) the solutions \( u(t, u_0) \) to (2.25) are bounded in \( L^\infty(\mathbb{R}^N) \) on any compact time interval uniformly with respect to \( u_0 \) varying in bounded subsets of \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \).

Proof. Note that (1.13) implies the relations
\[
\begin{align*}
  f(x, s) &\leq C(x)s + D(x), \ s \geq 0, \ x \in \mathbb{R}^N, \\
  f(x, s) &\geq C(x)s - D(x), \ s \leq 0, \ x \in \mathbb{R}^N.
\end{align*}
\]
Since the resolvent \((\lambda I - \Delta)^{-1}\) of \( \Delta \) in \( L^p(\mathbb{R}^N) \) is increasing for large \( \lambda \), the heat semigroup \( \{e^{\Delta t}\} \) is \textit{order preserving} in locally uniform spaces, (see [7, Proposition 5.22, Remark 5.11]).

Therefore, by monotonicity and comparison with the solutions of a linear problem in \( \dot{L}^p(\mathbb{R}^N) \)
\[
U_t = \Delta U + C(x)U + D(x), \ U(0) = |u_0|,
\]
we observe that \(|u(t, u_0)|\) must stay below \( U(t, |u_0|) \). The latter function is bounded in \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \) on compact subintervals of \((0, \infty)\) and uniformly for \( \|u_0\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)} \leq R \). We also remark that for any \( R > 0 \) there exists \( \tau_R > 0 \) such that solutions \( u(\cdot, u_0) \) to (2.25) with \( \|u_0\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)} \leq R \) are bounded in \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \) uniformly for \( t \in [0, \tau_R] \) and \( \|u_0\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)} \leq R \) (see [14, Proposition 2.3.1]). Since under our assumptions on \( \alpha \) and \( p \) we have that \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N) \), the required \( L^\infty(\mathbb{R}^N) \) estimate of \( u(\cdot, u_0) \) is straightforward. \(\square\)

Lemma 2.15 If additionally \( \text{Re}(-\Delta - C(\cdot)I) > 0 \), then the \( L^\infty(\mathbb{R}^N) \) estimate obtained in Lemma 2.14 can be strengthened from compact time intervals to the whole half line \([0, \infty)\) and, moreover, it remains asymptotically independent of initial condition \( u_0 \) varying in bounded subsets of \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \).

Proof. Since \( \text{Re} \sigma(-\Delta - C(\cdot)I) > 0 \), the solution to (2.26) can be decomposed as
\[
U = V + \Phi,
\]
where
\[
\Phi = (-\Delta - C(\cdot)I)^{-1}D \in \dot{W}^{2,p}_U(\mathbb{R}^N)
\]
and \( V \) solves the linear equation
\[
V_t = \Delta V + C(x)V, \ t > 0, \ x \in \mathbb{R}^N.
\]
We thus have that
\[
U = e^{(\Delta + C(\cdot)I)t}(U(0) - \Phi) + \Phi,
\]
and, recalling that the semigroup \( \{e^{(\Delta + C(\cdot)I)t}\} \) is exponentially decaying, we obtain the estimate of \( U \) in \( \dot{W}^{2\alpha,p}_U(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N) \) uniform on infinite time intervals away from \( t = 0 \). The latter estimate is next translated into \( L^\infty(\mathbb{R}^N) \) bound on \( u(\cdot, u_0) \) based on comparison methods. As a consequence we obtain both
\[
\sup_{t \geq 0} \sup_{\|u_0\|_{\dot{W}^{2\alpha,p}(\mathbb{R}^N)} \leq R} \|u(t, u_0)\|_{L^\infty(\mathbb{R}^N)} \leq C(R)
\]
\[
\limsup_{t \to \infty} \sup_{\|u_0\|_{W^{2,p}(\mathbb{R}^N)} \leq R} \|u(t, u_0)\|_{L^\infty(\mathbb{R}^N)} \leq \|\Phi\|_{L^\infty(\mathbb{R}^N)}. \quad (2.30)
\]

The remaining part of the proof of Theorem 1.5 is similar as in the case of Theorem 1.3. In particular, the existence of an attractor follows as before from Corollary 2.2.

3 Comparison with known results and examples

In the last fifteen years a number of problems in unbounded domains originating in applications have been intensively studied \cite{1, 6, 9, 12, 13, 14, 15, 16, 18, 19, 20, 23, 32, 33, 34, 45}. The references contain merely a sample list of the related publications. Observe that in none of these references the existence of an attractor for a scalar semilinear parabolic equation in the whole of \(\mathbb{R}^N\) is obtained, as in this paper, under the same assumptions as employed for problems in bounded domains. Also in contrary to the method used in a number of references devoted to the Cauchy problems in \(\mathbb{R}^N\), like e.g. \cite{9, 18, 19, 45}, a functional analytic setting and a consequent analytic semigroup approach in locally uniform spaces constitute in the present paper a main tool both to obtain local well posedness result and to derive sufficiently smooth estimates of the solutions. Such an approach is used in the monographs \cite{26, 2, 31, 14} and the present paper remains in the same vein.

Very recently, attractors for semilinear parabolic equations and systems have been investigated in \cite{13, 18, 19} and \cite{45}. Here we compare several known structure conditions on nonlinear terms with the ones in this paper.

**Example 3.1** As for the gradient dependent nonlinearities \(f = f(x, u, \nabla u)\) in scalar equations the main ‘structure’ condition of \cite{18, 19} reads:

\[
f(x, u, \nabla u) = -\lambda_0 u + f_0(u, \nabla u) + g(x), \quad \text{where } u f_0(u, \nabla u) \leq c \quad \text{and} \quad \lambda_0 > 0,
\]

plus the subquadratic growth restriction concerning \(\nabla u\). In \cite{18} a polynomial bound of \(|f|\) with respect to \(u\) as well as a strong ‘sign’ condition \(u f_0(u, \nabla u) \leq 0\) were also required. The latter two requirements were weakened in \cite{19}; however, in both mentioned publications some assumptions on the derivatives \(f'_u, f'_{\nabla u}\) are used.

The results of the present paper allow us to consider nonlinearities for which (3.1) is not satisfied, e.g.

\[
f(x, u, \nabla u) = a(x, u)|\nabla u|^{\gamma_0} + b(x, u),
\]

with \(0 \leq \gamma_0 < 2, a(x, s), b(x, s)\) locally Lipschitz with respect to \(s\) uniformly for \(x \in \mathbb{R}^N\) and Hölder continuous with respect to \(x\) uniformly for \(s\) varying in bounded subsets of \(\mathbb{R}\) and such that \(sb(x, s) \leq sh(s) < 0\) for \(|s| > s_0 > 0, x \in \mathbb{R}^N\).

In particular the Burgers equation

\[
u_t = u_{xx} + uu_x, \quad x \in \mathbb{R}, \quad t > 0,
\]

with \(\gamma_0 = 1\), \(a(x, u) = 1\), \(b(x, u) = u\), and \(\gamma_0 = 1\), \(a(x, s) = 1\), \(b(x, s) = s\) locally Lipschitz and Hölder continuous with respect to \(s\) and \(x\), respectively, for \(x \in \mathbb{R}^N\) and \(s \in \mathbb{R}\), the above conditions are satisfied. However, the solutions of (3.2) are not uniformly bounded. To overcome this difficulty, we will employ in the next section an additional hypothesis on the growth of the nonlinear terms at infinity.
is an important example for which (3.1) fails although Theorem 1.3 i) applies.

Even if a stabilizing and a suitable source term are added in (3.2) so that we have

\[ u_t = u_{xx} - \lambda_0 u + uu_x + g(x), \quad x \in \mathbb{R}, \quad t > 0, \tag{3.3} \]

condition (3.1) of [18] fails again, whereas the applicability of Theorem 1.3 ii) is straightforward.

**Example 3.2** For the non-gradient case, the assumptions in [45] come from the earlier publication [9] and read:

\[ f(u) = -\lambda_0 u + f_0(u) \quad \text{where} \quad \lambda_0 > 0, \]
\[ uf_0(u) \leq c, \quad f'_0(u) < \tilde{c} \]
\[ |f(u)| \leq C(1 + |u|^p) \quad \text{with} \quad p < p_0 = \frac{N}{N-4} \text{ if } N > 4 \text{ or } p \text{ arbitrarily large if } N \leq 4. \tag{3.4} \]

Note that in [9] the maximal exponent \( p_0 \) could not exceed \( \min\{\frac{4}{N}, \frac{2}{N-2}\} \) for \( N \geq 3 \), in which case a nonlinearity like \( u(1-|u|^{r-1}) \) in space dimensions \( N \geq 3 \) requires a severe growth restriction. In [13] the nonlinearity \( f = f(u) \) was much like in (3.4) above but the growth exponent \( p \) could be arbitrarily large independently of the space dimension \( N \).

Conditions (3.4) imply \( uf(u) \leq -\lambda_0 u^2 + c \) while, as shown in the present paper, a necessary and sufficient condition for the existence of an attractor is in fact the much weaker condition \( uf(u) < 0 \) for large \( |s| \) (see Corollary 1.4).

Dissipativeness assumption (1.13) appearing in Theorem 1.5 has already been exploited in [6] in the discussion of attractors for reaction-diffusion equations in ‘standard’ Lebesgue spaces. In the present (locally uniform space) setting the mentioned assumption becomes even more flexible in applications. Also in some special situations it is possible to observe that the solutions enter asymptotically a ‘smaller’ space consisting of integrable functions. Observe that it is easy to see, e.g. considering a nonzero constant initial data in the heat equation, that this latter property cannot be true in general.

**Example 3.3** Recall that a sample nonlinearity

\[ f(x, s) = m(x)s - s^3, \tag{3.5} \]

has been studied in [32] in a \( BUC(\mathbb{R}^N) \) setting under the requirement that the function \( m \in C^\mu(\mathbb{R}^N) \) was a difference of the continuous maps \( m^+ \) and \( m^- \), \( m^+ \in C_0(\mathbb{R}^N) \) and \( m^- \) is such that \( \{e^{(\Delta - m^-)t}\} \) is asymptotically decaying.

In [6] the same model example was considered in standard \( L^p(\mathbb{R}^N) \) spaces. It was then shown that the equation is dissipative in that space under the assumption that \( m \in L^p(\mathbb{R}^N), \quad p > \frac{N}{2}, \quad p \geq 2, \quad \text{the semigroup } \{e^{(\Delta - m^-)t}\} \text{ decays exponentially and} \quad \lim_{R \to \infty} \|m^+\|_{L^p(\mathbb{R}^N \setminus B(0,R))} = 0. \]
With the techniques of this paper, note that the assumptions in Theorem 1.5 are satisfied only assuming \( m \in \dot{L}^p(U IR^N) \), \( p > \frac{N}{2} \), \( p \geq 2 \). Indeed, the embedding \( \dot{W}^{2\alpha, p}(IR^N) \hookrightarrow L^\infty(IR^N) \), \( 2\alpha > \frac{N}{p} \), ensures that the corresponding Nemitskii map takes \( \dot{W}^{2\alpha, p}(IR^N) \) into \( \dot{L}^p(U IR^N) \) and is Lipschitz continuous on bounded sets.

On the other hand, note that (1.13) is satisfied with \( C \) equal to a negative constant and \( D(x) \) being a multiple of \( |m(x) - C|^\frac{3}{2} \) so that Theorem 1.5 applies.

In particular, the case \( m(x) = 1 \), i.e. the bistable equation in the Introduction, falls within the range of our results. The travelling waves for this model belong to the attractor and are actual connections. Note that much stronger attracting properties are obtained here than in [21, 22].

In the same spirit, with the results in this paper we can consider nonlinearities of the form,

\[
 f(x, u) = m(x)u - n(x)u^3 + g(x)
\]

with \( m \) and \( n \geq 0 \).

Then (1.13) is satisfied with \( C \) equal to a negative constant and \( D(x) \) is \( |g(x)| \) plus a multiple of \( \left( \frac{m(x) - C}{n(x)} \right)^2 \). Thus, Theorem 1.5 applies provided \( D \in \dot{L}^p(U IR^N) \).

**Example 3.4** In [6] a result analogous to Theorem 1.5 was given in standard \( L^p(IR^N) \) spaces. In fact if (1.13) is satisfied with

\[
 C \in \dot{L}^p(U IR^N) \quad \text{and} \quad D \in L^p(IR^N),
\]

where \( p > \frac{N}{2} \), \( p \geq 2 \), and the semigroup \( \{ e^{(\Delta + C(\cdot)I)t} \} \) is asymptotically decaying, then Theorem 1.5 and the results in [6] apply. Hence, the corresponding semigroup of global solutions is dissipative both in \( \dot{W}^{2\alpha, p}(IR^N) \subset L^\infty(IR^N) \) or in a Bessel potential space \( H^{2\alpha, p}(IR^N) \subset \dot{W}^{2\alpha, p}(IR^N) \), see [6]. Hence we denote here the corresponding attractors \( A_U \) and \( A \) respectively.

We show now that in fact \( A_U = A \). Note, however, \( A \) attracts in different norms solutions depending whether the initial data is in \( \dot{L}^p(U IR^N) \) or in \( L^p(IR^N) \). Also, note that the inclusion \( A \subset A_U \) is immediate.

Referring to the proof of Theorem 1.5 we observe that the absolute value of any solution to (2.25) will stay below \( U \), where

\[
 U = e^{(\Delta + C(\cdot)I)t}|u_0| + \int_0^t e^{(\Delta + C(\cdot)I)(t-s)}D ds, \quad t > 0.
\]

Note that \( e^{(\Delta + C(\cdot)I)t}|u_0| \) tends to zero in \( L^\infty(IR^N) \) as \( t \to \infty \) and, since \( 0 \leq D \in L^p(IR^N) \), then \( \int_0^t e^{(\Delta + C(\cdot)I)(t-s)}D ds \) converges in \( H^{2-\epsilon, p}(IR^N) \) to a certain \( 0 \leq \Phi \in H^{2p}(IR^N) \) (see (2.26), (2.27) and (2.28)).

This gives that for any \( u \in A_U \) we have \( |u(x)| \leq \Phi(x) \), for \( x \in IR^N \). In particular, \( A_U \) is bounded in \( L^p(IR^N) \cap L^\infty(IR^N) \).
With this, from (1.7), we have for every $u_0 \in A_U$ and $t > 0$

$$u(t, u_0) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(\cdot, u(s, u_0))
ds.$$ 

where $f(\cdot, u(s, u_0))$ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Hence, using that $A_U$ is invariant, we get from this, taking $t = 1$ that $A_U$ is actually bounded in $H^{2q,p}(\mathbb{R}^N)$. Now the invariance and attractivity of $A$ implies $A_U \subseteq A$.

Also systems of semilinear parabolic equations can be successfully treated within the approach of the present paper provided that a nonlinear vector field $f = (f_1, \ldots, f_N)$ satisfies appropriate quasi-monotonicity assumption (“Kamke condition”, see [39] or “Ważewski condition”, see [43]).

**Example 3.5** Let $y = (y_1, \ldots, y_N)$, $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_N) \in \mathbb{R}^N$. We say that

$$y \preceq \tilde{y} \quad \text{if} \quad y_j \leq \tilde{y}_j \quad \text{for} \quad j = 1, \ldots, N$$

and

$$y \prec \tilde{y} \quad \text{if} \quad y_j \leq \tilde{y}_j \quad \text{for} \quad j = 1, \ldots, N \quad \text{and} \quad y_i = \tilde{y}_i.$$ 

Let $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be locally Lipschitz continuous vector field which is moreover quasi-monotone increasing; that is for each $i = 1, \ldots, N$,

$$y \preceq \tilde{y} \quad \text{implies that} \quad f_i(y) \leq f_i(\tilde{y}).$$

Suppose that the system of ordinary differential equations

$$\dot{U} = f(U), \quad t > 0, \quad U(0) = U_0 \in \mathbb{R}^N,$$ 

has an attractor in $\mathbb{R}^N$ and consider the parabolic Cauchy problem

$$\begin{cases}
    u_t = \Delta u + f(u), \quad t > 0, \quad x \in \mathbb{R}^N, \\
    u(0) = u_0 \in [W^{1,p}_U(\mathbb{R}^N)]^N, \quad p > N.
\end{cases} \quad (3.7)$$

If $u(t, u_0)$ is a solution to (3.7) and $u_0(x)$ belongs for each $x \in \mathbb{R}^N$ to an ordered interval $[-M, M] \subseteq \mathbb{R}^N$, then by comparison arguments (see e.g. [11, Lemma 3]) the solution $u(t, u_0)$ will be estimated above and below by $U(t, \pm M)$ as long as $u(t, u_0)$ exists. This ensures that the solutions to (3.7) are bounded in $[L^\infty(\mathbb{R}^N)]^N$ uniformly for $t \geq 0$ and $[L^\infty(\mathbb{R}^N)]^N$-estimate is asymptotically independent of initial condition $u_0$ varying in bounded subsets of $[W^{1,p}_U(\mathbb{R}^N)]^N$. With such an estimate the proof of the existence of a bounded dissipative semigroup $\{T(t)\}$ corresponding to (3.7) on $[W^{1,p}_U(\mathbb{R}^N)]^N$ and the existence of a bounded \(\left([W^{1,p}_U(\mathbb{R}^N)]^N - [W^{1,p}_\rho(\mathbb{R}^N)]^N\right)\) attractor for $\{T(t)\}$, where $\rho$ is any translation of $\rho_0(x) = (1 + |x|^2)^{-\frac{N}{2}}$, goes the same way as in the case of the scalar equation.

Hence Corollary 1.4 can be formulated in the case of systems as well. Namely, for a locally Lipschitz continuous and quasi-monotone increasing vector field $f$ there exists a bounded \(\left([W^{1,p}_U(\mathbb{R}^N)]^N - [L^p_\rho(\mathbb{R}^N)]^N\right)\) attractor for (3.7), iff there exists an attractor for (3.6) in $\mathbb{R}^N$. 22
4 Appendix

In this section we recall some results reported in [7] that have been used before in this paper and prove some an additional property that has been used in the proof of Lemma 2.8. Moreover we give an additional characterization of the spaces $W^{s,p}_U(\mathbb{R}^N)$. In particular we show that for noninteger orders $\dot{W}^{s,p}_U(\mathbb{R}^N)$ and $W^{s,p}_U(\mathbb{R}^N)$ coincide.

As mentioned in the introduction $L^p_U(\mathbb{R}^N)$ is the Banach space of measurable functions $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying

$$\|\phi\|_{L^p_U(\mathbb{R}^N)} := \sup_{y \in \mathbb{R}^N} \|\phi\|_{L^p(B(y,1))} < \infty$$

for $1 \leq p \leq \infty$. The closed subspace of $L^p_U(\mathbb{R}^N)$ consisting of functions which are translation continuous with respect to $\|\cdot\|_{L^p_U(\mathbb{R}^N)}$, that is

$$\|\tau_y \phi - \phi\|_{L^p_U(\mathbb{R}^N)} \rightarrow 0 \text{ as } |y| \rightarrow 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations, is denoted $\dot{L}^p_U(\mathbb{R}^N)$.

Similarly, the Sobolev space, of integer order, constructed above $L^p_U(\mathbb{R}^N)$, is denoted $W^{k,p}_U(\mathbb{R}^N)$ and it is the Banach space consisting of elements $\phi \in W^{k,p}_{loc}(\mathbb{R}^N)$ satisfying, for $|\sigma| \leq k$, $D^\sigma \phi \in L^p_U(\mathbb{R}^N)$, with norm $\|\phi\|_{W^{k,p}_U(\mathbb{R}^N)} = \sum_{|\sigma| \leq k} \|D^\sigma \phi\|_{L^p_U(\mathbb{R}^N)}$, or equivalently

$$\|\phi\|_{W^{k,p}_U(\mathbb{R}^N)} := \sup_{y \in \mathbb{R}^N} \|\phi\|_{W^{k,p}(B(y,1))} < \infty.$$ 

Also, $\dot{W}^{k,p}_U(\mathbb{R}^N)$ can be defined as the space of all $\phi \in W^{k,p}_{loc}(\mathbb{R}^N)$ satisfying, for $|\sigma| \leq k$, $D^\sigma \phi \in \dot{L}^p_U(\mathbb{R}^N)$, or equivalently, as the closed subspace of all elements in $W^{k,p}_U(\mathbb{R}^N)$, which are translation continuous with respect to $\|\cdot\|_{W^{k,p}_U(\mathbb{R}^N)}$.

On the other hand, denoting by $\rho$ any translation of the weight $\rho_0(x) = (1 + |x|^2)^{-\nu}$, for $\nu > N/2$, the weighted space $L^p_\rho(\mathbb{R}^N)$ is defined as the Banach space of measurable functions such that

$$\|\phi\|_{L^p_\rho(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\phi(x)|^p \rho(x) \, dx < \infty.$$ 

In fact, as shown in [7, Section 4], a very wide class of weights could be considered in the discussion rather than $\rho_0$. However we stick to this example because it has been traditionally used in the literature.

Analogously the Sobolev space, of integer order, constructed on $L^p_\rho(\mathbb{R}^N)$, is denoted $W^{k,p}_\rho(\mathbb{R}^N)$ and it is the Banach space consisting of elements $\phi \in W^{k,p}_{loc}(\mathbb{R}^N)$ satisfying, for $|\sigma| \leq k$, $D^\sigma \phi \in L^p_\rho(\mathbb{R}^N)$, with norm $\|\phi\|_{W^{k,p}_\rho(\mathbb{R}^N)} = \sum_{|\sigma| \leq k} \|D^\sigma \phi\|_{L^p_\rho(\mathbb{R}^N)}$.

In summary, we have the following diagram, where all arrows indicate continuous inclusions

$$\begin{align*}
W^{k,p}_U(\mathbb{R}^N) &\hookleftarrow \dot{W}^{k,p}_U(\mathbb{R}^N) & W^{k,p}_\rho(\mathbb{R}^N) \\
\uparrow & & \uparrow \\
\dot{W}^{k+1,p}_U(\mathbb{R}^N) &\hookleftarrow W^{k+1,p}_U(\mathbb{R}^N) & W^{k+1,p}_\rho(\mathbb{R}^N)
\end{align*}$$
Lemma 4.1 For \( p \geq 2 \) and \( s \geq 0 \) the space \( \dot{W}_U^{s,p}(\mathbb{R}^N) \) coincides with

\[
V_{0,s} = \left\{ \phi \in \dot{H}_p^s(\mathbb{R}^N) : \sup_{y \in \mathbb{R}^N} \|\rho_y^\frac{1}{p} \phi\|_{H_p^s(\mathbb{R}^N)} < \infty, \lim_{|z| \to 0} \sup_{y \in \mathbb{R}^N} \|\rho_z^\frac{1}{p} (\tau_z \phi - \phi)\|_{H_p^s(\mathbb{R}^N)} = 0 \right\}. 
\]
Proof. Define the norm

\[ \| \phi \|_{V_0,s} = \sup_{y \in \mathbb{R}^N} \| \rho_y^\frac{1}{2} \phi \|_{H^s_{p}(\mathbb{R}^N)} \] (4.6)

which, by (4.2) and (4.3) is equivalent to the norm of \( \dot{W}^{s,p}_{U}(\mathbb{R}^N) \) and \( W^{s,p}_{U}(\mathbb{R}^N) \) if \( s \in \mathbb{N} \), respectively. In particular the case \( s \in \mathbb{N} \) follows from this property. Also, since \( C_{bd}^\infty(\mathbb{R}^N) \) is dense in \( \dot{W}^{s,p}_{U}(\mathbb{R}^N) \), for noninteger \( s \) it will be then enough to show that \( C_{bd}^\infty(\mathbb{R}^N) \) is dense in \( V_{0,s} \).

For this, following [17, \S 4.2.1, 4.2.2] we define, for fixed \( s \geq 0 \),

\[ \mathcal{F}(\phi) = \left( \sum_{j=0}^{\infty} 2^{2js} \left| F^{-1}(\psi_j F\phi) \right|^2 \right)^{\frac{1}{2}}, \] (4.7)

where \( F \) is a Fourier transform and \( \{ \psi_j \} \) is the dyadic resolution of unity.

Then, using mollifiers, we observe that for \( \phi \in V_{0,s} \), \( J_\varepsilon \ast \phi \in C_{bd}^\infty(\mathbb{R}^N) \) and

\[
\frac{1}{c} \| \rho_y^\frac{1}{2} (J_\varepsilon \ast \phi - \phi) \|_{H^s_{p}(\mathbb{R}^N)} \leq \| \rho_y^\frac{1}{2} \mathcal{F} (J_\varepsilon \ast \phi - \phi) \|_{L^p(\mathbb{R}^N)}
\]

\[
= \| \rho_y^\frac{1}{2} \left( \sum_{j=0}^{\infty} 2^{2js} \left| J_\varepsilon(z) \left[ F^{-1}(\psi_j F\phi)(x-z) - F^{-1}(\psi_j F\phi)(x) \right] \right| \right)^{\frac{1}{2}} \|_{L^p(\mathbb{R}^N)}
\]

\[
= \left( \int_{\mathbb{R}^N} \left\{ \sum_{j=0}^{\infty} 2^{2js} \int_{\mathbb{R}^N} J_\varepsilon(z) \left[ F^{-1}(\psi_j F\phi)(x-z) - F^{-1}(\psi_j F\phi)(x) \right] dz \right\}^\frac{p}{2} \rho_y(x)dx \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_{\mathbb{R}^N} \left\{ \int_{\mathbb{R}^N} J_\varepsilon(z) \mathcal{F}^2(\phi(x-z) - \phi(x)) \rho_y^\frac{2}{p}(x)dz \right\}^\frac{p}{2} \rho_y(x)dx \right)^{\frac{1}{p}}
\]

Applying then the generalized Minkowski inequality we obtain the estimate

\[
\frac{1}{c} \| \rho_y^\frac{1}{2} (J_\varepsilon \ast \phi - \phi) \|_{H^s_{p}(\mathbb{R}^N)} \leq \left( \int_{\mathbb{R}^N} \left\{ \int_{\mathbb{R}^N} J_\varepsilon(z) \mathcal{F}(\phi(x-z) - \phi(x)) \rho_y(x)dx \right\}^\frac{2}{p} dz \right)^{\frac{1}{2}}
\]

\[
= \left( \int_{\mathbb{R}^N} J_\varepsilon(z) \| \rho_y^\frac{1}{2} \mathcal{F}(\phi(\cdot - z) - \phi) \|_{L^p(\mathbb{R}^N)}^2 dz \right)^{\frac{1}{2}}
\]

\[
\leq \sup_{|z| < \varepsilon} \| \rho_y^\frac{1}{2} \mathcal{F}(\phi(\cdot - z) - \phi) \|_{L^p(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} J_\varepsilon(z)dz \right)^{\frac{1}{2}}
\]

\[
\leq \text{const. sup} \| \rho_y^\frac{1}{2} (\phi(\cdot - z) - \phi) \|_{H^s_{p}(\mathbb{R}^N)}.
\]
Hence
\[ \| J_\varepsilon * \phi - \phi \|_{V_0,s} \leq C \sup_{|z| < \varepsilon} \| (\phi(\cdot - z) - \phi) \|_{V_0,s}, \] (4.8)
which shows that \( J_\varepsilon * \phi \) converges to \( \phi \) as \( \varepsilon \to 0 \) for any \( \phi \in V_{0,s} \).

Note that inequality (4.8) generalizes for noninteger \( s \) the one obtained in [7, Lemma 4.9] for \( W^{k,p}_U(\mathbb{R}^N) \).

As a consequence we get the following description of \( \dot{W}^{s,p}_U(\mathbb{R}^N) \) spaces, which has been used in the proof of Lemma 2.8.

**Corollary 4.2** For \( p \geq 2 \) and \( s \geq 0 \) the space \( \dot{W}^{s,p}_U(\mathbb{R}^N) \) coincides with the closed subspace of \( W^{s,p}_U(\mathbb{R}^N) \) given by
\[ V_s = \{ \phi \in W^{s,p}_U(\mathbb{R}^N), \ (\tau_y \phi - \phi) \to 0 \text{ in } W^{s,p}_U(\mathbb{R}^N) \text{ as } y \to 0 \}. \]
In particular,
\[ W^{k,p}_U(\mathbb{R}^N) \hookrightarrow \dot{W}^{s,p}_U(\mathbb{R}^N) \text{ for } 0 \leq s < k, \ k \in \mathbb{N}, \ p \geq 2, \] (4.9)
and for noninteger \( s \) and \( p \geq 2 \)
\[ W^{s,p}_U(\mathbb{R}^N) = \dot{W}^{s,p}_U(\mathbb{R}^N). \]

**Proof.** From (4.4) and the fact that in \( \dot{W}^{s,p}_U(\mathbb{R}^N) \) translations are continuous in the norm of \( W^{s,p}_U(\mathbb{R}^N) \), note that \( \dot{W}^{s,p}_U(\mathbb{R}^N) \subset V_s \subset V_{0,s} \). Therefore the first assertion holds as a consequence of Lemma 4.1.

Now, to prove (4.9), we show that \( W^{k+1,p}_U(\mathbb{R}^N) \hookrightarrow \dot{W}^{s,p}_U(\mathbb{R}^N) \) for \( s \in (k, k+1) \). For this note that, from the characterization above, it is enough to prove the continuity of translations, with respect to \( W^{s,p}_U(\mathbb{R}^N) \)-norm, for \( w \in W^{k+1,p}_U(\mathbb{R}^N) \). By [7, Remark 4.5] we have \( W^{k+1,p}_U(\mathbb{R}^N) \hookrightarrow \dot{W}^{k,p}_U(\mathbb{R}^N) \), whereas \( W^{s,p}_U(\mathbb{R}^N) = [W^{k,p}_U(\mathbb{R}^N), W^{k+1,p}_U(\mathbb{R}^N)]_\theta \) whenever \( \theta \in (0, 1) \) and \( s = k + \theta \in (k, k+1) \). Then, for \( w \in W^{k+1,p}_U(\mathbb{R}^N) \), by the interpolation inequality we obtain
\[ \| \tau_y w - w \|_{W^{s,p}_U(\mathbb{R}^N)} \leq C \| \tau_y w - w \|_{W^{k+1,p}_U(\mathbb{R}^N)}^{\theta} \| \tau_y w - w \|_{W^{k,p}_U(\mathbb{R}^N)}^{1-\theta} \]
\[ \leq C \| \tau_y w - w \|_{W^{k+1,p}_U(\mathbb{R}^N)}^{\theta} \rightarrow 0 \]
as \( y \to 0 \).

Finally, we prove \( \dot{W}^{s,p}_U(\mathbb{R}^N) = W^{s,p}_U(\mathbb{R}^N) \) for noninteger \( s \). Observe that the inclusion \( \dot{W}^{s,p}_U(\mathbb{R}^N) \subset W^{s,p}_U(\mathbb{R}^N) \) is obvious.

For this just observe that if \( k \in \mathbb{N} \) and \( k < s < k+1 \), then by properties of the complex interpolation, \( W^{k+1,p}_U(\mathbb{R}^N) \subset W^{s,p}_U(\mathbb{R}^N) \) is dense in \( W^{s,p}_U(\mathbb{R}^N) \), see [42, Theorem 1.9.3 (c)] and we get the result. \( \square \)

**References**


[34] A. Mielke, G. Schneider, Attractors for modulation equations on unbounded domains—existence and comparison, Nonlinearity 8 (1995), 743-768.


