CONTINUOUS DEPENDENCE & STABILIZATION OF SOLUTIONS
OF THE DEGENERATE SYSTEM IN TWO-PHASE FILTRATION

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The filtration of two immiscible fluids in a heterogeneous
anisotropic porous medium is described by a system of nonlinear
differential equations including an eventual degenerate parabolic
equation for the saturation $S(x,t)$ of one of the phases and
an elliptic equation for the "reduced" pressure $p(x,t)$:

\[
\begin{align*}
\frac{\partial S}{\partial t} &= \text{div} \left( k_0(x)\alpha(\theta)^n \nabla \theta + k_1(x, S)\nabla p + f_0(x,S) \right), \\
\text{div}(k(x,S)\nabla p + f(x,S)) &= 0,
\end{align*}
\]

$(x,t) \in \Omega \times (0,T),$ where $\Omega$ is a bounded and multiple-connected domain with the boundary $\Gamma.$ The derivation of such a system and the meaning of each of the coefficients can be found in [1].

For system (1), (2) we consider the following initial-boundary problem
\[ s(x,t) = s^0(x,t), \quad p(x,t) = p^0(x,t), \quad (x,t) \in \Gamma_{1T} = \Gamma_1 \times (0,T), \quad (3) \]
\[ (K \psi + f) \n = -Q(x,t), \quad (K_0 \psi \delta + K_1 \nu \psi + f) \n = -Q_1(x,t), \quad (4) \]
\[ (x,t) \in \Gamma_{2T} = \Gamma_2 \times (0,T), \]
\[ s(x,0) = s^0(x,0), \quad x \in \Omega. \quad (5) \]

Here \( \Gamma = \Gamma_1 + \Gamma_2; \ n \) is the unit outward normal vector to \( \Gamma; \)
\( p^0, s^0, Q, Q_1 \) are the prescribed functions.

The solvability of problem (1)-(5) has been considered by numerous authors; see references in the monograph [1].

The uniqueness of generalized solutions of this problem was proved in [2] and in [3] for \( \Gamma = \Gamma_1 \).

This paper presents a complete account of the brief article [4].

1. Continuous dependence result.

In [1] the proof is made for existence of generalized solution of the problem (1)-(5) in the following sense:

\[ s(x,t), p(x,t) \] are the bounded measurable functions,

\[ s(x,t) \in [0,1], \quad p(x,t) \in L_{\infty}(\Omega_t) \cap L_{\infty}(0,T; W^2_2(\Omega)), \]

\[ s(x,t) = \int_0^1 \sqrt{a(\xi)} \, d\xi \in L_{\infty}(0,T; W^2_2(\Omega)) \]

satisfy (3) in the sense of traces from \( W^2(\Omega); \)

for all \( \psi(x,t), \phi(x,t), (\psi, \phi) \in W^2(\Omega_t), \quad \phi(x,t) = 0, \]

\( (x,t) \in \Gamma_{1T}, \quad \phi(x,t) = 0, \quad x \in \Omega \) and almost all \( t \in (0,T) \) the following integral identities are fulfilled:

\[ -(m_s, \phi)_{\Omega_t} + (m_s, Q_1)_{\Omega_t} - (K_0 \psi \delta + K_1 \nu \psi + f)_{\Omega_t} - (Q, \phi)_{\Gamma_{2T}} = 0, \quad (6) \]

\[ (K \psi + f, \psi)_{\Omega_t} + (Q, \psi)_{\Gamma_{2T}} = 0, \quad (7) \]

where

\[ (u,v)_{\Omega_t} = \int_{\Omega_t} u \nabla v \, dx, \quad (u,v)_{\Omega_0} = \int_{\Omega_0} u \nabla v \, dx, \quad \Omega_t = \Omega \times (0,t), \quad \Gamma_{2T} = \Gamma_2 \times (0,T). \]

In the present article we shall assume the extra condition

\[ \nu p \in L_{\infty}(\Omega_t) \]

which has been obtained in [1] under some additional conditions.

Introduce the function

\[ c(s) = \int_0^s c(t) \, dt \]

Let \( \omega^i = (s^i, p^i), \quad i = 1,2 \) denote any generalized solutions of problem (1)-(5) associated to the data \( a^i, c^i, m^i, s^0, p^0, f^i \).

\[ f^i = K^i, K_0^i, K_1^i, Q^i, Q_1^i. \]

One of the key assumptions of our results is the one concerning the diffusion coefficients and the boundary data on \( \Gamma_1; \) it is stated as the following alternative:

\[ |c^1(s^2) - c^2(s^2)| \leq M \int_0^1 c^1(t \, s^1 + (1-t) s^2) \, dt \]

for some \( M > 0 \) and \( q \in (0,2) \), and

\[ c^1(s^1) = c^2(s^2), \quad (x,t) \in \Gamma_{1T}, \]

either there exist \( M > 0 \) such that

\[ a^1(s) < M^2 s^2(s), \quad s, \psi \in (0,1) \quad and \quad s^0 = s_0^i, \quad (x,t) \in \Gamma_{1T}. \quad (8') \]

Theorem 1. Assume, that conditions (8) or (8') holds. Assume also, that there exist \( M > 0 \) such that

\[ m^1 < m^2, \quad (K^1_0, K^1_1, f^1_0) < M, \quad (f^2_1, f^2_0, f^2_0) < M \]

\[ |K^1_0(x) - K^0(x)| \leq M, \quad 0 < c^j \in C[0,1], \quad j = 1,2, \quad (9) \]

\[ K^j_0(x) \in C(\Omega), \quad \text{meas} \quad \Gamma_1 > 0, \]

then for \( \omega^i = (s^i, p^i) \) with \( |\nu p^i| \leq M \) the following estimate holds

\[ (s^1 - s^2, c^1(s^1) - c^2(s^2))_{\Omega_t} + |p^1 - p^2|_{\Omega_0} \leq C (M, T) H, \quad (10) \]

where

\[ H = |m^1 - m^2|_{L^\infty(\Omega)} + |s^0 - s^2|_{L^\infty(\Omega)}, \quad \Gamma_{1T}, \quad |p^1 - p^2|_{L^\infty(\Omega)}, \Gamma_{1T} + |Q^1 - Q^2|_{L^\infty(\Omega)}, \Gamma_{1T} + |K^1_0 - K^2_0|_{L^\infty(\Omega)}, \Gamma_{1T} + |K^1_1 - K^2_1|_{L^\infty(\Omega)} + \int_0^T |\nabla^i - \nabla^0|^2_{C^{0,1}} \, dt, \quad (11) \]

\[ H = |m^1 - m^2|_{L^\infty(\Omega)} + |s^0 - s^2|_{L^\infty(\Omega)}, \quad \Gamma_{1T}, \quad |p^1 - p^2|_{L^\infty(\Omega)}, \Gamma_{1T} + |Q^1 - Q^2|_{L^\infty(\Omega)}, \Gamma_{1T} + |K^1_0 - K^2_0|_{L^\infty(\Omega)} + |K^1_1 - K^2_1|_{L^\infty(\Omega)} + \int_0^T |\nabla^i - \nabla^0|^2_{C^{0,1}} \, dt, \quad (11) \]

\[ H = |m^1 - m^2|_{L^\infty(\Omega)} + |s^0 - s^2|_{L^\infty(\Omega)}, \quad \Gamma_{1T}, \quad |p^1 - p^2|_{L^\infty(\Omega)}, \Gamma_{1T} + |Q^1 - Q^2|_{L^\infty(\Omega)}, \Gamma_{1T} + |K^1_0 - K^2_0|_{L^\infty(\Omega)} + |K^1_1 - K^2_1|_{L^\infty(\Omega)} + \int_0^T |\nabla^i - \nabla^0|^2_{C^{0,1}} \, dt, \quad (11) \]
Here \( V = \{0, 1\} \cdot \Omega \) and \( q = q \) if \((8')\) takes place.

Proof. From the notion of generalized solution we deduce, after integration by parts, that
\[
\begin{align*}
(m' \{s' - s^2\}, \varphi)_{\Omega_t} + (c(1) - c(2), dtu(K_1 \varphi))_{\Omega_t} + \\
(k_1'(x, s') \varphi^2 - k_1'(x, s') \varphi)_{\Omega_t} + \int_0^t \left( f_0'(x, s') \varphi + f_1'(x, s') \right) dt \varphi_{\Omega_t} = \\
- (c(1) - c(2), dtu(K_1 \varphi))_{\Omega_t} + (c(1) - c(2), K_1 \varphi \cdot \hat{n})_{\Gamma_{1T}} - \\
((m' - m^2) s', \varphi)_{\Omega_t} + ((k_0' - k_0) \omega^2 s', \varphi)_{\Omega_t} + \\
\left( (k_1'(x, s') - k_0' \varphi)^2 \varphi, \varphi \right)_{\Omega_t} + \\
((m' - m^2) s, \varphi)_{\Omega_t} - (m' \{s - s^2\}, \varphi)_{\Omega_t} + (q - q^2, \varphi)_{\Gamma_{2T}} = E(\varphi),
\end{align*}
\]

Analogously,
\[
\begin{align*}
(k_1'(x, s') \varphi^2 - k_1'(x, s') \varphi)_{\Omega_t} + f_1'(x, s') - f_1'(x, s') \varphi)_{\Omega_t} = \\
(q - q^2, \varphi)_{\Gamma_{2T}} + (k_1'(x, s') - k_0' \varphi)^2 \varphi + f_1'(x, s') - f_1'(x, s') \varphi)_{\Omega_t} = \\
F(t, \varphi).
\end{align*}
\]

As in (12) we integrate (13) on \((0, t)\) and sum up it with (11); then, after substitution \( t = T - t \), we obtain the following identities for the functions
\[
E = K_1'(x, s'), D = K_1'(x, s').
\]

Now let us assume that
\[
L_1(\varphi, \psi) = h(x, t), \quad L_2(\varphi, \psi) = g(x, t)
\]
and consider initial boundary-value problem (14)-(15). It is possible to show that this problem has a unique solution when \( \varepsilon > 0 \) and conditions of Theorem 1 takes place. As in (2) it may be shown that if there exist \( M > 0 \) such that
\[
\frac{\|g\|_1}{\|h\|_1 + |A_1| + |C_1|} / \sqrt{A_0} \leq M,
\]
then for its solution the following estimate holds
\[
\|\psi\|_{z, \omega, \Omega}^2 + \|\psi\|_{z, \omega, \Omega}^2 + \|\varphi\|_{z, \omega, \Omega}^2 \leq C_4(\varepsilon, M, M, T, \varepsilon).
\]

With \( h = (1 - f(s_1 - s_0^2))h(x, y, s_1, s_2), g = (p - f(p_1 - p_0^2)) \),
\[
\int_{T_1} F(t, \psi) dt + (p, 1)_{\Omega_t} = 0,
\]
the equality (13) takes the form
\[
\begin{align*}
(m' s, c(1)'(s') - c(1)'(s'))_{\Omega_t} + \|p\|_{z, \omega, \Omega_t}^2 = E(\varphi) + \int F(t, \psi) dt + (p, 1)_{\Omega_t} - \\
\int (p_1 - p_0^2) dt + (m' s, c(1)'(s') - c(1)'(s'))_{\Omega_t} \int (s_1 - s_0^2) dt - \\
(\lambda, 1)_{\Omega_t} = 0,
\end{align*}
\]
and the coefficients and the boundary conditions are given by
\[
A = c(1)'(s') - c(1)'(s') / m' (s' - s^2) + \varepsilon = A_0 + \varepsilon, \quad \varepsilon > 0.
\]

\[
- A_1 = \varphi (K_1'(x, s') - K_1'(x, s') \varphi + f_1'(x, s') \varphi) / m' (s' - s^2),
\]
\[
- C_1 = \varphi (K_1'(x, s') - K_1'(x, s') \varphi + f_1'(x, s') \varphi) / m' (s' - s^2),
\]
\[
E = K_1'(x, s'), D = K_1'(x, s').
\]

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Assume that condition (8) is valid. Then the second term in the right-hand part of (11) is equal to zero and on the basis of the method suggested in (5) the first term is estimated as follows

\[ |(c'-c')_2 - c-c_2, dtv(K^{v\phi}_t)| \leq \frac{\int \left| \frac{c-c_2}(s^2)-c-c_2 (s^2) \right| \left| \nabla \left( s^2 \right) \right|}{\sqrt{A}} \left| \nabla \left( K^{v\phi}_t \right) \right| \leq \left( A^{1-\eta} \right)^{1/2} \left( \frac{G_2}{|\Omega|} \right)^{1/2} \left( \frac{1}{|\Omega|} \right)^{1/2} \left| \nabla \left( K^{v\phi}_t \right) \right| = J. \]

For the case of (8') the sum of the first two terms in \( E(\phi) \) may be represented as

\[ \left( \nabla (c'-c')_2 - c-c_2, \nabla K^{v\phi}_t \right)_{\Omega_t} = J. \]

From the facts that

\[ u^t = \int_0^1 \sqrt{a'(\xi)} \ d\xi \in L_2 \{ 0, T; W^1_2 (\Omega) \} \]

and

\[ \left| \nabla u^t(s^2) - \nabla u^t(s^2) \right| = \sqrt{ \left| (a'(s^2)-a'(s^2)) \right| \left( \frac{1}{\sqrt{a'(s^2)}} \right) } \left| \nabla u^t \right| \leq C_2 \left( |\Omega| \right)^{1/2} \left| \nabla \left( K^{v\phi}_t \right) \right| \]

we obtain

\[ J \leq C_2 \left( |\Omega| \right)^{1/2} \left( \frac{G_2}{|\Omega|} \right)^{1/2} \left| \nabla \left( K^{v\phi}_t \right) \right| = J. \]

The other terms in \( E(\phi) \) and \( F(t) \) are estimated using (16). The obtained estimates make it possible to go over to the limit when \( \varepsilon \to 0 \) in (19). Theorem 1 is proved.

Let us study now stability of \( S(x, t) \) in \( L_2(\Omega) \) with respect to the variation of the initial data (5).

Let two solutions \( \Omega' \) and \( \Omega'' \) of the problem (1)-(5) differ only by the initial data (5) and \( S, \Psi \) are defined analogously theorem 1.

Theorem 2. Let \( u^t \), \( t=1, 2 \), be generalized solutions of the problems (1)-(5), differing only by the initial data, \( \Omega' = (\Omega_{ij}^t) \) and the conditions of theorem 1 hold. Then there hold estimates

\[ \| S(x, t) - s(x, t) \| \leq C \| S(x, 0) - s(x, 0) \|, \quad \forall t \in T. \]

Proof. Analogously theorem 1 we deduce equality

\[ \| S(x, t) \|_{L^2(\Omega)} = \int_0^t + (S, L_1 (\phi, \Psi))_{\Omega \setminus \left( \partial \Omega \setminus \Omega \right)} \leq \varepsilon (S, dtv(K^{v\phi}_t))_{\Omega_t}, \]

where \( L_1, L_2 \) are defined as in theorem 1.

Consider initial-boundary problem (14)-(15) with

\[ h = 0, \quad g = 0, \quad \Psi(x, t) = \phi(x, t). \]

Similarly (2) may be shown that for this conditions the next estimates take place (after returning to the starting time \( t=T-t \))

\[ \| \Psi(x, t) \|_{L^2(\Omega)} \leq C \| \Psi(x, T) \|_{L^2(\Omega)}, \quad \forall t \in T. \]

Letting \( \varepsilon \to 0 \) in (16) we get the equality

\[ (S(x, 0), \Psi(x, 0))_{\Omega} = (S(x, T), \Psi(x, T))_{\Omega}. \]

Let \( \Delta \) denotes the Laplacian and \( \xi_k(x) k=1, 2, \ldots \) be the solutions of the problems

\[ \left( \Delta + \lambda_k \right) \xi_k = 0, \quad \xi_k \in L_2(\Omega) ; \quad \left( \xi_k, \xi_k \right) = \delta_k, \]

\[ \left. \xi_k \right|_{\Gamma_1} = 0, \quad \left. \nabla \xi_k \right|_{\Gamma_2} = 0. \]

Then any function \( f \in L_2(\Omega) \) may be represented in the form

\[ f(x) = \sum_{k=1}^{\infty} f_k \xi_k(x), \quad f_k = \int_{\Omega} f \xi_k. \]

Let \( \Delta^{-1} \) denotes operator inverse to \( \Delta \) under the boundary conditions (17). Then there valid the formulas

\[ -(\delta, \Delta^{-1} \delta)_{\Omega} = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\lambda_k^2} \| \delta \|^2 \_{L^2(\Omega)} = \| \delta \|^2 \_{L^2(\Omega)} \]

Indeed,

\[ -(\delta, \Delta^{-1} \delta)_{\Omega} = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\lambda_k^2} \| \delta \|^2 \_{L^2(\Omega)} = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\lambda_k^2} \| \delta \|^2 \_{L^2(\Omega)} = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\lambda_k^2}. \]
\[ \sup_{\Omega} \left\langle (s, h) \right\rangle^2_{\Omega} = \sup_{\Omega} \left\langle s, h \right\rangle^2_{\Omega} + \sup_{\Omega} \left\langle s, \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} h_k \right\rangle^2_{\Omega} \]

\[ \|v\|^2_{L^2(\Omega)} = (v, v)_{\Omega} = (D^2 v, v)_{\Omega} \leq \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k^{1/2}} \]

\[ \|D_{x}^2 s\|^2_{L^2(\Omega)} = (D^2 v, D^2 v)_{\Omega} = \sum_{k=1}^{\infty} \lambda_k h_k^2 \leq \frac{1}{\lambda_k^{1/2}} \sum_{k=1}^{\infty} \lambda_k h_k^2 \leq 1. \]

Theorem 2 is proved.

2. Stabilization of generalized solutions.

Consider stationary solution of problem (1)-(5). This solution \((s_0(x), p_0(x))\) for all \((\varphi, \psi) \in W^1_0(\Omega), \varphi = 0, \psi = 0, x \in \Gamma_1\) satisfy the integral identities

\[ (K_0 s_0 \varphi + \int_\Omega f(x, \varphi) \psi - R_1 \varphi)_{\Gamma_2} = 0, \]

\[ (K s_0 \varphi + f(x, \varphi) \psi - R \psi)_{\Gamma_2} = 0, \]

and boundary conditions

\[ s_0(x) = 1, \quad p_0(x) = p^0(x), \quad x \in \Gamma_1. \]

The proof of Lemma 1 may be found in [6].

Consider the problem

\[ \frac{d\varphi}{dt} + A(x, t) \psi = -\lambda A \varphi, \]

where \(A(x, t)\) is the symbol of the inverse function to \(\Phi\). The function \(Q, Q_i\) in (4).

If \(R = R_1\), there exists the solution in the form

\[ s_0(x) = 1, \quad x \in \Omega, \]

and function \(p_0\) is a generalized solution of the boundary-value problem

\[ d\psi(K(x, t) \varphi + f(x, t)) = 0, \]

\[ (K(x, t) \varphi + f(x, t)) \cdot n = -R, \quad x \in \Gamma_2, \]

\[ p_0 = p^0(x), \quad x \in \Gamma_1. \]

Since the coefficients equations of system (1)-(2) have the properties [1, p.212]

\[ K_1(x, t) = K(x, t), \quad f(x, t) = f_0(x, t). \]

Note 1. The solution of (20)-(21) corresponds in terms of physics to the case of a complete displacement of oil initially occupying a part of the volume \(\Omega\) by water.

Here it is proved that the generalized solution \((s, p)\) of degenerate problem (1)-(5) is convergent to the solution \((I, P_0)\) from (20)-(21) when \(t\) infinitely increasing.

Lemma 1. Let the non-negative functions \(A_i(t) \in L_i([0, \infty))\) and \(y(t)\) be connected by the differential inequality

\[ \frac{dy}{dt} + g(y) \leq A_i(t), \quad y(0) = y_0 > 0, \]

where the function \(g(z)\) is such that

\[ g(z) > 0, z > 0, \quad \frac{dz}{dt} = \frac{1}{g(z)} > 0, \quad \frac{dz}{dt} = \Phi(z) > 0, z > 0, z_0 \leq y_0. \]

Then

\[ y \leq H(t) \left[ 1 + \int_0^t A_i(t) \right] = E(t) > 0, \quad t > 0, \]

where \(H(t)\) is the symbol of the inverse function to \(\Phi\).
compatibility.

Let the function $\omega(x)$ be the solution of problem
\begin{equation}
\text{div}(K\nabla \omega) = \alpha_1, \quad x \in \Omega, \quad \alpha > 0, \quad \omega = 0, \quad x \in \Gamma_1, \quad K_0 \nabla \omega \cdot n = - \alpha_1, \quad x \in \Gamma_2. \tag{24}
\end{equation}

By the maximum principle $\omega \leq 0$, $x \in \Omega \setminus \{0\}$. \hfill \[7\]

Lemma 2. Assume that $|B|/A < M_1$, where the constant $M_1$ is independent of $t$, $x$, $m$, $\Gamma_1 > 0$. Then there exist a number $\lambda > 0$, such that for all $\lambda < \lambda_0$ any solution of problem (23) satisfies the maximum principle in the form
\begin{equation}
0 \leq \varphi(x, t) \leq \max_{\Omega} \varphi_0 + t. \tag{25}
\end{equation}

If the function $\varphi_0(x, t)$ satisfies the inequality
\begin{equation}
\varphi_0(x, t) \leq \omega (1 - \exp(\alpha_1 \min_\Omega \omega) / \min_\Omega \omega, \tag{26}
\end{equation}

where $\omega$ is taken from (24), then for $\lambda < \lambda_0$ next estimate is held

\begin{equation}
- \lambda \exp(\alpha_1 \omega) K_0 \nabla \omega \cdot n \leq K_0 \nabla \varphi \cdot n \leq 0, \tag{27}
\end{equation}

and $\varphi_0$ can be chosen such that $\varphi_0^1(x, t) \in L_2(\Omega)$, $q \in (0, 1)$. In addition, for all $\eta \in L_2(\Omega)$ holds
\begin{equation}
\langle K_0 \nabla \varphi, \eta \rangle \leq M_1(\eta), \quad \langle K_0 \nabla \varphi, \eta \rangle \leq \sqrt{M_1(\eta)} \|\eta\|_{L_2(\Omega)}. \tag{28}
\end{equation}

Proof. Substituting $t = T - t$ from (23) results in the problem
\begin{equation}
L\omega = \varphi_t - A \text{div}(K_0 \nabla \varphi) - \lambda \exp(\alpha_1 \omega) - \lambda \alpha A \varphi = 0, \tag{29}
\end{equation}

$\varphi = 0$, $(x, t) \in \Gamma_{1T}$, $K_0 \nabla \omega \cdot n = 0$, $(x, t) \in \Gamma_{2T}$, $\varphi(x, 0) = \varphi_0(x) \geq 0$, $(x, t) \in \Gamma_T$.

Its unique solvability in a class of functions $C^{2+\alpha, 1/2+\alpha/2}(\Omega)$ under assumption from (23) was proved in [8], [9].

It will be shown that $\varphi \geq 0$. Determine the function $u_1(x, t)$ by the equality $\varphi = \exp(\alpha_1 t + \beta) u_1$, where $\beta$ is the solution of the problem
\begin{equation}
\text{div}(K_0 \nabla \beta) = \gamma_1, \quad K_0 \nabla \beta \cdot n = \gamma, \quad x \in \Gamma_1, \quad \gamma_1 \text{ meas } \Omega = \gamma \text{ meas } \Gamma
\end{equation}

with constants $\gamma_1, \gamma, \alpha > 0$. According to the maximum principle for solutions of parabolic equations [8] may be shown that $u_1 \geq 0$ in $\Omega$ for sufficiently large values of $\alpha$ and, therefore, $\varphi \geq 0$.

Introduce the function
\begin{equation}
\varphi_+ = \exp(\alpha_1 \omega) - \exp(\alpha_1 R), \tag{30}
\end{equation}

where $\omega$ satisfies (24) and the constant $R > 0$. It is obvious that
\begin{equation}
L\chi + \delta \chi = \exp(-\delta t)(\omega + \varphi_+), \quad \chi = \exp(-\delta t)(\varphi + \varphi_+). \tag{31}
\end{equation}

Let us prove that for sufficiently large values of $0$, $\alpha_1$, $R$, the function $\chi < 0$, $(x, t) \in \eta_T$. Calculate $L\varphi$, and estimate the above expression
\begin{equation}
\exp(-\alpha_1 \omega) \varphi_+ < - A \left(\alpha_1 + \frac{\alpha_1 B}{\sqrt{A}} \varphi \omega + \alpha_1 K_0 \nabla \omega \cdot n - \lambda \exp(\alpha_1 (R - \omega))\right) \leq \tag{32}
\end{equation}

\begin{equation}
\leq - A \left(\alpha_1^{1/2} + \frac{\alpha_1}{2} \frac{M_1}{2} - 1\right) < 0, \quad \alpha_1 > 1 + \frac{M_1}{2}. \tag{33}
\end{equation}

In this estimate the Cauchy inequality was used and it was assumed that
\begin{equation}
\lambda = \lambda_0 = \exp(\alpha_1 h), \quad h > R - \min_\Omega \omega. \tag{34}
\end{equation}

Thus, if $0 > \lambda_0 > M_1$, there is no the positive maximum of $\chi$ inside $\Omega_T$. For $t = 0$ and $R = \alpha_1^{1/2} \ln(\|\varphi_0\|_C(\Omega) + 1)$ the function $\chi < 0$. Since $\omega = 0$ on $\Gamma_1$, then $\chi < 0$ on $\Gamma_{1T}$. As $K_0 \nabla \omega \cdot n < 0$ on $\Gamma_{2T}$, on this part of the boundary the function $\chi$ has no maximum. The first estimate of lemma has been proved.

The right-hand side inequality in (26) follows from the fact that $\varphi > 0$ and $\varphi = 0$ on $\Gamma_{1T}$.

Let us prove that under condition (25) the maximum of the function $\varphi + \varphi_+ \in \eta_T$. Actually, since
\begin{equation}
L(\varphi + \varphi_+) < 0, \quad K_0 \nabla (\varphi + \varphi_+) \cdot n < 0, \quad (x, t) \in \Gamma_{2T}, \tag{36}
\end{equation}

the function $\varphi + \varphi_+$ has the negative maximum at $t = 0$ or on $\Gamma_{1T}$. From (25) it follows that $\varphi_0 \in L - \max(\alpha_1 \omega)$, hence $(\varphi + \varphi_+)|_{t=0} \leq \varphi_0 \in L - \max(\alpha_1 \omega) \leq \varphi_0 \in \eta_T$. Thus, the function $\varphi + \varphi_+$ has the maximum value on $\Gamma_{1T}$. Since this value is independent of $(x, t) \in \Gamma_{1T}$, we have $K_0 \nabla (\varphi + \varphi_+) \cdot n > 0$ for $(x, t) \in \Gamma_{1T}$. This proves the estimate (26).

The estimates (27) may be received analogously [2].
Equation (24) multiplied by \((-\omega)^{-q}\) and integrated over \(\Omega\)
\((1 - q) \langle K, \varphi \rangle_{\Omega} \rangle_{\Omega} = \alpha(1, -\omega)^{-q} \Omega \cup \Gamma_2\).

It follows from (24) that \(\omega < 0\) inside \(\Omega\). Further, the boundary condition on \(\Gamma_2\) and the Zaremba-Giraud principle shows that 
\(|\varphi_{\Omega}| \neq 0\) on \(\Gamma\). Hence the summation of the function \((-\omega)^{-q}\),
\(q \in (0, 1)\) follows from the latter equality.

Thus, the function \(\varphi_0(x, t)\) can be chosen such that it satisfies (25) and \(\varphi_0^1 \in L_Q^q(\Omega)\).

Let us define
\[ W = \int_0^1 \alpha(t) dt. \]

Let
\[ W(s)(x, t) \in L_1(\Gamma_1 \times (0, \infty)), \quad Q - R_1, \quad Q - R \in L_1(\Gamma_1 \times (0, \infty)). \]

For
\[ t \to \infty, \quad s \to 0, \quad \alpha, \alpha^2, \quad \alpha^3 \]

and there exist such functions \(f, \varphi\)
\[ W(s) \geq f(1-s), \quad \frac{d^2 f(y)}{dy^2} \geq 0, \quad f(0) = \varphi(0), \quad f(1) = \varphi(1), \quad y \in [0, N_2], \]

where \(N_2\) are the constant and the function \(\varphi(y)\) satisfies conditions of lemma 1.

Theorem 3. Let conditions (28), (29) be fulfilled. Then for any generalized solutions of problems (1)-(5) and (20), (21) the following equality holds
\[ \lim_{t \to \infty} \left( \frac{1}{N_2} \right)^{1/2} + \left( \frac{1}{N_2} \right)^{1/2} \varphi_{\Omega} \right)_{\Omega} = 0. \]

Proof. Differentiation of identity (6) in \(t\) and subsequent subtraction of (18) give the following equality for almost all \(t \in (0, T)\)
\[ \begin{align*}
\frac{d}{dt}(\text{mp}, 1-s)_{\Omega} - (\text{mp}, 1-s)_{\Omega} + (K, \varphi_0 W(s), \varphi W)_{\Omega} - (Q, 1, \varphi)_{\Omega} + \\
(1-s)_{\Omega} - (1-s)_{\Omega} + (K, \varphi_0 W(s), \varphi W)_{\Omega} = 0.
\end{align*} \]

Here we assume that \(s_0 = 1\).

Analogically
\[ ((K, 1) - K), \varphi_0 W(s), \varphi W)_{\Omega} + (Q, 1, \varphi)_{\Omega} + \\
(1-s)_{\Omega} - (1-s)_{\Omega} + (K, \varphi_0 W(s), \varphi W)_{\Omega} = 0. \]

Let \(\varphi = \psi\) and \(K_0 \varphi \psi = 0\) on \(\Gamma_2\). Then subtraction of (31) from (30) with the allowance for (22) and integration by parts in the third term of (30) give the equality
\[ \frac{dt}{dt}(\text{mp}, 1-s)_{\Omega} = (\text{mp}, 1-s)_{\Omega} + A dU(K_0 \varphi \psi) + \alpha^2 \varphi \psi \varphi_{\Omega} + (Q, 1, \varphi)_{\Omega} + \\
(1-s)_{\Omega} - (1-s)_{\Omega} + (K, \varphi_0 W(s), \varphi W)_{\Omega} - (Q, 1, \varphi)_{\Omega} + \\
(1-s)_{\Omega} - (1-s)_{\Omega} + (K, \varphi_0 W(s), \varphi W)_{\Omega} = 0. \]

The function \(\tilde{B}\) is approximated by \(B\), at substituting \(\varphi\) by its smooth averaging \(\varphi_{\Omega}\). The solution of problem (23) with the coefficients \(A, B\) is substituted into the latter equality for \(\varphi\).

Then, using the results of lemma 2, we estimate the integrals in the right-hand side of the obtained equality. As a result, we derive the inequality
\[ \frac{dy}{dt} + \lambda W_{\Omega}(\text{mp}, W(s))_{\Omega} \leq A \left( \lambda \right), \]

where
\[ A = N_2(\text{mp}, 1-s)_{\Omega} + (W(s), \varphi)_{\Omega} + N_2(\text{mp}, 1-s)_{\Omega} + (W(s), \varphi)_{\Omega} + N_2(\text{mp}, 1-s)_{\Omega} + \lambda < \lambda_0, \quad N_2 = \text{const}. \]

From (29), (32) and the Jensen inequality for the function \(f(y)\) it follows that
\[
\frac{dy}{dt} + \lambda y^{-1} g(y) \leq A_y(t),
\]
\[0 < y_0 \leq \bar{y}_0 = \mu \text{ meas } \Omega (\|\phi_0\|_{C(\Omega)} + 1).
\]
Taking into account estimates of lemma 2 in coefficient \(A_y\) the limiting transition in parameters \(\nu, \varepsilon\) are made. After it according to lemma 1 we get
\[y(T) = (\mu \phi_0, 1-s(T, T), ) \leq B(T).
\]
From the Holder inequality and the latter estimate at \(\phi_0^{-1} \in L^q(\Omega)\) it follows that
\[\|1-S(T, T)\|_{1, \Omega} \leq \left[ \left( \mu \|\phi_0^{-1}\|_{C(\Omega)} B(T) \right)^{\frac{1}{q+1}} + 0 \right]. T \rightarrow \infty.
\]
Assume that \(\phi = \bar{\phi}_0 + p \ln\). Using the Cauchy inequality the fact that \(|p|, |\nabla p|\) are bounded uniformly, we have
\[\|\nabla (p_0 - p)\|_{2, \Omega} \leq N_4 (\|1-s\|_{1, \Omega} + \|1-q\|_{1, \Omega}).
\]
If \(\text{ meas } \Gamma > 0\), then \(\|p_0 - p\|_{2, \Omega} \leq N_5 \|\nabla (p_0 - p)\|_{2, \Omega} \). Here \(N_4, N_5\) are constant independent on \(t\). Theorem has been proved.

**Remark 1.** Conditions of Lemma 1 and (29) are fulfilled, for example, by such functions \(g(y), f(y)\) that
\[g, f \sim cy^\gamma, r_0 > 2, r_1 \int_0^{r_1} \exp(-\frac{r_1}{\xi}) d\xi, r_2 \int_0^{(\ln(r_1))^{-1}} d\xi, y > 0,
\]
where \(r_1, \xi, s = 0, 1, 2, 3, 4, \) are constant.

A behaviour of the coefficient
\[\alpha(s) = (1-s)^{r_0-1}, \exp(-\frac{1}{1-s}), |\ln(1-s)|^{-1}, s > 1
\]
corresponds to the mentioned functions.

**REFERENCES**


