Divisors on real curves

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In this note, a real algebraic curve is a smooth proper geometrically integral scheme over \( \mathbb{R} \) of dimension 1.

If \( X \) is a real curve, by \( X_{\mathbb{C}} \) we denote the base extension of \( X \) to \( \mathbb{C} \) and by \( \mathbb{R}(X) \) is the function field of \( X \). Let \( g \) denote the genus and \( J \) the Jacobian of \( X \). The group \( \text{Div}(X) \) (resp. \( \text{Div}(X_{\mathbb{C}}) \)) of divisors on \( X \) (resp. \( X_{\mathbb{C}} \)) is the free abelian group on the closed points of \( X \) (resp. \( X_{\mathbb{C}} \)). The Galois group \( \text{Gal}(\mathbb{C}/\mathbb{R}) \) acts on the complex variety \( X_{\mathbb{C}} \) and also on \( \text{Div}(X_{\mathbb{C}}) \), we always indicate the action by a bar. We will say that a closed point \( P \) of \( X \) is real if the residue field at \( P \) is \( \mathbb{R} \) and \( P \) is complex if the residue field at \( P \) is \( \mathbb{C} \) i.e. \( P = Q + \bar{Q} \) with \( Q \) a closed point of \( X_{\mathbb{C}} \). Let \( \text{Pic}(X) \) denote the Picard group of \( X \) which is the quotient of \( \text{Div}(X) \) by the subgroup of principal divisors i.e. divisors of elements in \( \mathbb{R}(X) \). It is well known that \( \text{Pic}^0(X) \) the subgroup of divisors of degree 0 modulo rational equivalence can be identified with \( J(\mathbb{R}) \) since the set of real points \( X(\mathbb{R}) \) will always be assumed to be non empty. We will denote by \( J(\mathbb{R})_0 \) the connected component of the identity of \( J(\mathbb{R}) \). We always write \([D]\) for the class of a divisor \( D \) in \( \text{Pic}(X) \) and \( \ell(D) = \dim_{\mathbb{R}}(\mathcal{L}(D)) \).

The set of real points \( X(\mathbb{R}) \) decomposes into finitely many connected components. Let always \( s \) denote their number. By Harnack’s Theorem we know that \( 1 \leq s \leq g + 1 \). A curve with \( g + 1 - k \) real connected components is called an \( M - k \)-curve.

For \( d \geq 0 \), let \( S^dX \) denote the symmetric \( d \)-fold product of \( X \). We have an usual map \( \phi_d : (S^dX)(\mathbb{R}) \to \text{Pic}^d(X) \) with \( \text{Pic}^d(X) \) representing divisor classes represented by \( \text{Gal}(\mathbb{C}/\mathbb{R}) \)-invariant elements of degree \( d \).

We are interested in bounding in terms of \( g \) and \( s \) the following invariants of the curve:

**Definitions 0.1** (i) Let \( N(X) \) be the smallest integer \( n \geq 1 \) such that for any real closed point \( P \), for any \( \alpha \in J(\mathbb{R}) \) there exist \( P_1, \ldots, P_n \in X(\mathbb{R}) \) such that \( \alpha = \sum_{i=1}^{n} [P_i - P] \). If such integer doesn’t exist we set \( N(X) = +\infty \) (resp. \( N_c(X) = +\infty \)).
(ii) Let $M(X)$ be the smallest integer $m \geq 1$ such that for any real closed point $P$, for any $\alpha \in J(\mathbb{R})_0$ there exist some complex closed points $Q_1, \ldots, Q_m$ such that $\alpha = \sum_{i=1}^m [Q_i - 2P]$. If such integer doesn’t exist we set $M_r(X) = +\infty$.

By a result of Scheiderer we know that $N(X)$ is finite. Statement (ii) may be seen as an effective version of the E.P.T (Everybody proved this theorem) of Bröcker. The aim of the first part of the note is to find lower and upper bounds on these invariants.

We may prove that $M(X)$ is finite using Riemann-Roch Theorem.

We obtain the following general theorem for $M(X)$:

**Theorem 0.2** We have

$$\max[1, \frac{1}{2}(g - 1)] \leq M_r(X) \leq 2g + 1$$

if $g$ is odd, and

$$\max[1, \frac{1}{2}g] \leq M_r(X) \leq 2g + 1$$

if $g$ is even.

**Proof**: We only give the proof for the upper bound. Let $P$ a real closed point of $X$ and $\alpha \in J(\mathbb{R})_0$. Since $J(\mathbb{R})_0$ is a divisible group, there is $\beta \in J(\mathbb{R})_0$ such that $2\beta = \alpha$. By Riemann-Roch Theorem, the map

$$\phi_d : (S^d X)(\mathbb{R}) \to \text{Pic}^d(X)$$

is surjective for $d \geq g$. Hence the divisor $\beta + [gP] = [D]$ with $D$ an effective divisor of degree $g$. Then

$$\beta = [D + (g + 1)P] - [(2g + 1)P].$$

By Riemann-Roch Theorem we may show that the divisor $D + (g + 1)P$ is very ample as a complex divisor, and also as a real divisor since $D + (g + 1)P \in \text{Div}(X)$. Let $\psi$ denote the embedding of $X$ in $\mathbb{P}^{g+1}_\mathbb{R}$ associated to $D + (g + 1)P$. Let $S$ be the quadric hypersurface of $\mathbb{P}^{g+1}_\mathbb{R}$ with equation $x_0^2 + \ldots + x_{g+1}^2 = 0$, $2D + 2(g + 1)P$ is linearly equivalent to the effective divisor $D'$ of degree $2(2g + 1)$ obtained by intersecting $S$ and $X$. Since $S(\mathbb{R}) = \emptyset$, the support of $D'$ is complex. Hence

$$\alpha = [D'] - [2(2g + 1)P]$$

and the proof is done. \qed

For curves with $g \leq 1$, we have a more precise result:

**Theorem 0.3** Let $X$ be a real rational curve or a real elliptic curve, then

$$M_r(X) = 1.$$
We give now some geometric applications of the previous results. Let \( X \subseteq \mathbb{P}^n_{\mathbb{R}} \) be a real curve. We will say that \( X(\mathbb{R}) \) is affine in \( \mathbb{P}^n_{\mathbb{R}} \) if there exists a real hyperplane \( H \) such that \( H(\mathbb{R}) \cap X(\mathbb{R}) = \emptyset \).

**Proposition 0.4** Let \( X \subseteq \mathbb{P}^n_{\mathbb{R}}, \ n \geq 2 \), be a nondegenerate linearly normal curve of degree \( d \) such that \( X(\mathbb{R}) \) has no pseudo-line. If \( X \) satisfies one of the two following conditions

(i) \( X \) is rational or elliptic,

(ii) \( g \geq 2 \) and \( d \geq 4g + 2 \),

then \( X(\mathbb{R}) \) is affine in \( \mathbb{P}^n_{\mathbb{R}} \).

We give some upper bounds on \( N_r(X) \), \( N_c(X) \) in terms of \( g \) for \( M \) and \( M - 1 \)-curves.

**Theorem 0.5** Let \( X \) be a \( M \)-curve or a \( M - 1 \)-curve, we have

\[ N_r(X) \leq 2g - 1. \]

We extend this result to \( M - 2 \)-curves under the assumption of a conjecture of Huisman on unramified curves.

In the second part of the paper we study linear systems on real curves that contain a divisor which support is totally real. We show that linear systems which are extremal for the real Clifford inequality satisfy the previous condition. In particular, we look at linear systems on real hyperelliptic curves and prove a new Clifford inequality for such curves. We recall that a divisor \( D \) is special if \( \ell(K - D) > 0 \) with \( K \) denoting the canonical divisor.

**Theorem 0.6** Let \( X \) be a real hyperelliptic curves such that \( s \neq 2 \), \( D \in \text{Div}(X) \) an effective and special divisor of degree \( d \). Then

\[ \dim |D| \leq \frac{1}{2}(d - \delta(D)), \]

with \( \delta(D) \) the number of connected components \( C \) of \( X(\mathbb{R}) \) such that the degree of the restriction of \( D \) to \( C \) is odd.

We also determine explicitly the real hyperelliptic curves with \( s = 2 \) for which the previous inequality is no longer valid.

In the last part of the paper, we study the existence of \( g^1_d \) i.e. linear systems of degree \( d \) and dimension 1 for a real curve \( X \). We introduce some invariants: let

\[ \rho_C(X) = \inf\{d \in \mathbb{N} | \ X_C \ has \ a \ g^1_d \} \]

and

\[ \rho_R(X) = \inf\{d \in \mathbb{N} | \ X \ has \ a \ g^1_d \}. \]

We obtain the following theorem:
**Theorem 0.7** We have

(i) $\rho_R(X) = \rho_C(X)$ if $1 \leq \rho_C(X) \leq 2$ or $X_C$ has an odd number of $g^1_{\rho_C(X)}$.

(ii) $\rho_C(X) \leq \rho_R(X) \leq 2\rho_C(X) - 2$ else.

(iii) $\rho_R(X) \leq g$ if $g \geq 2$.

We prove that the bound we give on $\rho_R(X)$ is optimal when $2 \leq g \leq 4$. 