Boundary control and homogenization: optimal climatization through smart double skin boundaries

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Dedicated to the unforgettable Olga A. Oleinik (1925-2001)

Abstract. We consider the homogenization of an optimal control problem in which the control $v$ is placed on a part $\Gamma_0$ of the boundary and the spatial domain contains a thin layer of “small particles”, very close to the controlling boundary, and a Robin boundary condition is assumed on the boundary of those “small particles”. This problem can be associated with the climatization modeling of Bioclimatic Double Skin Façades which was developed in modern architecture as a tool for energy optimization. We assume that the size of the particles and the parameters involved in the Robin boundary condition are critical (and so they justify the occurrence of some “strange terms” in the homogenized problem). The cost functional is given by a weighted balance of the distance (in a $H^1$-type metric) to a prescribed target internal temperature $u_T$ and the proper cost of the control $v$ (given by its $L^2(\Gamma_0)$ norm). We prove the (weak) convergence of states $u_\varepsilon$ and of the controls $v_\varepsilon$ to some functions, $u_0$ and $v_0$, respectively, which are completely identified: $u_0$ satisfies an artificial boundary condition on $\Gamma_0$ and $v_0$ is the optimal control associated to a limit cost functional $J_0$ in which the “boundary strange term” on $\Gamma_0$ arises. This information on the limit problem makes much more manageable the study of the optimal climatization of such double skin structures.

Key words: optimal control, homogenization, thin layer of “small particles”, critical case, cost functional convergence

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1. Introduction

A well-known energy optimization technique in modern architecture is the theory of smart façade systems (also called as Bioclimatic Double Skin Façades) in which climatization takes place by means of active glass windows (see, e.g., [4], [2] and [1]). Today, there are different types of active glass in the market: LCD Liquid Crystal, Gasochromic, SPO suspended particles, Electrochromic, etc. See, e.g., the case of fluids and windows in [2] and [8].

From the mathematical view point, many different climatization models have been proposed in the literature: see, for instance, Chapter 1 of the excellent book by Duvaut and Lions [14]. Some studies on internal climatization and homogenization can be found in [24]. In this paper, we will analyze a simplified formulation of Double Skin Façades in which there is an active flux control \( v_\varepsilon \), located in a part \( \Gamma_0 \) of the boundary, and a kind of celosia (latticed windows called in this way in Spanish) traditionally made of masonry, wood, or a combination of these materials. We assume that the celosia is formed by a set of periodical small thermostats, of period \( \varepsilon > 0 \), located in an internal thin layer located very close to the controlling boundary. So, \( \varepsilon \) represents a small parameter related to the characteristic celosia.

Our simplified optimal control problem assumes that the state of the system \( u_\varepsilon \) (the internal temperature) satisfies a Poisson equation in the internal domain \( \Omega_\varepsilon \) of \( \mathbb{R}^n \), with \( n \geq 3 \), which is defined as the external domain to the set of periodical small thermostats (here represented by a set of \( \varepsilon \)-periodically balls) on whose contours \( S_\varepsilon \) a given climatization law (represented by a Robin boundary condition with a large parameter \( \varepsilon^{-\gamma} \) as coefficient, where \( \gamma = \frac{n-1}{n-2} \)) takes place. Non-symmetrical shapes, and/or the case \( n = 2 \), can also be considered thanks to the techniques presented in [13], but for the sake of simplicity in the presentation we will not develop it here. We assume that any thermostat has a critical radius \( a_\varepsilon \), where \( a_\varepsilon = C_0 \varepsilon^\alpha \) and \( \alpha = \frac{n-4}{n-2} \). As in many other frameworks (see many examples and references in the monograph [9]), this critical size leads to the occurrence of strange terms in the homogenized problem (in contrast with what happens for other possible sizes).

It is assumed that the cost functional \( J_\varepsilon(v_\varepsilon) \) is given by a weighted balance of the distance (in a \( H^1(\Omega_\varepsilon) \) type metric) to a prescribed target internal temperature \( u_T \) and the proper cost of the control \( v \) (given by its \( L^2(\Gamma_0) \) norm). Our main result proves the (weak) convergence of solutions \( u_\varepsilon \) and of the controls \( v_\varepsilon \) to some functions, \( u_0 \) and \( v_0 \), respectively, which are completely identified: \( u_0 \) satisfies the Poisson equation in the whole domain \( \Omega \) (which we assume to be a bounded open set with \( \partial \Omega \) of class \( C^1 \)) and \( v_0 \) is the boundary optimal control but in an artificial boundary condition in which the thermostats effects are located on the own controllability boundary \( \Gamma_0 \). Moreover, we prove the convergence of the cost functional \( J_\varepsilon(v_\varepsilon) \) to a new cost functional \( J_0(v_0) \) in which the boundary strange term arises. This information on the limit problem makes much more manageable the study of the optimal climatization of such double skin structures.
In the last section, we consider the pure homogenization process (without any control, \( v_\varepsilon \equiv 0 \)) and prove the convergence of the energies: an information which is stronger than the mere weak convergence \( u_\varepsilon \to u_0 \) in \( H^1(\Omega) \).

This paper complements the scope considered by previous papers in the literature concerning optimal control problems in which the controls are located in different parts of the spatial domain (see \([22], [23], [21], [12] \) and \([13]\)).

2. Problem statement. Adjoint optimality problem

Let \( \Omega \subset \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n > 0 \} \) be a bounded open set of \( \mathbb{R}^n, n \geq 3 \), with \( \partial \Omega \) of class \( C^2 \), and \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), is assumed to be of class \( C^1 \), where \( \Gamma_0 = \partial \Omega \cap \{ x_n = 0 \} \neq \emptyset \) is the \((n-1)\)-dimensional domain on the plane \( x_n = 0 \) which represents the controlling boundary, \( \Gamma_1 = \partial \Omega \setminus \Gamma_0 \). Define \( Y_0 = (-1/2, 1/2)^{n-1} \times (0, 1) \), \( Y_\varepsilon^j = \varepsilon Y_0 + \varepsilon j, j = (j_1, \ldots, j_{n-1}, 0) \), \( j_i \in \mathbb{Z}, i = 1, \ldots, n-1 \). We denote by \( P_\varepsilon^j \) the center of the cube \( Y_\varepsilon^j \), \( G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j \), where \( G_0 \) is the unit ball with the center coinciding with the center \((0, \ldots, 0, 1/2)\) of the cube \( Y_0 \) and \( a_\varepsilon = C_0 \varepsilon^n \) with \( \alpha = (n-1)/(n-2) \). We define \( \Upsilon_\varepsilon = \{ j \in \mathbb{Z}^n : j = (j_1, \ldots, j_{n-1}, 0), Y_\varepsilon^j \subset \Omega \} \). It is easy to see (as in Chapter 6 of \([9]\)) that \( |\Upsilon_\varepsilon| \approx d \varepsilon^{1-n}, d = \text{const} > 0 \), where \( |\Upsilon_\varepsilon| \) denotes the cardinality of the set of isolated points \( \Upsilon_\varepsilon \).

We introduce the sets

\[
G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j, \quad S_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} \partial G_\varepsilon^j, \quad \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad \partial \Omega_\varepsilon = S_\varepsilon \cup \Gamma_0 \cup \Gamma_1.
\]

The set \( G_\varepsilon \) represents the \( \text{celosía} \) or double skin. It is localized as a subset of \( \Omega \cap \{ x \in \mathbb{R}^n : x_n \in (0, \varepsilon) \} \) (see Figure 1).

For an arbitrary function \( v \in L^2(\Gamma_0) \), we denote by \( u_\varepsilon(v) \in H^1(\Omega_\varepsilon, \Gamma_1) \) the solution of the problem

\[
\begin{aligned}
-\Delta u_\varepsilon(v) &= f, & x &\in \Omega_\varepsilon, \\
\partial_{\nu} u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) &= 0, & x &\in S_\varepsilon, \\
\partial_{\nu} u_\varepsilon(v) &= v, & x &\in \Gamma_0, \\
u_\varepsilon(v) &= 0, & x &\in \Gamma_1,
\end{aligned}
\]  

(1)

where \( f \in L^2(\Omega) \), \( a(x) \in C^\infty(\overline{\Omega}) \), \( a(x) \geq a_0 = \text{const} > 0 \), and the notation \( \partial_{\nu} g \) represents the partial derivative along the outward unit normal vector \( \nu \) to the boundary. Here, the space \( H^1(\Omega_\varepsilon, \Gamma_1) = \{ w \in H^1(\Omega_\varepsilon) \text{ such that } w = 0 \text{ on } \Gamma_1 \} \).

We assume to be given a target function \( u_T \in H^1(\Omega) \) and we consider the cost functional

\[
J_\varepsilon : L^2(\Gamma_0) \to \mathbb{R},
\]

given by

\[
J_\varepsilon(v) = \frac{\eta}{2} \int_{\Omega_\varepsilon} B(x) \nabla(u_\varepsilon(v) - u_T) \nabla(u_\varepsilon(v) - u_T) dx + \frac{N}{2} \| v \|^2_{L^2(\Gamma_0)} ,
\]  

(2)
where the weighted balance is defined through the arbitrary positive constants $\eta, N$, and the $H^1$-metric is defined by the $n \times n$ symmetric matrix $B(x) = (b_{ij}(x))$ such that

$$\lambda_i |\xi|^2 \leq b_{ij}(x)\xi_i\xi_j \leq \lambda_2 |\xi|^2,$$  \hspace{1cm} (3)

for a.e. $x \in \Omega$, $\lambda_i = const > 0$, $i = 1, 2$, $B \in C^1(\overline{\Omega})^{n \times n}$.

By well-known results (see e.g. [20], [15], [25]), there exists a unique optimal control $v_\varepsilon \in L^2(\Gamma_0)$

$$J_\varepsilon(v_\varepsilon) = \min_{v \in L^2(\Gamma_0)} J_\varepsilon(v).$$ \hspace{1cm} (4)

One of the main goals of this paper is to find the limit as $\varepsilon \to 0$ of the optimal control $v_\varepsilon$, of the associate state $u_\varepsilon(v_\varepsilon)$ and of the cost functional value $J_\varepsilon(v_\varepsilon)$.

In order to characterize the optimal control $v_\varepsilon$, we will study the particularization of the abstract version of the Pontryagin maximum principle applied to elliptic PDEs mentioned in Section 1.3 of Lions [20].

**Proposition 1.** Let $u_T \in H^1(\Omega)$, $v_\varepsilon \in L^2(\Gamma_0)$ and $u_\varepsilon(v_\varepsilon) \in H^1(\Omega_\varepsilon, \Gamma_1)$ be the target state, the optimal control and the associate optimal state, respectively. Let $P_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1)$ be
the unique solution of the problem

\[
\begin{aligned}
&\Delta P_\varepsilon = \text{div}(B(x)\nabla (u_\varepsilon - u_T)), \quad x \in \Omega_\varepsilon, \\
&\partial_\nu P_\varepsilon - (B(x)\nabla (u_\varepsilon - u_T), \nu) + \varepsilon^{-\gamma} a(x) P_\varepsilon = 0, \quad x \in S_\varepsilon, \\
&\partial_\nu P_\varepsilon - (B(x)\nabla (u_\varepsilon - u_T), \nu) = 0, \quad x \in \Gamma_0, \\
&P_\varepsilon = 0, \quad x \in \Gamma_1.
\end{aligned}
\]

Then, the optimal control \( v_\varepsilon \) is given by

\[
v_\varepsilon = -\frac{\eta}{N} P_\varepsilon.
\]

**Proof.** Since \( v_\varepsilon \) is the optimal control, we know that for any other control \( v \in L^2(\Gamma_0) \)

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} (J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)) = 0.
\]

It is easy to see that if, for a given \( \lambda \in \mathbb{R} \), we define

\[
\theta_\varepsilon = \frac{1}{\lambda} (u_\varepsilon(v_\varepsilon + \lambda v) - u_\varepsilon(v_\varepsilon)),
\]

then \( \theta_\varepsilon \) is a weak solution to the problem

\[
\begin{aligned}
&\Delta \theta_\varepsilon = 0, \quad x \in \Omega_\varepsilon, \\
&\partial_\nu \theta_\varepsilon + \varepsilon^{-\gamma} a(x) \theta_\varepsilon = 0, \quad x \in S_\varepsilon, \\
&\partial_\nu \theta_\varepsilon = v, \quad x \in \Gamma_0, \\
&\theta_\varepsilon = 0, \quad x \in \Gamma_1.
\end{aligned}
\]

Then, we have

\[
0 = \lim_{\lambda \to 0} \frac{1}{\lambda} \left( J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon) \right) = \eta \int_{\Omega_\varepsilon} B(x)\nabla (u_\varepsilon(v_\varepsilon) - u_T)\nabla \theta_\varepsilon dx + N \int_{\Gamma_0} v_\varepsilon v d\xi. \tag{8}
\]

We recall that if \( P_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1) \) is a solution of (5), then we have the integral identity

\[
\int_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla \phi dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon \phi ds = \int_{\Omega_\varepsilon} B(x)\nabla (u_\varepsilon - u_T)\nabla \phi dx, \tag{9}
\]

for an arbitrary function \( \phi \in H^1(\Omega_\varepsilon, \Gamma_1) \). Then, we can set \( \phi = \theta_\varepsilon \) in it and use \( P_\varepsilon \) as a test function in the integral identity for the problem (7). Subtracting the one from the other, we get

\[
\eta \int_{\Omega_\varepsilon} B(x)\nabla (u_\varepsilon(v_\varepsilon) - u_T)\nabla \theta_\varepsilon dx + N \int_{\Gamma_0} v_\varepsilon v d\xi = \int_{\Gamma_0} (\eta P_\varepsilon + N v_\varepsilon) v d\xi = 0,
\]

from which we deduce that \( v_\varepsilon = -\frac{\eta}{N} P_\varepsilon. \)

In order to get the homogenization (as \( \varepsilon \to 0 \)) we will use the usual \( H^1 \)-extensions of functions \( u_\varepsilon \) and \( P_\varepsilon \) to \( \Omega \) which we denote by \( \bar{u}_\varepsilon \) and \( \bar{P}_\varepsilon \) (see, e.g., Section 3.1.1 of [9] and its references).

Then, from the properties of the extension operator (see [26]) and estimates (19), we have
Theorem 1. Let $f \in L^2(\Omega)$, $u_T \in H^1(\Omega)$ and let $(u_\varepsilon, P_\varepsilon) \in H^1(\Omega, \Gamma_1)^2$ be the weak solution of the coupled system

\[
\begin{aligned}
-\Delta u_\varepsilon &= f, \quad x \in \Omega_\varepsilon, \\
\Delta P_\varepsilon &= div(B(x) \nabla (u_\varepsilon - u_T)), \quad x \in \Omega_\varepsilon, \\
\partial_\nu u_\varepsilon + \varepsilon^{-\gamma} a(x) u_\varepsilon &= 0, \quad x \in S_\varepsilon, \\
\partial_\nu P_\varepsilon - (B(x) \nabla (u_\varepsilon - u_T)) \nu + \varepsilon^{-\gamma} a(x) P_\varepsilon &= 0, \quad x \in S_\varepsilon, \\
\partial_\nu u_\varepsilon &= -\frac{\eta}{N} P_\varepsilon, \quad x \in \Gamma_0, \\
\partial_\nu P_\varepsilon - (B(x) \nabla (u_\varepsilon - u_T)) \nu &= 0, \quad x \in \Gamma_0, \\
u_\varepsilon &= P_\varepsilon = 0, \quad x \in \Gamma_1.
\end{aligned}
\]

(10)

Then,

\[
\| \tilde{u}_\varepsilon \|_{H^1(\Omega, \Gamma_1)} \leq C, \quad \| \tilde{P}_\varepsilon \|_{H^1(\Omega, \Gamma_1)} \leq C,
\]

(11)

and thus there exists some subsequences (still denoted as the original ones) such that

\[
\tilde{u}_\varepsilon \rightharpoonup u_0, \quad \tilde{P}_\varepsilon \rightharpoonup P_0, \quad \text{weakly in } H^1(\Omega, \Gamma_1),
\]

(12)

as $\varepsilon \to 0$, for some $(u_0, P_0) \in H^1(\Omega, \Gamma_1)^2$.

Proof. We start by getting some a priori estimates for $u_\varepsilon$ and $P_\varepsilon$. From the integral identity for the function $P_\varepsilon$, we derive

\[
\int_{\Omega_\varepsilon} |\nabla P_\varepsilon|^2 dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon^2 ds = \int_{\Omega_\varepsilon} B(x) \nabla (u_\varepsilon - u_T) \nabla P_\varepsilon dx \leq \frac{1}{2} \| \nabla P_\varepsilon \|^2_{L^2(\Omega_\varepsilon)} + C \| \nabla (u_\varepsilon - u_T) \|^2_{L^2(\Omega_\varepsilon)}.
\]

(13)

The constant $C$ doesn’t depend on $\varepsilon$ here and below. From here, we conclude

\[
\| \nabla P_\varepsilon \|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \| P_\varepsilon \|^2_{L^2(S_\varepsilon)} \leq C \| \nabla (u_\varepsilon - u_T) \|^2_{L^2(\Omega_\varepsilon)}.
\]

(14)

From the integral identity for the function $u_\varepsilon$, we have

\[
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla P_\varepsilon dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) u_\varepsilon P_\varepsilon ds = \int_{\Omega_\varepsilon} f P_\varepsilon dx - \frac{\eta}{N} \int_{\Gamma_0} P_\varepsilon^2 d\tilde{x}.
\]

(15)

From the integral identity for the function $P_\varepsilon$, we get

\[
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla P_\varepsilon dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon u_\varepsilon ds = \int_{\Omega_\varepsilon} B(x) \nabla (u_\varepsilon - u_T) \nabla u_\varepsilon dx.
\]

(16)

Subtracting the equality (15) from (16), we get

\[
\frac{\eta}{N} \int_{\Gamma_0} P_\varepsilon^2 d\tilde{x} + \int_{\Omega_\varepsilon} B(x) \nabla (u_\varepsilon - u_T) \nabla (u_\varepsilon - u_T) dx = \int_{\Omega_\varepsilon} f P_\varepsilon dx - \int_{\Omega_\varepsilon} B(x) \nabla u_T \nabla (u_\varepsilon - u_T) dx.
\]

(17)

This equality and the condition (3) imply

\[
\| P_\varepsilon \|^2_{L^2(\Gamma_0)} \leq C(\| \nabla u_T \|^2_{L^2(\Omega)} + \| f \|^2_{L^2(\Omega)}), \quad \| \nabla (u_\varepsilon - u_T) \|^2_{L^2(\Omega_\varepsilon)} \leq C(\| \nabla u_T \|^2_{L^2(\Omega)} + \| f \|^2_{L^2(\Omega)}).
\]

(18)
From (14) and (18), we derive
\[
\| u_\varepsilon \|_{H^1(\Omega, \Gamma_1)} \leq C, \quad \| P_\varepsilon \|_{H^1(\Omega, \Gamma_1)} \leq C. \tag{19}
\]
Since \( \tilde{u}_\varepsilon \) and \( \tilde{P}_\varepsilon \), are the \( H^1 \)-extensions of \( u_\varepsilon \) and \( P_\varepsilon \) to \( \Omega \), from the properties of the extension operator (see, e.g., [26]) and estimates (19), we get the estimates (11) and thus we conclude the weak convergences indicated in (12) for some subsequences. 

In order to identify the limit problem satisfied by the pair \((u_0, P_0)\), we need several auxiliary results.

3. Auxiliary statements

For \( j \in \Gamma_\varepsilon \), we introduce the function \( w^j_\varepsilon(x) \) as being the unique solution to the capacity boundary value problem
\[
\begin{aligned}
\Delta w^j_\varepsilon &= 0, \quad x \in T^{j}_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\
w^j_\varepsilon &= 1, \quad x \in \partial G^j_\varepsilon, \\
w^j_\varepsilon &= 0, \quad x \in \partial T^{j}_{\varepsilon/4},
\end{aligned}
\tag{20}
\]
where \( T^{j}_{\varepsilon/4} \) is the ball of radius \( \varepsilon/4 \) with center in the point \( P^j_\varepsilon \) (the center of the cube \( Y^j_\varepsilon \)). It is easy to see (Section 3.1.5.1 of [9]) that
\[
w^j_\varepsilon(x) = \frac{|x - P^j_\varepsilon|^2 - (\varepsilon/4)^2}{a^2_\varepsilon - (\varepsilon/4)^2}.
\]
Define the extension function
\[
W_\varepsilon(x) = \begin{cases} 
w^j_\varepsilon(x), & x \in T^{j}_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \; j \in \Gamma_\varepsilon, \\
1, & x \in \bigcup_{j \in \Gamma_\varepsilon} \overline{G^j_\varepsilon}, \\
0, & x \in \Omega \setminus \bigcup_{j \in \Gamma_\varepsilon} T^{j}_{\varepsilon/4}.
\end{cases}
\tag{21}
\]
It is clear that we have \( W_\varepsilon \rightharpoonup 0 \), weakly in \( H^1_0(\Omega) \), as \( \varepsilon \to 0 \).

**Lemma 1.** Let \( \phi \in C^1(\overline{\Omega}, \Gamma_1) \). Then,
\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} (B \nabla W_\varepsilon, \nabla W_\varepsilon) \phi dx = \frac{C_{n-2}(n-2)\omega_n}{n} \int_{\Gamma_0} \text{tr} B(0, \hat{x}) \phi(0, \hat{x}) d\hat{x}, \tag{22}
\]
where \( \text{tr} B(x) = \sum_{j=1}^n b_{jj}(x) \) is the trace of the matrix \( B(x) \) and \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

**Proof.** Note that if \( \widetilde{B} \) is the matrix with the constant elements (with respect to the \( y \)-variable), then
\[
\int_{S^0_1} (\widetilde{B} y, y) ds = \int_{T^0_1} \text{div}(\widetilde{B} y) dy = \text{tr} \widetilde{B} |T^0_1| = \text{tr} \widetilde{B} \omega_n, \tag{23}
\]
where \( S^0_1, T^0_1 \) are the unit sphere and the unit ball with the center at the coordinate’s origin.
Using equality (23), if \( r = \sqrt{\sum_{j=1}^{n}(x_i - P_{e,i}^j)^2} \), we get

\[
\left. \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} (B(x)\nabla W_\varepsilon, \nabla W_\varepsilon)\phi(x)dx = \lim_{\varepsilon \to 0} \sum_{j \in \mathcal{J}_\varepsilon} \phi(P_{j}^e) \int_{G_\varepsilon} (B(P_{j}^e)\nabla w_{\varepsilon}^j, \nabla w_{\varepsilon}^j)dx = \right. \\
= \lim_{\varepsilon \to 0} \sum_{j \in \mathcal{J}_\varepsilon} a_{\varepsilon}^{-2(n-2)}(n-2)^2\phi(P_{j}^e) \sum_{k,l=1}^{n} \int_{G_\varepsilon} b_{kl}(P_{j}^e)(x_l - P_{e,i,l}, x_k - P_{e,i,k})r^{-2n}dr = \\
= \lim_{\varepsilon \to 0} \sum_{j \in \mathcal{J}_\varepsilon} a_{\varepsilon}^{-2(n-2)}(n-2)^2\phi(P_{j}^e) \int_{a_\varepsilon}^{2\varepsilon} r^{1-n}dr \int_{S_\varepsilon} (B(P_{j}^e)y, y)ds = \\
= \lim_{\varepsilon \to 0} \sum_{j \in \mathcal{J}_\varepsilon} a_{\varepsilon}^{-2(n-2)}(n-2)^2\phi(P_{j}^e) \int_{a_\varepsilon}^{2\varepsilon} r^{1-n}dr = \\
= C_0^{n-2}(n-2)^{\omega_n/2} \lim_{\varepsilon \to 0} \sum_{j \in \mathcal{J}_\varepsilon} \phi(P_{j}^e) \int_{\Gamma_\varepsilon} trB(0, \hat{x})\phi(0, \hat{x})d\hat{x},
\]

which proves the result. 

We consider now the auxiliary boundary value problem

\[
\begin{aligned}
\Delta h_\varepsilon &= div(B\nabla W_\varepsilon), \quad x \in \Omega_\varepsilon, \\
\partial_\nu h_\varepsilon - (B\nabla W_\varepsilon, \nu) + \varepsilon^{-\gamma}a(x)h_\varepsilon &= 0, \quad x \in S_\varepsilon, \\
\partial_\nu h_\varepsilon - (B\nabla W_\varepsilon, \nu) &= 0, \quad x \in \Gamma_0, \\
h_\varepsilon &= 0, \quad x \in \Gamma_1.
\end{aligned}
\]

As usual, we say that function \( h_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_1) \) is a weak solution of problem (24) if it satisfies the integral identity

\[
\int_{\Omega_\varepsilon} \nabla h_\varepsilon \nabla \phi dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x)h_\varepsilon \phi ds = \int_{\Omega_\varepsilon} (B\nabla W_\varepsilon, \nabla \phi)dx,
\]

where \( \phi \) is an arbitrary function from the space \( H^1(\Omega_\varepsilon, \Gamma_1) \). Setting \( \phi = h_\varepsilon \) in (20) and using the properties of function \( W_\varepsilon \), we derive an estimate \( \| h_\varepsilon \|_{H^1(\Omega, \Gamma_1)} \leq C \). Therefore, there exists a subsequence (we preserve the notation of the original one) such that

\[
\tilde{h}_\varepsilon \rightharpoonup h_0, \text{ weakly in } H^1(\Omega, \Gamma_1), \quad \varepsilon \to 0.
\]

The following theorem identifies the limit function \( h_0 \).

**Theorem 2.** The function \( h_0 \) defined in (26) is a weak solution of the boundary value problem

\[
\begin{aligned}
\Delta h_0 &= 0, \quad x \in \Omega, \\
\partial_\nu h_0 + A_1 \frac{a(x)}{a(x) + C_n} h_0 &= -A_2 \frac{trB(x)a(x)}{a(x) + C_n}, \quad x \in \Gamma_0, \\
h_0 &= 0, \quad x \in \Gamma_1,
\end{aligned}
\]

where \( A_1(n) = (n-2)C_0^{n-2}\omega_n, \quad A_2(n) = \frac{A_1(n)}{n}, \quad C_n = \frac{n-2}{C_0} \).
Proof. We take as test function in the integral identity (25) the function \( \phi = W_\varepsilon \psi \), where \( \psi \in C^\infty(\Omega, \Gamma_1) \) (this is called an oscillating test function according to the general Tartar’s method) and we get

\[
\int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (h_\varepsilon \psi) dx + \varepsilon^{-\gamma} \int_{\partial \Omega_\varepsilon} a(x) h_\varepsilon \psi ds = \int_{\Omega_\varepsilon} (B(x) \nabla W_\varepsilon, \nabla W_\varepsilon) \psi dx + \beta_\varepsilon,
\]

where \( \beta_\varepsilon \to 0, \varepsilon \to 0 \).

From (28), we derive

\[
\sum_{j \in \Theta_\varepsilon} \int_{\partial \Omega_\varepsilon} \partial_\nu w^j_\varepsilon h_\varepsilon \psi ds + \varepsilon^{-\gamma} \int_{\partial \Omega_\varepsilon} a(x) h_\varepsilon \psi ds +
\]

\[
+ \sum_{j \in \Theta_\varepsilon} \int_{\partial \Gamma_\varepsilon/4} \partial_\nu w^j_\varepsilon h_\varepsilon \psi ds = \frac{C_0^{n-2}(n-2)\omega_n}{n} \int_{\Gamma_0} tr B(\hat{x}) \psi(\hat{x}) d\hat{x}.
\]

Hence, it follows that

\[
e^{-\gamma} \int_{\partial \Omega_\varepsilon} \left( a(x) + \frac{n-2}{C_0} \right) h_\varepsilon \psi ds =
\]

\[
- \sum_{j \in \Theta_\varepsilon} \int_{\partial \Gamma_\varepsilon/4} \partial_\nu w^j_\varepsilon h_\varepsilon \psi ds + \frac{C_0^{n-2} \omega_n (n-2)}{n} \int_{\Gamma_0} tr B(\hat{x}) \psi(\hat{x}) d\hat{x} + \beta_\varepsilon.
\]

We set \( \psi(x) = \frac{a(x)}{a(x) + C_n} v(x) \), where \( v \in C^\infty(\Omega, \Gamma_1), C_n = \frac{n-2}{C_0} \), and get

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{\partial \Omega_\varepsilon} a(x) h_\varepsilon v ds =
\]

\[
= C_0^{n-2}(n-2)\omega_n \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} h_0 v d\hat{x} + \frac{C_0^{n-2}(n-2)\omega_n}{n} \int_{\Gamma_0} \frac{tr B(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} v(\hat{x}) d\hat{x}.
\]

Therefore, from (25) and (31), we obtain the integral identity for the function \( h_0 \)

\[
\int_{\Omega} \nabla h_0 \nabla \phi dx + A_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} h_0 \phi d\hat{x} + \frac{A_1}{n} \int_{\Gamma_0} \frac{tr B(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} \phi(\hat{x}) d\hat{x} = 0.
\]

This implies the statement of the theorem. ■

4. The main result

So, in order to obtain the characterization of \( u_0 \) and \( P_0 \), we have to pass to the limit in the identity (9).
Theorem 3. Let $n \geq 3$, $\alpha = \gamma = \frac{n-1}{n-2}$ and let $(u_\varepsilon, P_\varepsilon)$ be a weak solution of the coupled system (10). Then, the pair $(u_0, P_0)$ defined in (12) is a weak solution of the system

\[
\begin{cases}
-\Delta u_0 = f, & x \in \Omega, \\
\Delta P_0 = \text{div}(B \nabla (u_0 - u_T)), & x \in \Omega, \\
\partial_n u_0 + A_1 \frac{a(x)}{a(x) + C_n} u_0 = -\frac{\eta}{N} P_0, & x \in \Gamma_0, \\
\partial_n P_0 - (B(x) \nabla (u_0 - u_T), \nu) + A_1 \frac{a(x)}{a(x) + C_n} P_0 - A_2 \frac{\text{tr} B(x) a^2(x)}{(a(x) + C_n)^2} u_0 = 0, & x \in \Gamma_0, \\
u_0 = P_0 = 0, & x \in \Gamma_1,
\end{cases}
\]

where $\text{tr} B(x) = \sum_{j=1}^{n} b_{jj}(x)$ is the trace of the matrix $B(x)$, $A_1(n) = (n - 2) C_0^{n-2} \omega_n$, $A_2(n) = \frac{A_1(n)}{n}$, $C_n = \frac{n - 2}{C_0}$ and $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$. In addition, if we define the functional

\[
J_0(v) = \frac{\eta}{2} \int_{\Omega} B \nabla (u_0(v) - u_T) \nabla (u_0(v) - u_T) dx + \frac{N}{2} \int_{\Gamma_0} v^2 d\hat{x} \\
+ \frac{\eta A_1}{2n} \int_{\Gamma_0} \text{tr} B(\hat{x}) \left( \frac{a(\hat{x})}{a(\hat{x}) + C_n} \right)^2 u_0^2(v) d\hat{x},
\]

then

\[
\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0),
\]

where $v_0 = -\frac{\eta}{N} P_0$ on $\Gamma_0$. In particular, $v_\varepsilon \rightharpoonup v_0$ weakly in $L^2(\Gamma_0)$, and $v_0$ is the optimal control of the problem

\[
J_0(v_0) = \inf_{v \in L^2(\Gamma_0)} J_0(v),
\]

associated to the state problem

\[
\begin{cases}
-\Delta u_0(v) = f, & x \in \Omega, \\
\partial_n u_0 + A_1 \frac{a(x)}{a(x) + C_0} u_0(v) = v, & x \in \Gamma_0, \\
u_0(v) = 0, & x \in \Gamma_1.
\end{cases}
\]

Proof. Let us find the homogenized boundary value problem for the function $u_0$. We set $\phi = W_\varepsilon \psi$ in the integral identity for $u_\varepsilon$ and get

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) u_\varepsilon(x) \phi ds = A_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 \phi d\hat{x}.
\]

Hence, $u_0 \in H^1(\Omega, \Gamma_1)$ satisfies integral identity

\[
\int_{\Omega} \nabla u_0 \nabla \phi dx + A_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 \phi d\hat{x} = \int_{\Omega} f \phi dx - \frac{\eta}{N} \int_{\Gamma_0} P_0 \phi d\hat{x},
\]

for any $\phi \in H^1(\Omega, \Gamma_1)$. From here, we derive that $u_0$ is a weak solution of the problem
\[
\left\{
\begin{array}{ll}
-\Delta u_0 = f, & x \in \Omega, \\
\partial_n u_0 + A_n \frac{a(x)}{a(x) + C_n} u_0 = -\frac{\eta}{N} P_0, & x \in \Gamma_0, \\
u_0 = 0, & x \in \Gamma_1.
\end{array}
\right.
\]

Next, let us obtain the limit problem satisfied by \(P_0\). Let \(\psi \in C^\infty(\overline{\Omega}, \Gamma_1)\). We take \(\phi = W_\epsilon \psi\) as a test function in the integral identity for the function \(P_\epsilon\) and get

\[
\int_{\Omega_\epsilon} \nabla P_\epsilon \nabla (W_\epsilon \psi) \, dx + \varepsilon^{-1} \int_{S_\epsilon} a(x) P_\epsilon \psi \, ds = \int_{\Omega_\epsilon} B(x) \nabla (u_\epsilon - u_T) \nabla (W_\epsilon \psi) \, dx. \tag{36}
\]

In order to pass to the limit, as \(\varepsilon \to 0\), in the right-hand side of the identity (35), we set \(\phi = u_\epsilon \psi\) in the integral identity (21) and \(\phi = h_\epsilon \psi\) in the integral identity for \(u_\epsilon\). Then, subtracting one from the other, we get

\[
\int_{\Omega_\epsilon} B(x) \nabla W_\epsilon \nabla (u_\epsilon \psi) \, dx = \int_{\Omega_\epsilon} \nabla h_\epsilon \nabla (u_\epsilon \psi) \, dx - \int_{\Omega_\epsilon} \nabla u_\epsilon \nabla (h_\epsilon \psi) \, dx + \int_{\Omega_\epsilon} f h_\epsilon \psi \, dx - \frac{\eta}{N} \int_{\Gamma_0} P_\epsilon h_\epsilon \psi \, dx. \tag{37}
\]

Identity (37) implies

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\epsilon} B \nabla W_\epsilon \nabla (u_\epsilon \psi) \, dx = \int_{\Omega} \nabla h_0 \nabla \psi u_0 \, dx - \int_{\Omega} h_0 \nabla u_0 \nabla \psi \, dx + \int_{\Omega} f h_0 \psi \, dx - \frac{\eta}{N} \int_{\Gamma_0} P_0 h_0 \psi \, d\hat{x}. \tag{38}
\]

We now use the fact that we already know the problems satisfied by \(u_0\) and \(h_0\). So, we get

\[
\int_{\Omega} (\nabla h_0, \nabla \psi) u_0 \, dx - \int_{\Omega} h_0 \nabla u_0 \nabla \psi \, dx = \int_{\Omega} \nabla h_0 \nabla (u_0 \psi) \, dx - \int_{\Omega} \nabla u_0 \nabla (h_0 \psi) \, dx - \frac{A_1}{n} \int_{\Gamma_0} \frac{\text{tr} B(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} u_0 \psi \, d\hat{x} - \int_{\Omega} f h_0 \psi \, dx + \frac{\eta}{N} \int_{\Gamma_0} P_0 \psi \, d\hat{x}. \tag{39}
\]

Comparing expressions (38) and (39), we conclude

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\epsilon} B \nabla W_\epsilon \nabla (u_\epsilon \psi) \, dx = -\frac{A_1}{n} \int_{\Gamma_0} \frac{\text{tr} B(\hat{x}) a(\hat{x})}{a(\hat{x}) + C_n} u_0 \psi \, d\hat{x}. \tag{40}
\]

We take \(W_\epsilon \psi\), with \(\psi \in C^\infty(\overline{\Omega}, \Gamma_1)\), as a test function in the integral identity for \(P_\epsilon\) and get

\[
\int_{\Omega_\epsilon} \nabla W_\epsilon \nabla (P_\epsilon \psi) \, dx + \varepsilon^{-1} \int_{S_\epsilon} a(x) P_\epsilon \psi \, ds = \int_{\Omega_\epsilon} B \nabla W_\epsilon \nabla (u_\epsilon \psi) \, ds + \kappa_\varepsilon, \tag{41}
\]

where \(\kappa_\varepsilon \to 0\) as \(\varepsilon \to 0\).
From the definition of the function \( W_\varepsilon \) and its properties, we can transform the left-hand side of equality (41) in the following way

\[
\sum_{j \in \Gamma_\varepsilon} \int_{\partial \Omega \cup \partial \Gamma_{\varepsilon_j}} \partial_\nu w_\varepsilon^j P_\varepsilon \psi ds + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon \psi ds = \frac{A_1}{n} \int_{\Gamma_0} tr B(\hat{x}) \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 \psi d\hat{x} + \kappa_\varepsilon,
\]

where \( \kappa_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

The left-hand side of the last equality takes the form

\[
\varepsilon^{-\gamma} \int_{S_\varepsilon} (a(x) + C_n) P_\varepsilon \psi ds + \sum_{j \in \Gamma_\varepsilon} \int_{\partial \Gamma_{\varepsilon_j}} \partial_\nu w_\varepsilon^j P_\varepsilon \psi ds + \theta_\varepsilon,
\]

where \( \theta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

From (40)-(42), we deduce

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) P_\varepsilon \phi ds = A_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} P_0 \phi d\hat{x} - \frac{A_1}{n} \int_{\Gamma_0} tr B(\hat{x}) \left( \frac{a(\hat{x})}{a(\hat{x}) + C_n} \right)^2 u_0 \phi d\hat{x},
\]

where \( \varphi \) is an arbitrary function from \( C^\infty(\Omega, \Gamma_1) \). Passing to the limit in the integral identity for \( P_\varepsilon \) we get the theorem’s statement.

Now we will find the limit as \( \varepsilon \to 0 \) of the cost functional

\[
J_\varepsilon(v_\varepsilon) = \frac{\eta}{2} \int_{\Omega_\varepsilon} B \nabla (u_\varepsilon(v_\varepsilon) - u_T) \nabla (u_\varepsilon(v_\varepsilon) - u_T) dx + \frac{N}{2} \| v_\varepsilon \|_{L^2(\Gamma_0)}^2.
\]

We take \( \phi = u_\varepsilon \) in the integral identity for \( P_\varepsilon \) and \( \phi = P_\varepsilon \) in the integral identity for \( u_\varepsilon \).

Then, we subtract the one from the other and pass to the limit as \( \varepsilon \to 0 \):

\[
A \equiv \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} B(x) \nabla (u_\varepsilon - u_T) \nabla u_\varepsilon dx = \int_{\Omega} f P_0 dx - \frac{\eta}{N} \int_{\Gamma_0} P_0^2 d\hat{x}.
\]

Taking into account the integral identity for the limit of the adjoint problem, we get

\[
A = \int_{\Omega} \nabla u_0 \nabla P_0 dx + A_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 P_0 dx = \int_{\Omega} B \nabla (u_0 - u_T) \nabla u_0 dx + \frac{A_1}{n} \int_{\Gamma_0} tr B(\hat{x}) \left( \frac{a(\hat{x})}{a(\hat{x}) + C_n} \right)^2 u_0^2 d\hat{x}.
\]

From here, we obtain (33). Since the trace is a continuous operator on \( H^1(\Omega_\varepsilon, \Gamma_1) \) we get that \( v_\varepsilon \to v_0 \) weakly in \( L^2(\Gamma_0) \). Since we have that \( v_0 = -\frac{\eta}{N} P_0 \) on \( \Gamma_0 \) and this is the optimality condition associated to the control problem (34) (the proof is an easy variation of Proposition 1), then \( v_0 \) is the unique optimal control problem associated to the convex cost functional \( J_0(v) \).

**Remark 1.** It is not too difficult to prove that the coupled system (32) has only one weak solution \((u_0, P_0)\). For instance, one indirect proof can be obtained through the strict convexity of the functional \( J_0 \). In particular, this implies that the weak convergence obtained
in Theorem 1 holds for any subsequences of the original ones (since the limit \((u_0, P_0)\) is unique).

**Remark 2.** Such as it is detailed explained in the book [9], it can be proved that the choice of the scales and parameters \(\alpha = \gamma = \frac{n-1}{n-2}\) is the reason to get an anomalous boundary behavior on \(\Gamma_0\) (for bigger size of the elements of the lattice we get different coefficients). This phenomenon was called in the literature as the appearance of a “strange term” and in several papers it was associated to a certain “measure” \(\mu\). One of the merits of Theorem 3 is to show that the “strange term” is a certain completely identified function on \(\Gamma_0\).

**Remark 3.** Similar problems arise in many different applications, especially in the field of Chemical Engineering (see, e.g., [17] and Chapter 5 of [9]). Some models in Climatology also use the identification as a final boundary condition the limit of a thin layer on which there are some suitable balances of differential equations and transmissions conditions (see the so called energy balance models coupled with a deep ocean in [11]). Problems quite similar to the one considered in this paper arise also in Elasticity (see, e.g., [5]).

**Remark 4.** Many generalizations and applications seem possible and some of them will be developed in some future works by the authors: i) Non-symmetrical shapes can also be treated thanks to the techniques presented in [13], ii) The case of non-periodic lattices (under the assumption of “stationary and ergodic” random media) can be considered as in the framework treated in [3], [18], [6] and [19] (see many other references in Appendix C of the book [9]), iii) Optimal control problems for semilinear equations and/or nonlinear boundary conditions could be approached as, for instance, in [7]. iv) The extension of the techniques of this paper can be also applied to the consideration of several parabolic problems (see, e.g., Appendix A of [9]) and its references. v) by passing to the limit when parameter \(N \to +\infty\) it is possible to get some results on the approximate controllability with internal observation (the \(H^1\) norm of \((u_e(v) - u_T)\) can be made as small as wanted): see [16].

5. **Convergence of the energy for the problem without control**

In this last Section, we consider the boundary value problem without any control (i.e. problem (1) with \(v \equiv 0\))

\[
\begin{align*}
-\Delta u_e(v) &= f, & x \in \Omega_e, \\
\partial^\nu u_e(v) + \varepsilon^{\nu} a(x) u_e(v) &= 0, & x \in S_e, \\
\partial^\nu u_e(v) &= 0, & x \in \Gamma_0, \\
u_e(v) &= 0, & x \in \Gamma_1,
\end{align*}
\]

where \(f \in L^2(\Omega), a(x) \in C^\infty(\Omega), a(x) \geq a_0 = \text{const} > 0\). The homogenization techniques of previous sections can be easily adapted to prove that \(\tilde{u}_e \to u_0\) weakly in \(H^1(\Omega_e, \Gamma_1)\) and
that \( u_0 \) is a weak solution of the problem

\[
\begin{align*}
-\Delta u_0(v) &= f, & x &\in \Omega, \\
\partial_n u_0 + A_1 \frac{a(x)}{a(x) + C_0} u_0(v) &= 0, & x &\in \Gamma_0, \\
u_0(v) &= 0, & x &\in \Gamma_1.
\end{align*}
\]

Our main goal now is to prove that the consideration of an artificial complementary system (formally corresponding to the case \( v \equiv 0 \) and \( u_T \equiv 0 \) and \( B = I \)) allows to prove the convergence of the corresponding energies.

**Theorem 4.** Let \( u_\varepsilon \) be the solution of (46) with \( v \equiv 0 \). Let \( u_0 \in H^1_0(\Omega) \) be the weak limit of the extension \( P_\varepsilon u_\varepsilon \). Then, we have the convergence of the energy

\[
\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \to \int_{\Omega} |\nabla u_0|^2 dx + A_1 \int_{\Gamma_0} \left( \frac{a(\hat{x})}{a(\hat{x}) + C_n} \right)^2 u_0^2 d\hat{x}.
\]

**Proof.** We consider the auxiliary problem

\[
\begin{align*}
\Delta P_\varepsilon &= \Delta u_\varepsilon, & x &\in \Omega_\varepsilon, \\
\partial_n P_\varepsilon - \partial_n u_\varepsilon + \varepsilon^{-\gamma} a(x) P_\varepsilon &= 0, & x &\in S_\varepsilon, \\
\partial_n P_\varepsilon - \partial_n u_\varepsilon &= 0, & x &\in \Gamma_0, \\
P_\varepsilon &= 0, & x &\in \Gamma_1.
\end{align*}
\]

(48)

As in the proof of Theorem 3, we get that \( \tilde{P}_\varepsilon \to P_0 \) weakly in \( H^1(\Omega, \Gamma_1) \) as \( \varepsilon \to 0 \), with \( P_0 \) the weak solution of the problem

\[
\begin{align*}
\Delta P_0 &= \Delta u_0, & x &\in \Omega, \\
\partial_n P_0 - \partial_n u_0 + A_1 \frac{a(x)}{a(x) + C_n} P_0 - A_1 \frac{a^2(x)}{(a(x) + C_n)^2} u_0 &= 0, & x &\in \Gamma_0, \\
P_0 &= 0, & x &\in \Gamma_1,
\end{align*}
\]

(49)

where \( A_1(n) = (n - 2)C_0^{n-2} \omega_n, \ C_n = \frac{n-2}{C_0} \) and \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

From the variational formulation of the problem (46), taking \( P_\varepsilon \) as a test function, we have

\[
\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} a(x) u_\varepsilon P_\varepsilon ds = \int_{\Omega_\varepsilon} f P_\varepsilon dx.
\]

Similarly, from the variational formulation of the problem (49) on \( P_\varepsilon \), taking \( u_\varepsilon \) as a test function, we derive

\[
\int_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla u_\varepsilon dx + \varepsilon^{-k} \int_{S_\varepsilon} a(x) P_\varepsilon u_\varepsilon ds = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.
\]
Thus, we have
\[
\int_{\Omega} |\nabla u_\varepsilon|^2 \, dx = \int_{\Omega} fP_\varepsilon \, dx \to \int_{\Omega} fP_0 \, dx = \\
= \int_{\Omega} \nabla u_0 \nabla P_0 \, dx + A_1 \int_{\Gamma_0} \frac{a(\hat{x})}{a(\hat{x}) + C_n} u_0 P_0 \, d\hat{x} = \\
= \int_{\Omega} |\nabla u_0|^2 \, dx + A_1 \int_{\Gamma_0} \left( \frac{a(\hat{x})}{a(\hat{x}) + C_n} \right)^2 u_0^2 \, d\hat{x},
\]
which ends the proof. 

**Remark 5.** It can be proved (for instance, by adapting the arguments presented in Section 4.7.1.4 of [9]) that if we know that \( u_0 \in W^{1,\infty}(\Omega) \) then we can get some results implying the strong convergence of \( u_\varepsilon \) plus a suitable “correction term”. Notice that the conclusion presented in the proof of Theorem 4 follows different ideas.

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