Definable families of definable stratified sets

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Abstract

We study definable mappings $f : A \to R^l$, where A is a definable Whitney stratified set and f, restricted to each stratum, is a submersion. Here, "definable" means "definable in an o-minimal structure expanding a real closed field R". Such a mapping can be seen as a definable family of definable stratified sets $\{A_t\}_{t \in R^l}$, where $A_t = f^{-1}(t)$. If f is proper, we prove that this family is trivial.

To prove our result, we use the *definable spectrum*, introduced by M. Coste, and we study the *generic fibers* of definable families at points in the definable spectrum. We use also the notion of *elementary extension* of o-minimal structures and we construct *definable models* of definable sets and mappings.

1 Main result

In this note, we prove some results on definable families of definable Whitney stratified sets. Let us fix some definitions.

A *family* of sets, parametrized by R^p, is a subset $X \subset R^p \times R^n$. The *fiber* of the family at a point $t \in R^p$ is just $X_t = \{x \in R^n : (t, x) \in X\}$.

An interesting particular case is the following. Let $N \subset R^n$ and $P \subset R^p$ be two subsets and $g : N \to P$ a mapping. We can consider its graph

$$\Gamma_g = \{(x, y) \in R^p \times R^n : x = g(y)\} \subset R^p \times R^n$$

that can be seen as family of subsets, whose fibers are just $g^{-1}(t)$.

We are interested in *definable* families of *definable* subsets. Here, "definable" means "definable in an o-minimal structure expanding a real closed field R". In particular, we study definable families of *definable Whitney stratified sets.*

Definition 1.1 *Let $A \subset R^n$ be a definable subset. We say that A is a $\mathcal{D}^r$ Whitney stratified set if we have a finite decomposition $A = \bigcup_i X_i$, where each X_i is a $\mathcal{D}^r$ manifold and $\{X_i\}$ is a Whitney stratification.*

In the above definition (and in the following), $\mathcal{D}^r$ means "C^r and definable". We will always consider $0 < r < \infty$. In what follows, all the definable subsets will be *locally closed*.

Definition 1.2 *Let A and B be $\mathcal{D}^r$ Whitney stratified subsets. A $\mathcal{D}^r$ stratified homeomorphism is a definable homeomorphism $h : A \to B$ such that for each stratum $X \subset A$ there exists a stratum $Y \subset B$ such that $h(X) \subset Y$ and $f_{|X} : X \to Y$ is a $\mathcal{D}^r$ diffeomorphism.*

The main theorem in this note is:

Theorem 1.3 *Let $A \subset R^n$ be $\mathcal{D}^r$ Whitney stratified subset. Let $f : A \to R^l$ be a proper definable map such that $f_{|X} : X \to R^l$ is a $\mathcal{D}^r$ surjective submersion, for each strata $X \subset A$. Then f is $\mathcal{D}^r$ trivial, that is, there exist a point $t_0 \in R^l$ and a $\mathcal{D}^r$ stratified homeomorphism $h = (h_0, f) : A \to f^{-1}(t_0) \times R^l$ such that $h_0 = id$ on $f^{-1}(t_0)$.*

This result is a definable version of Thom's first isotopy lemma.

2 The Tools

To prove our main result, we can not use integration of vector fields (for example, it is clear that integration of semialgebraic functions does not produce necessarily semialgebraic functions).

We substitute integration of vector fields by two constructions. The first one is the definable spectrum and the study of generic fibers of families of definable subsets. The second one is the "construction of definable models", that carries us to define the concept of elementary extension.

The *definable spectrum* was introduced by M. Coste in [1]. A brief sketch of this concept goes as follows. We consider a compactification $\widetilde{R^p}$ of R^p. The elements $\alpha \in \widetilde{R^p}$ are the ultrafilters of definable subsets of R^p. We can associate to each element $\alpha \in \widetilde{R^p}$ a real closed field $k(\alpha)$ and an o-minimal structure $\mathcal{S}(\alpha)$ expanding $k(\alpha)$.

Given a definable family $X \subset R^p \times R^n$, we can define X_α, that is a definable subset of $k(\alpha)^n$. In fact, $\mathcal{S}_n(\alpha) = \{X_\alpha : X \in \mathcal{S}_{p+n}\}$, for $n \geq 1$. An interesting aspect of this concept is that we can relate properties of the generic fiber X_α with properties of the family X on "big" subsets of the parameter space R^p. For example, we can transfer properties of the family to properties of the generic fiber:

Proposition 2.1 *Let $X \subset R^p \times R^n$ be a definable subset such that, for each $b \in R^p$, X_b is a $\mathcal{D}^r$ submanifold of R^n. Then, for each $\alpha \in \widetilde{R^p}$, X_α is a $\mathcal{D}^r$ submanifold of $k(\alpha)^n$.*

More interesting is to transfer properties of the generic fiber to the original family:

Proposition 2.2 *Let $X \subset R^p \times R^n$ be a definable subset. Let us consider an element $\alpha \in \widetilde{R^p}$ such that X_α is a $\mathcal{D}^r$ submanifold of $k(\alpha)^n$. Then there exists a $\mathcal{D}^r$ submanifold M of R^p such that $M \in \alpha$, $X \cap (M \times R^n)$ is a $\mathcal{D}^r$ submanifold of $M \times R^n$ and the projection $\pi : X \cap (M \times R^n) \to M$ is a submersion.*

(The proof of these results can be found in [3]).

The second tool we use is the *construction of definable models*. First, we need the definition of *elementary extension*.

Consider two o-minimal structures $(R', \mathcal{S}')$ and $(R, \mathcal{S})$ such that $R' \subset R$. We say that $(R, \mathcal{S})$ is an *elementary extension* of $(R', \mathcal{S}')$, $(R', \mathcal{S}') \prec (R, \mathcal{S})$, if, for each n, there exists an *extension mapping* $\mathcal{S}'_n \to \mathcal{S}_n : A' \mapsto A'_R$ such that

1. $A'_R \cap R'^n = A'$ for each $A' \in \mathcal{S}'_n$,
2. $A' \mapsto A'_R$ commutes with boolean operations,
3. $(A' \times B') = A'_R \times B'_R$ for each $A' \in \mathcal{S}'_n$, $B' \in \mathcal{S}'_m$,
4. if $\pi' : R'^{n+1} \to R'^n$ and $\pi : R^{n+1} \to R^n$ are the projections on the first n coordinates, and $A' \in \mathcal{S}'_{n+1}$, then $\pi(A'_R) = \pi'(A')_R$,
5. if A' is semialgebraic, A'_R is the usual extension of semialgebraic sets,
6. for each $B \in \mathcal{S}_m$, there exists $n \in \mathbb{N}$, $A' \in \mathcal{S}'_{m+n}$ and $a \in R^n$ such that $B = \{x \in R^m : (x, a) \in A'_R\}$.

For example, if $R' \subset R$ is a real closed field extension and $\mathcal{S}$ (resp. $\mathcal{S}'$) is the o-minimal structure formed by the semialgebraic subsets on R (resp. on R'), then $(R', \mathcal{S}') \prec (R, \mathcal{S})$. Another important example is the following. If $(R, \mathcal{S})$ is an o-minimal structure expanding the real closed field R and $\alpha \in \widetilde{R^p}$, then $(k(\alpha), \mathcal{S}(\alpha))$ is an elementary extension of $(R, \mathcal{S})$.

We are interested in elementary extensions because there exists a relation between the triviality of definable families and the construction of definable models of definable subsets on "smaller" o-minimal structures (see [2], [3]). For example, to prove the following

Theorem 2.3 *Let N be a $\mathcal{D}^r$ manifold and let $g : N \to R$ a proper surjective $\mathcal{D}^r$ submersion. Then g is $\mathcal{D}^r$ trivial;*

it is enough to prove:

Theorem 2.4 *Let $(R', \mathcal{S}') \prec (R, \mathcal{S})$ be an elementary extension. Let $X \subset R^n$ be a $\mathcal{D}^r_{\mathcal{S}}$ manifold. Then there exists a $\mathcal{D}^r_{\mathcal{S}'}$ manifold Y such that Y_R is $\mathcal{D}^r_{\mathcal{S}}$ diffeomorphic to X.*

Let us show the key step in the proof of Theorem 2.3. Considering the graph, we can take the family $X \subset R \times R^n$ that verifies $X_t = g^{-1}(t)$ for each $t \in R$. Consider $\alpha \in \widetilde{R}$ and take the generic fiber X_α. By Theorem 2.4 there exists a model $X' \subset R^n$ such that

$$X_\alpha \simeq X'_{k(\alpha)}$$

But $X'_{k(\alpha)}$ is just the generic fiber at α of the trivial family $R \times X'$:

$$X'_{k(\alpha)} = (R \times X')_\alpha$$

Then there exists a definable subset $M \in \alpha$ such that

$$M \times X' \simeq X \cap (M \times R^n)$$

This means that X is trivial on M. We finish by a compactness argument. $\square$

To prove our main result, we will need the following

Theorem 2.5 *Let $A \subset R^n$ be a $\mathcal{D}^r$ Whitney stratified subset. Then, we can find a definable $\mathcal{D}^r$ Whitney stratified subset $A' \subset R'^{n'}$ such that there exists a $\mathcal{D}^r$ stratified homeomorphism $\phi : A'_R \to A$.*

3 The Proofs

We will prove our theorems by *induction on the depth of the stratification of A*. Let $A = (A, \Sigma)$ be a stratified set. Consider $X \in \Sigma$. Then we can define

$$\text{depth}_\Sigma(X) = \sup\{n : \exists \text{ a chain of strata } X = X_0 < X_1 < \ldots < X_n\}.$$

The depth of the stratification (A, Σ) is

$$\text{depth}(\Sigma) = \sup\{\text{depth}_\Sigma(X) : X \in \Sigma\}.$$

If the stratification is clearly defined, we will write $\text{depth}(A)$ instead of $\text{depth}(\Sigma)$.

If $\text{depth}(A) = 0$, then A is a $\mathcal{D}^r$ manifold, and hence Theorems 1.3 and 2.5 for $\text{depth}(A) = 0$ are proved in [4].

Assume that Theorems 1.3 and 2.5 are proved for $\text{depth}(A) < k$.

Proof of Theorem 2.5 for $\text{depth}(A) = k > 0$. Let Y be an stratum, maximal for the depth, that is, $\text{depth}(Y) = k$. As we are interested in what happens in a neighbourhood of this kind of strata, we can assume that Y is the unique stratum with this property (in other words, we can make the same construction for all strata of maximal depth at the same time). Let us take $X = A \setminus Y$, and let (T, π, ρ) be a tube for Y. That is, T is a neighbourhood of Y in R^n, $\pi : T \to Y$ is a retraction, and $\rho : T \to R$ is a "distance function". We observe that depth $X \leq k - 1$.

Shrinking T if necessary, we can assume that $(\pi, \rho) : T \cap X \to Y \times (0, 1)$ is proper and that this mapping, restricted to each stratum, is a submersion. By a generalized version of Theorem 1.3 (for depth $< k$), we have a stratified homeomorphism $\alpha = (\alpha_0, \rho) : T \cap X \to F \times (0, 1)$, where $F = \rho^{-1}(t_0) \cap X$ for certain $t_0 \in (0, 1)$, $\alpha_{0|F} = id$ and $\pi \circ \alpha_0 = \pi$.

Consider now the definable sets

$$\widehat{T} = \{((1 - t)\pi(x), tx, t) : x \in F, t \in (0, 1)\}$$

and

$$\widehat{Y} = \{(y, 0, 0) : y \in Y\}$$

Clearly $\widehat{Y} \subset \mathrm{adh}(\widehat{T})$ and we observe that $\mathrm{adh}(\widehat{T}) \cap \{t = 0\} = \widehat{Y}$.

We have that $\widehat{Y} \cup \widehat{T}$ is a Whitney stratified set. To check this, we just observe that below mapping $\omega_1 : T \cap X \to \widehat{T}$ is defined by $\omega_1 = \theta \circ \alpha$, where $\theta : F \times (0,1) \to \widetilde{T}$ is given by $\theta(x,t) = ((1-t)\pi(x), tx, t)$.

We consider the mapping

$$\omega_1 : T \cap X \longrightarrow \widehat{T}$$

$$x \mapsto ((1 - \rho(x))\pi(x), \rho(x)\alpha_0(x), \rho(x))$$

This mapping is in fact a stratified homeomorphism and we can extend it in a natural way to an stratified homeomorphism $\omega : (T \cap X) \cup Y \to \widehat{T} \cup \widehat{Y}$.

Now, we consider again our mapping $(\pi, \rho) : T \cap X \to Y \times (0,1)$, which is proper and submersive restricted to each stratum. We have that $\mathrm{depth}(T \cap X) < k$, hence, by Theorem 1.3 (in fact,a generalization for mappings, see [4]), we have a model over R'. That is, there exists $\psi' : T' \to B'$ and a commutative diagram

$$\begin{array}{ccc} T'_R & \xrightarrow{\delta_1} & T \cap X \\ \downarrow{\scriptstyle \psi'_R} & & \downarrow{\scriptstyle (\pi,\rho)} \\ B'_R & \xrightarrow{\beta} & Y \times (0,1). \end{array}$$

Let Y' be a model of Y over R' and $\gamma : Y'_R \to Y$ a $\mathcal{D}^r$ diffeomorphism. Then we can consider the $\mathcal{D}^r$ diffeomorphism $(\gamma \times id)^{-1} \circ \beta : B'_R \to Y'_R \times (0,1)$. Hence there exists a diffeomorphism $\beta' : B' \to Y' \times (0,1)$ over R' such that $\beta'_R = (\gamma \times id)^{-1} \circ \beta$. We obtain the following diagram

$$\begin{array}{ccccc} & & T'_R & \xrightarrow{\delta_1} & T \cap X \\ & \swarrow{\scriptstyle (\beta' \circ \psi')_R} & \downarrow{\scriptstyle \psi'_R} & & \downarrow{\scriptstyle (\pi,\rho)} \\ Y' \times (0,1) & \xrightarrow[\beta'^{-1}_R]{} & B'_R & \xrightarrow[\beta_1]{} & Y \times (0,1) \end{array}$$

(observe that $(\beta' \circ \psi')_R = \beta'_R \circ \psi'_R$). Defining $(\pi', \rho') = \beta' \circ \psi'$ and $\delta_2 = \beta_1 \circ \beta'^{-1}_R$, we obtain a proper submersion $(\pi', \rho') : T' \to Y' \times R'$ over R', and diffeomorphisms $\delta_1 : T'_R \to T \cap X$ and $\delta_2 : Y'_R \times R \to Y \times R$, such that $(\pi, \rho) \circ \delta_1 = \delta_2 \circ (\pi'_R, \rho'_R)$.

Now we repeat above argument over R' for (π', ρ'). We can construct $\widehat{T'}$ and $\widehat{Y'}$ as above, and we prove that $T \cap A \simeq (\widehat{Y'} \cup \widehat{T'})_R$, that is, we have a model for $T \cap A$. By induction hypothesis, we can assume that there exist a $\mathcal{D}^r_{S'}$ stratified subset X' and a $\mathcal{D}^r_S$ stratified homeomorphism for X to X'_R. Hence we finish our proof by gluing both models.

Proof of Theorem 1.9. We have that, for each $t \in R^l$, $A_t = f^{-1}(t)$ is a $\mathcal{D}^r$ Whitney stratified definable subset. Hence, for $\alpha \in \widetilde{R^l}$, A_α is a $\mathcal{D}^r_{S(\alpha)}$ Whitney stratified definable subset (defined on $k(\alpha)$). By Theorem 2.5, there exists a $\mathcal{D}^r_S$ Whitney stratified definable subset A'

such that its extension $A'_{k(\alpha)}$ is homeomorphic (by a definable $\mathcal{D}^r$ stratified homeomorphism) to A_α. Hence there exists a $\mathcal{D}^r$ submanifold $M \subset R^l$ such that $A' \times M$ is homeomorphic to $A \cap (M \times R^l)$, that is, f is $\mathcal{D}^r$ trivial over M.

Finally, we can repeat the argument done in [3]. Let us assume that $l = 1$ (the case $l > 1$ is similar, but technically more complicated). By the compactness of the definable spectrum and above observation, we obtain that $R = I_1 \cup \ldots \cup I_q$, where, for each I_i, f is $\mathcal{D}^r$ trivial over I_i, and I_i is an open interval or a singleton. Using a *local argument*, we can assume that each I_i is an open interval. We finish the proof gluing these local trivializations in order to obtain a global one. □

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