Elliptic Equations and Steiner Symmetrization

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Abstract

We give some comparison results for elliptic equations by using Steiner symmetrization. © 1996 John Wiley & Sons, Inc.

1. Introduction

Let us consider the following Dirichlet problem in an open, bounded subset G of $\mathbb{R}^N$

$$
(1.1)\qquad \begin{cases} -\sum_{i,j=1}^{N} (a_{ij}(x)u_{x_i})_{x_j} = f & \text{in } G \\ u = 0 & \text{on } \partial G\,, \end{cases}
$$

with the ellipticity condition

$$
(1.2)\qquad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqq |\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^N\,.
$$

It is well-known that any L^p or even any Orlicz norm of the solution u of (1.1) can be sharply estimated by the same norm of the solution v of the following spherically symmetric problem

$$
(1.3)\qquad -\Delta v = f^{\#} \quad \text{in } B\,, \qquad v = 0 \quad \text{on } \partial B\,;
$$

here B is the ball of $\mathbb{R}^N$ centered at zero, with the same measure as G, and $f^{\#}$ is the Schwarz-symmetrized f, that is, a spherically symmetric function, decreasing with respect to $|x|$, such that

$$
\text{meas}\,\{x : |f(x)| > t\} = \text{meas}\,\{x : f^{\#}(x) > t\} \qquad \forall\, t \in \mathbb{R}^+\,.
$$

This comparison result was first proven by G. Talenti [18] (see also [21]). It has since been generalized by introducing lower-order terms ([3], [6], [7], [10], [14],

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[20]), by weakening the ellipticity condition (1.2) ([5], [17]), and by applying the result to nonlinear operators ([3], [15], [19]) and parabolic operators ([3], [4], [8], [16]).

In nearly all the papers referred to above, the proofs integrate the equation on a generic level set of the solution u. Then by employing classical inequalities such as the isoperimetric inequality [11] and Hardy's inequality on rearrangements [12], one obtains a differential inequality satisfied by the function

$$u^*(s) = \sup\{t \geqq 0 : \mu(t) > s\} , \tag{1.4}$$

where μ is the distribution function of u. The function (1.4), known as the decreasing rearrangement of u, is the unique function, decreasing on $[0, +\infty)$ with the same distribution function as u. At this point, by means of classical maximum principles, the rearrangement u^* of u and the rearrangement v^* of the solution v of a suitable spherically symmetric problem can be compared. More precisely, we have

$$u^*(s) \leqq v^*(s) \tag{1.5}$$

or the weaker inequality

$$\int_0^s u^* \leqq \int_0^s v^* . \tag{1.6}$$

In any case, (1.5) or (1.6) imply the desired estimates of L^p or Orlicz norms of u.

On one hand, these results make the problem of determining a priori estimates easier by turning it into a one-dimensional problem; on the other hand, by this symmetrization process, the differential problem may lose properties that arise from the symmetry of the data with respect to a group of variables. In order to preserve this kind of symmetry, it is useful to check whether comparison results hold when a partial symmetrization such as the Steiner symmetrization is used.

To investigate this possibility, we need to introduce some notation.

Let u be a function from an open, bounded set $G \subset \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$. For any $y \in \mathbb{R}^m$ such that

$$G_y = \{x \in \mathbb{R}^n : (x, y) \in G\} \neq \emptyset$$

we consider the function

$$x \in G_y \to u(x, y) \in \mathbb{R} . \tag{1.7}$$

We denote by $u^*(\cdot, y)$ the decreasing rearrangement of function (1.7) (see (1.4)). Moreover, we set

$$u^\#(x, y) = u^*(c_n|x|^n, y) , \tag{1.8}$$

where c_n is the measure of the unit ball of $\mathbb{R}^n$. Function (1.8) can be seen as the Schwarz-symmetrized function of (1.7); (1.8) is spherically symmetric with respect to the x-variable, decreasing as a function of $|x|$: It is the Steiner rearrangement of u with respect to x. We denote by $G^{\#}$ the Steiner-symmetrized version of G; that is, the set whose indicator function is $(1_G)^{\#}$.

Now we can state the main result of the paper:

THEOREM 1.1. *Let*

$$Lu = -\sum_{i,j=1}^{n}(a_{ij}(x,y)u_{x_i})_{x_j} - \sum_{h,k=1}^{m}(b_{hk}(y)u_{y_h})_{y_k} - \sum_{i=1}^{n}\sum_{h=1}^{m}(c_{ih}(y)u_{x_i})_{y_h} - \sum_{i=1}^{n}\sum_{h=1}^{m}(d_{hi}(y)u_{y_h})_{x_i},$$

and let u be the weak solution of the Dirichlet problem

$$Lu = f \quad \text{in } G, \qquad u = 0 \quad \text{on } \partial G. \tag{1.9}$$

We assume the following:

(i) *coefficients a_{ij}, b_{hk}, c_{ih}, d_{hi}, and f belong to $L^\infty(G)$;*

(ii) *(ellipticity condition) there exists $\nu > 0$ such that*

$$\begin{aligned}&\sum_{i,j=1}^{n} a_{ij}(x,y)\xi_i\xi_j + \sum_{h,k=1}^{m} b_{hk}(y)\eta_h\eta_k \\ &\quad + \sum_{i=1}^{n}\sum_{h=1}^{m} c_{ih}(y)\xi_i\eta_h + \sum_{i=1}^{n}\sum_{h=1}^{m} d_{hi}(y)\eta_h\xi_i \geqq |\xi|^2 + \nu|\eta|^2 \\ &\forall\, (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m \qquad a.e.\ (x,y) \in G;\end{aligned} \tag{1.10}$$

(iii) *$G = G' \times G''$ with G', G'' open, bounded subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.*

Moreover, let v be the weak solution of the problem

$$\begin{cases} -\Delta_x v - \sum_{h,k=1}^{m}(b_{hk}(y)v_{y_h})_{y_k} = f^{\#} & \text{in } G^{\#} \\ v = 0 & \text{on } \partial G^{\#}. \end{cases} \tag{1.11}$$

In this case $G^{\#} = B \times G''$, where B is the ball of $\mathbb{R}^n$ centered on zero whose measure is $|G'|_n$, the Lebesgue measure of G'.

Then we have, for any $y \in \mathbb{R}^m$,

$$\int_0^s u^*(\sigma,y)\,d\sigma \leqq \int_0^s v^*(\sigma,y)\,d\sigma \quad \forall\, s \in [0,|G'|_n], \tag{1.12}$$

where $u^*(\cdot,y)$, $v^*(\cdot,y)$ are the decreasing rearrangements of the solutions u, v of problems (1.9), (1.11), *respectively.*

Inequality (1.12) can be used to estimate any Orlicz norm of $u(\cdot, y)$ by the same norm of $v(\cdot, y)$. For instance, we have

$$\sup_{x\in G'} |u(x,y)| \leqq \sup_{x\in B} v(x,y) .$$

We mention that Theorem 1.1 generalizes a result of [4]. However, in that paper a totally different method is used: indeed, by means of simple properties of the fundamental solution of the heat equation and semigroups, it is possible to get a comparison result for parabolic operators and then for elliptic operators.

Some of the results included in this paper have been announced in [1].

Finally, we mention that similar results are in Bandle–Kawhol [9], where the Laplace operator is considered; the authors use a discretization method.

2. Preliminary Results

Clearly, by a simple approximation process, it is enough to prove Theorem 1.1 in the case when the coefficients and the data are analytic, an assumption we shall always make below.

Let G be an open, bounded subset of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$; let u be a function defined on $\overline{G}$. We assume $u > 0$ on G and $u = 0$ on ∂G; moreover, let u be analytic on G

We introduce the following function

$$F(s,y) = \int_0^s u^*(\sigma, y)\, d\sigma \quad y \in \mathbb{R}^m,\ s \in [0, |G_y|_n] . \tag{2.1}$$

Since $u(\cdot, y)$ has no flat zone (u is in fact analytic), we can write function (2.1) as follows:

$$F(s,y) = \int_{G_y(s)} u(x,y)\, dx , \tag{2.2}$$

where

$$G_y(s) = \{x \in G_y : u(x,y) > u^*(s,y)\} .$$

The integral (2.2) is computed on a level set of function (1.7) whose measure is s. Formula (2.2) is a consequence of both the fact that (1.7) has no flat zone and the following property of rearrangements: *If w is a measurable function on a set Ω and ω denotes a generic subset of Ω, then*

$$\int_0^s w^*(\sigma) d\sigma = \sup_{|\omega|=s} \int_\omega w(x)\, dx . \tag{2.3}$$

We are interested in differentiating function (2.1) with respect to y. To this end, we begin by stating the following result:

LEMMA 2.1. *We have*

$$\frac{\partial F}{\partial y_i} = \int_{G_y(s)} \frac{\partial u}{\partial y_i}\, dx \tag{2.4}$$

$$(2.5)\quad \left(\frac{\partial^2 F}{\partial y_i \partial y_y}\right)_{i,j} \geqq \left(\int_{G_y(s)} \frac{\partial^2 u}{\partial y_i \partial y_j}\, dx\right)_{i,j} \quad \textit{in the sense of matrices;}$$

(2.4), (2.5) *are to be understood in the sense of distribution.*

Remark 2.2. Formula (2.4) has been obtained by Bandle [8] and later, under weaker hypotheses, by Mossino–Rakotoson [16].

Remark 2.3. If we leave out regularity matters, formulae (2.4) and (2.5) can be easily explained. Fix a value $\overline{y}$ of the variable y and set

$$(2.6)\qquad \overline{F}(y) = \int_{G_{\overline{y}}(s)} u(x, y)\, dx\ .$$

In (2.6) the domain of integration does not depend on y, so it is easy to differentiate $\overline{F}$ under the integral sign. On the other hand, since $|G_y(s)|_n = s$, by (2.3) we have

$$\overline{F}(y) \leqq F(s, y)\ , \quad \overline{F}(\overline{y}) = F(s, \overline{y})\ ..$$

Then, we obtain (2.4) and (2.5) by easy considerations.

Inequality (2.5) will be stated in a more general form; indeed, as a consequence of Lemma 2.1, we will give a formula (see Remark 2.8) with the exact expression of the second derivatives of F. Nevertheless, we prefer to give a direct and simple proof of formula (2.5).

Proof of Lemma 2.1: For (2.4), we can obviously consider only the case $n = 1$. Given a value $\overline{y}$ of the variable y and a value $s \in (0, |G_{\overline{y}}|_n)$, it is possible to find an open set $A \subseteq \mathbb{R}^n$ and an interval $I \subseteq R$ such that $\overline{y} \in I$, $A \times I \subseteq G$, and the sets

$$\{(x, y) \in G\ :\ u(x, y) > u^*(s, y)\}$$

are contained in $A \times I$ for $y \in I$. If y, $y + \Delta y \in I$, we have

$$\begin{aligned} &F(s, y + \Delta y) - F(s, y)\\ &\quad = \int_{G_{y+\Delta y}(s)} u(x, y + \Delta y)\, dx - \int_{G_y(s)} u(x, y)\, dx\\ &\quad \geqq \int_{G_y(s)} u(x, y + \Delta y)\, dx - \int_{G_y(s)} u(x, y)\, dx\ . \end{aligned}$$

Here the inequality is a consequence of the fact that

$$\begin{aligned} \int_{G_{y+\Delta y}(s)} u(x, y + \Delta y) dx &= \sup_{|\Omega| = s} \int_\Omega u(x, y + \Delta y)\, dx\\ &\geqq \int_{G_y(s)} u(x, y + \Delta y)\, dx\ , \end{aligned}$$

by (2.3), since $|G_y(s)|_n = s$.

Setting $\Delta F = F(s, y + \Delta y) - F(s, y)$ and $\Delta u = u(x, y + \Delta y) - u(x, y)$, we get

$$(2.7)\qquad \begin{aligned} &\frac{\Delta F}{\Delta y} \geqq \int_{G_y(s)} \frac{\Delta u}{\Delta y}\, dx \qquad \text{if } \Delta y > 0\,, \\ &\frac{\Delta F}{\Delta y} \leqq \int_{G_y(s)} \frac{\Delta u}{\Delta y}\, dx \qquad \text{if } \Delta y < 0\,. \end{aligned}$$

If $\varphi(y) \in C_0^\infty(I)$ is a nonnegative function, by (2.7) we can write

$$(2.8)\qquad \begin{aligned} &\int \varphi(y) \frac{\Delta F}{\Delta y}\, dx \geqq \int \varphi(y)\, dy \int_{G_y(s)} \frac{\Delta u}{\Delta y}\, dx \quad \text{if } \Delta y > 0\,, \\ &\int \varphi(y) \frac{\Delta F}{\Delta y}\, dx \leqq \int \varphi(y)\, dy \int_{G_y(s)} \frac{\Delta u}{\Delta y}\, dx \quad \text{if } \Delta y < 0\,. \end{aligned}$$

When Δy goes to zero, (2.8) becomes

$$-\int \frac{\partial \varphi}{\partial y} F\, dy = \int \varphi\, dy \int_{G_y(s)} \frac{\partial u}{\partial y}\, dx\,,$$

which is (2.4).

For (2.5) we have to prove that

$$\frac{\partial^2}{\partial t^2} F(s, y + t\lambda)_{|_{t=0}} \geqq \int_{G_y(s)} \frac{\partial^2}{\partial t^2} u(x, y + t\lambda)_{|Pt=0}\, dx\,,$$

for any $\lambda \in \mathbb{R}^m$, in the sense of distribution. Obviously we can assume $m = 1$. So we have, if y, $y + \Delta y$, $y - \Delta y \in I$,

$$\begin{aligned} &\frac{F(s, y + \Delta y) + F(s, y - \Delta y) - 2F(s, y)}{\Delta y^2} \\ &= \frac{1}{\Delta y^2} \left\{ \int_{G_{y+\Delta y}(s)} u(x, y + \Delta y)\, dx + \int_{G_{y-\Delta y}(s)} u(x, y - \Delta y)\, dx \right. \\ &\qquad \left. - 2 \int_{G_y(s)} u(x, y)\, dx \right\} \\ &\geqq \frac{1}{\Delta y^2} \int_{G_y(s)} \left[u(x, y + \Delta y) + u(x, y - \Delta y) - 2u(x, y) \right]\, dx\,. \end{aligned}$$

In the last inequality we used property (2.3) once again. If $\varphi(y)$ is a nonnegative function in $C_0^\infty(I)$, we obtain

$$\begin{aligned} &\int \frac{F(s, y + \Delta y) + F(s, y - \Delta y) - 2F(s, y)}{\Delta y^2} \varphi(y)\, dy \\ &\geqq \int \varphi(y)\, dy \int_{G_y(s)} \frac{u(x, y + \Delta y) + u(x, y - \Delta y) - 2u(x, y)}{\Delta y^2}\, dx\,. \end{aligned}$$

By letting Δy go to zero, we get (2.5).

Now we are going to give an exact formula for the second derivatives of function (2.2). Let us introduce some further notation.

If we fix $y \in \mathbb{R}^m$, by Sard's lemma the level set

$$\{(x, y) \in G \ : \ u(x, y) = t\} \tag{2.9}$$

is noncritical for a.e. $t \in \mathbb{R}$, which means that $|\nabla_x u| \neq 0$ at all points that belong to (2.9). Furthermore, (2.9) is the boundary of the set

$$\{(x, y) \in G \ : \ u(x, y) > t\} \ . \tag{2.10}$$

We denote the set (2.10) by $G(s, y)$ and its boundary (2.9) by $\Gamma(s, y)$, where s is the n-dimensional measure of (2.10). Obviously, for a.e. $(x, y) \in G$ there exists a (noncritical) level $\Gamma(s, y)$ through (x, y); then we can define on G the following function:

$$\mathscr{U}(x, y) = u(x, y) - u^*(s, y) \ , \tag{2.11}$$

where s is such that $(x, y) \in \Gamma(s, y)$. If s is fixed, for any $\overline{y} \in \mathbb{R}^m$ such that $\Gamma(s, \overline{y})$ is noncritical we have

$$\mathscr{U}(x, \overline{y}) = 0 \quad \text{and} \quad |\nabla_x \mathscr{U}|(x, \overline{y}) \neq 0 \quad \text{on } \Gamma(s, \overline{y}) \ .$$

So, by the classical implicit function theorem, in a neighborhood of $\Gamma(s, \overline{y})$, the set

$$\{(x, y) \in G \ : \ u(x, y) = u^*(s, y)\} \tag{2.12}$$

is a manifold of dimension $N - 1$ crossing the level $\Gamma(s, \overline{y})$. It is easy to see that, for $|\Delta y|$ sufficiently small, the linear manifold $y = \overline{y} + \Delta y$ crosses (2.12) along the level $\Gamma(s, \overline{y} + \Delta y)$; so the manifold (2.12) describes how the levels (2.9) must move from $\Gamma(s, \overline{y})$ in order that they be the boundary of sets as in (2.10) with the same measure s.

It is useful to introduce the manifold (2.12) in another way. On any noncritical level set (2.9) we define the following vector fields:

$$\mathscr{A}^h(\alpha_i^h(x, y), \delta_k^h) \qquad (i = 1, \ldots, n; \ h, k = 1, \ldots, m) \ , \tag{2.13}$$

where δ_k^h is the usual Kronecker symbol and

$$\alpha_i^h(x, y) = \alpha^h(x, y) \frac{u_{x_i}}{|\nabla_x u|} \ , \tag{2.14}$$

with

$$\alpha^h(x, y) = \frac{1}{|\nabla_x u|} \left\{ \frac{\displaystyle\int_{\Gamma(s,y)} \frac{u_{y_h}}{|\nabla_x u|} H_{n-1}(dx)}{\displaystyle\int_{\Gamma(s,y)} \frac{1}{|\nabla_x u|} H_{n-1}(dx)} - u_{y_h} \right\} . \tag{2.15}$$

From (2.14) and (2.15) we easily deduce that

$$\sum_{i=1}^{n} \alpha_i^h u_{x_i} + u_{y_h} = \text{const} \quad \text{on } \Gamma(s,y) \,, \tag{2.16}$$

where the constant value is

$$\phi^h(s,y) = \frac{\displaystyle\int_{\Gamma(s,y)} \frac{u_{y_h}}{|\nabla_x u|} H_{n-1}\, dx}{\displaystyle\int_{\Gamma(s,y)} \frac{1}{|\nabla_x u|} H_{n-1}\, dx} \,. \tag{2.17}$$

Moreover, we have

$$\int_{\Gamma(s,y)} \alpha^h(x,y) H_{n-1}\, dx = 0 \,. \tag{2.18}$$

Remark 2.4. Formulae (2.14) and (2.15) allow us to define the vector field (2.13) at any point that belongs to a noncritical level (2.9). Moreover, the components α_i^h are continuous.

Now we consider the systems of differential equations

$$\frac{dx_1}{\alpha_1^h(x,y)} = \cdots = \frac{dx_n}{\alpha_n^h(x,y)} = dy_h \qquad (h = 1, \ldots, m) \,. \tag{2.19}$$

If $(\bar{x}, \bar{y}) \in \Gamma(s, \bar{y})$, let

$$x_i = x_i(\bar{x}, y_h) \qquad (i = 1, \ldots, n) \tag{2.20}$$

be the integral path of (2.19) satisfying the initial condition

$$x_i(\bar{x}, \bar{y}_h) = \bar{x}_i \,,$$

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$. The curve (2.20) moves from $(\bar{x}, \bar{y}) \in \Gamma(s, \bar{y})$ perpendicularly to $\Gamma(s, \bar{y})$ (see (2.14)) on the linear manifold $\{y_k = \bar{y}_k,\ k \neq h\}$. As $\bar{x}$ runs on $\Gamma(s, \bar{y})$ these paths describe a manifold of dimension $n+1$ that is the intersection of (2.12) and the linear manifold $\{y_k = \bar{y}_k,\ k \neq h\}$. In order to prove this statement, we have to verify the following facts:

(a) If y is sufficiently close to $\bar{y}$, the set

$$\Gamma(y) = \{(x_i(\bar{x}, y), y)\} \,, \tag{2.21}$$

where $\bar{x}$ is such that $(\bar{x}, \bar{y}) \in \Gamma(s, \bar{y})$, is a level set (2.9) for a suitable value of t depending on y and s; moreover, the level (2.21) is noncritical.

(b) The level (2.21) is the boundary of a set

$$\{(x,y) \in G \;:\; u(x,y) > t\} \tag{2.22}$$

whose measure is s. We denote by $G(y)$ the projection of (2.22) on the manifold $y = 0$.

For the time being we prove (a), since (b) is a consequence of a formula we will prove later. From (2.16), (2.17), and (2.19) we get

$$\begin{aligned} \frac{d}{dy} u(x_i(\overline{x}, y), y) &= \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial y} + \frac{\partial u}{\partial y} \\ &= \sum_{i=1}^{n} \alpha_i u_{x_i} + u_y = \phi(s, y) , \end{aligned} \tag{2.23}$$

where we write α_i, ϕ instead of α_i^h, ϕ^h since we are considering the case $m = 1$ for simplicity. From (2.23) we deduce that the values of u along the paths (2.20) do not depend on the initial point $(\overline{x}, \overline{y}) \in \Gamma(s, \overline{y})$, since the initial values $u(x_i(\overline{x}, \overline{y}), \overline{y})$ do not depend on $\overline{x}$. Hence we get (a).

Now let g be a C^1 function. We fix a noncritical level $\Gamma(s, \overline{y})$, denote by $\Gamma(y)$ and $G(y)$ the sets given by the above procedure (see (2.21) and (2.22)), and consider the function

$$y \to \int_{G(y)} g(x, y)\, dx . \tag{2.24}$$

We are interested in a formula for the derivatives of (2.24) with respect to the variables y_h. To this aim we state the following result:

THEOREM 2.5. *If $g \in C^1$ we have*

$$\frac{\partial}{\partial y_h} \int_{G(y)} g(x, y)\, dx = \int_{G(y)} \frac{\partial g}{\partial y_h}\, dx - \int_{\Gamma(y)} g\, \alpha^h H_{n-1}\, dx , \tag{2.25}$$

where the functions in (2.25) *are to be evaluated at $y = \overline{y}$.*

Remark 2.6. We deduce statement (b) as a consequence of (2.25). If we set $g \equiv 1$ in (2.25), we get by (2.18)

$$\frac{\partial}{\partial y_h} |G(y)|_n = - \int_{\Gamma(y)} \alpha^h H_{n-1}\, dx = 0 ;$$

then $|G(y)|_n = \text{const}$. So the levels $\Gamma(y)$ are the boundaries of sets whose measure is s, that is the measure of the "initial" set $G(s, \overline{y})$. We point out that condition (2.18) is made to this sole end. This property of sets $G(y)$ allows us to relate the function (2.1) to

$$\int_{G(y)} u\, dx .$$

Indeed, from property (2.3) we get

$$F(s, y) = \int_0^s u^*(\sigma, y)\, d\sigma = \int_{G(y)} u\, dx , \tag{2.26}$$

since $G(y) = \{x : u(x, y) > u^*(s, y)\}$.

Remark 2.7. From (2.25) and (2.26) we have

$$\frac{\partial}{\partial y_h} F(s, y) = \frac{\partial}{\partial y_h} \int_{G(y)} u\, dx = \int_{G(y)} \frac{\partial u}{\partial y_h}\, dx - \int_{\Gamma(y)} u\, \alpha^h H_{n-1}\, dx\,.$$

Then, since u is constant on $\Gamma(y)$, (2.18) gives

$$\frac{\partial}{\partial y_h} F(s, y) = \frac{\partial}{\partial y_h} \int_0^s u^*(\sigma, y)\, d\sigma = \int_{G(y)} \frac{\partial u}{\partial y_h}\, dx\,,$$

which is (2.4).

Remark 2.8. By means of (2.25) we can make formula (2.5) precise. Indeed, we have for $y = \overline{y}$

$$\begin{aligned} \int_{G(\overline{y})} \frac{\partial^2 u}{\partial y_h \partial y_k}\, dx &= \frac{\partial}{\partial y_h} \int_{G(y)} \frac{\partial u}{\partial y_k}\, dx + \int_{\Gamma(\overline{y})} \frac{\partial u}{\partial y_k} \alpha^h H_{n-1}\, dx \qquad \text{(by (2.25))} \\ &= \frac{\partial^2}{\partial y_h \partial y_k} \int_{G(y)} u\, dx + \frac{\partial}{\partial y_h} \int_{\Gamma(y)} u\, \alpha^k H_{n-1}\, dx \\ &\quad + \int_{\Gamma(\overline{y})} \frac{\partial u}{\partial y_k} \alpha^h H_{n-1}\, dx\,, \qquad \text{(by (2.25) again)}\,. \end{aligned}$$

Since u is constant on $\Gamma(y)$, by (2.18) and (2.26) we have

$$\int_{G(\overline{y})} \frac{\partial^2 u}{\partial y_h \partial y_k}\, dx = \frac{\partial^2}{\partial y_h \partial y_k} \int_0^s u^*(\sigma, y)\, d\sigma + \int_{\Gamma(\overline{y})} \frac{\partial u}{\partial y_k} \alpha^h H_{n-1}\, dx\,.$$

From (2.16) and (2.17) we get

$$\begin{aligned} \int_{\Gamma(\overline{y})} u_{y_k} \alpha^h H_{n-1}\, dx &= -\int_{\Gamma(\overline{y})} \left[\sum_{i=1}^n \alpha_i^k u_{x_i} - \phi^k \right] \alpha^h H_{n-1}\, dx \\ &= -\int_{\Gamma(\overline{y})} \alpha^k \alpha^h |\nabla_x u| H_{n-1}\, dx + \int_{\Gamma(\overline{y})} \phi^k \alpha^h H_{n-1}\, dx \qquad \text{(by (2.14))} \\ &= -\int_{\Gamma(\overline{y})} \alpha^k \alpha^h |\nabla_x u| H_{n-1}\, dx\,, \end{aligned}$$

where in the last step we used (2.18) again and the fact that ϕ^k is constant on $\Gamma(\overline{y})$. So we obtain

$$\int_{G(\overline{y})} \frac{\partial^2 u}{\partial y_h \partial y_k}\, dx = \frac{\partial^2}{\partial y_h \partial y_k} \int_0^s u^*(\sigma, y)\, d\sigma - \int_{\Gamma(\overline{y})} \alpha^h \alpha^k |\nabla_x u| H_{n-1}\, dx\,, \tag{2.27}$$

from which we can easily deduce (2.5), since $\{\alpha^h \alpha^k\}$ is semidefinite positive.

Remark 2.9. We point out that, when $\Gamma(y)$ moves from $\Gamma(\overline{y})$ on the manifold (2.12), the values of u on $\Gamma(y)$ can change; however, by (2.23), these values do not depend on y iff $\phi(s,y) \equiv 0$. This observation is useful in the study of stationary points of the energy functional

$$E(u) = \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dx = \int_\Omega |\nabla u|^2\, dx\,, \tag{2.28}$$

when u belongs to $H_0^1(\Omega)$ (Ω = bounded open set of $\mathbb{R}^n$) and runs in a class of functions with a fixed rearrangement. Indeed, let $\{u(x,t)\}$ be a family of functions defined in Ω such that $t \in (-\delta, \delta)$, $u^*(s,t) = u^*(s,0)$, and $u(x,0) = u(x)$, where u is a stationary point of (2.28). We have

$$\left[\frac{d}{dt}\int_\Omega |\nabla_x u(x,t)|^2\, dx\right]_{t=0} = -2\int_\Omega \Delta u\, u_t\, dx = 0\,. \tag{2.29}$$

Now we apply (2.16) and (2.17) to the function $u(x,t)$ defined in $\Omega \times (-\delta, \delta)$. If $\Gamma(0) = \{x : u(x) = u^*(s)\}$ is a noncritical level of u, let $\Gamma(t)$ be the level of $u(x,t)$ defined as in (2.21). By Remark 2.6, $\Gamma(t)$ is the boundary of the level set $\{x : u(x,t) > u^*(s)\}$. Hence $u(x,t)$ assumes the value $u^*(s)$ on $\Gamma(t)$ for any t, so $\phi \equiv 0$. Then, from (2.16), (2.17) we get

$$\frac{\partial u}{\partial t} = -\alpha|\nabla_x u|\,, \tag{2.30}$$

where α satisfies

$$\int_{\Gamma(0)} \alpha H_{n-1}\, dx = 0\,.$$

From (2.29) and (2.30) and by the classical co-area formula we obtain

$$\int_0^{+\infty} d\tau \int_{u=\tau} \Delta u\, \alpha H_{n-1}\, dx = 0$$

for any function α with mean value zero on all noncritical levels. So Δu must be constant on any noncritical level of u (see Laurence–Stredulinsky [13]).

Proof of Theorem 2.5: For simplicity, we assume $m = 1$. In this case we can write the map (2.20) in the following way:

$$x_i = x_i(\overline{x}, y), \qquad J(i = 1, \ldots, n)\,, \tag{2.31}$$

where $\overline{x}$ runs on the projection $\overline{\Gamma}$ of $\Gamma(\overline{y})$ on the plane $y = 0$. Obviously we can continue (2.31) in such a way that it can be viewed as a smooth transformation (with respect to x and y) from $G(\overline{y})$ onto $G(y)$; moreover, for $y = \overline{y}$

$$x_i(\overline{x}, y) = \overline{x}_i \quad \text{if } \overline{x} \in G(\overline{y})\,. \tag{2.32}$$

If we set

$$B_i(\overline{x}, y) = \frac{\partial x_i}{\partial y}(\overline{x}, y) \quad \text{when } \overline{x} \in G(\overline{y}) , \tag{2.33}$$

we have

$$B_i(\overline{x}, y) = \alpha_i(x(\overline{x}, y), y) \quad \text{if } \overline{x} \in \overline{\Gamma} , \tag{2.34}$$

since the paths $(x_i(\overline{x}, y))$ are solutions of system (2.19) if $\overline{x}$ is fixed on $\overline{\Gamma}$ and $y \in (\overline{y} - \delta, \overline{y} + \delta)$. If we denote by J the Jacobian of the map from $G(\overline{y})$ onto $G(y)$ defined above, we have

$$\int_{G(y)} g(x, y)\, dx = \int_{G(\overline{y})} g(x(\overline{x}, y), y)\, J(\overline{x}, y)\, d\overline{x} ,$$

and then

$$\begin{aligned} \frac{\partial}{\partial y} \int_{G(y)} g(x, y)\, dx \\ = \int_{G(\overline{y})} \left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \frac{\partial x_i}{\partial y} + \frac{\partial g}{\partial y} \right) J(\overline{x}, y)\, d\overline{x} + \int_{G(\overline{y})} g \frac{\partial J}{\partial y}\, d\overline{x} . \end{aligned} \tag{2.35}$$

Now we compute the right-hand side in (2.35) for $y = \overline{y}$. By the classical rule

$$\frac{\partial J}{\partial y}(\overline{x}, \overline{y}) = \sum_{i=1}^{n} \frac{\partial}{\partial \overline{x}_i} B_i(\overline{x}, \overline{y}) ,$$

the last integral in (2.35) becomes

$$-\int_{G(\overline{y})} \sum_{i=1}^{n} \frac{\partial g}{\partial \overline{x}_i} B_i\, d\overline{x} - \int_{\partial G(\overline{y})} g \sum_{i=1}^{n} B_i \frac{u_{x_i}}{|\nabla_x u|} H_{n-1}\, dx .$$

Since (2.32) gives

$$\int_{G(\overline{y})} \frac{\partial g}{\partial y} J(\overline{x}, \overline{y})\, d\overline{x} = \int_{G(\overline{y})} \frac{\partial g}{\partial y}\, dx ,$$

we get from (2.34) and (2.14)

$$\frac{\partial}{\partial y} \int_{G(y)} g(x, y)\, dx = \int_{G(\overline{y})} \frac{\partial g}{\partial y}\, dx - \int_{\Gamma(\overline{y})} g\alpha H_{n-1}\, dx \quad \text{for } y = \overline{y} ,$$

which is (2.25).

Formula (2.25) holds also if we disregard the special expression (2.15) of the functions α^h. However, in this situation we give up property (2.18); by this condition we essentially aim at the property of the sets $G(y)$ contained in Remark 2.4 (see statement (b)) and its applications to second-order elliptic equations (see Section 3).

For instance, if we consider a function u defined in $\Omega \subset \mathbb{R}^n$ and replace (2.15) by $\alpha = 1/|\nabla u|$, (2.15) becomes

$$\frac{d}{dt} \int_{u>t} g(x)\, dx = -\int_{u=t} g \frac{1}{|\nabla u|} H_{n-1}\, dx ,$$

from which we can deduce the classical co-area formula, which can be seen as a particular case of (2.25). Thus we can prove (2.25), proceeding as in the proof of the co-area formula. Now we sketch this alternative proof.

We have

$$\begin{aligned} &\int_{G(y)} g(x,y)\, dx - \int_{G(\bar{y})} g(x,\bar{y})\, dx \\ &= \int_{G(y)\setminus G(\bar{y})} [g(x,y) - g(x,\bar{y})]\, dx - \int_{G(\bar{y})\setminus G(y)} [g(x,y) - g(x,\bar{y})]\, dx \\ &\quad + \int_{G(\bar{y})} [g(x,y) - g(x,\bar{y})]\, dx . \end{aligned}$$

Obviously we have

$$\lim_{y\to\bar{y}} \int_{G(\bar{y})} \frac{g(x,y) - g(x,\bar{y})}{y - \bar{y}}\, dx = \int_{G(\bar{y})} \frac{\partial g}{\partial y}\, dx .$$

On the other hand, by using arguments from the proof of the co-area formula, we could see that

$$\begin{aligned} &\int_{G(y)\setminus G(\bar{y})} [g(x,y) - g(x,\bar{y})]\, dx - \int_{G(\bar{y})\setminus G(y)} [g(x,y) - g(x,\bar{y})]\, dx \\ &= (y - \bar{y}) \int_{\partial G(\bar{y})} \alpha g H_{n-1}\, dx + o(|y - \bar{y}|) . \end{aligned}$$

Hence we again obtain (2.25).

3. Symmetrization Results

We begin by proving Theorem 1.1. We fix y and consider a noncritical value $u^*(s,y) = t$. By integrating equation (1.9) on the set $G(y) = \{x : u(x,y) > t\}$ we get

$$\begin{aligned} &-\int_{G(y)} \sum_{i,j=1}^{n} (a_{ij}(x,y) u_{x_i})_{x_j}\, dx - \int_{G(y)} \sum_{h,k=1}^{m} (b_{hk}(y) u_{y_h})_{y_k}\, dx \\ &-\int_{G(y)} \sum_{i=1}^{n} \sum_{h=1}^{m} (c_{ih}(y) u_{x_i})_{y_h}\, dx - \int_{G(y)} \sum_{h=1}^{m} \sum_{i=1}^{n} (d_{hi}(y) u_{y_h})_{x_i}\, dx \\ &= \int_{G(y)} f\, dx . \end{aligned} \tag{3.1}$$

Let us denote by I_1, I_2, I_3, I_4 the integrals on the left side in (3.1), respectively. For simplicity we use the convention on repeated indices, so we write:

$$I_1 = \int_{\Gamma(y)} a_{ij}(x,y) u_{x_i} u_{x_j} \frac{1}{|\nabla_x u|} H_{n-1}\, dx\, . \tag{3.2}$$

For I_2 we apply formula (2.25); so

$$\begin{aligned} I_2 &= -\frac{\partial}{\partial y_k} \int_{G(y)} b_{hk}(y) u_{y_h} dx - \int_{\Gamma(y)} b_{hk}(y) u_{y_h} \alpha^k H_{n-1}\, dx \\ &= -\frac{\partial}{\partial y_k} \left(b_{hk}(y) \frac{\partial}{\partial y_h} \int_{G(y)} u\, dx \right) - \int_{\Gamma(y)} b_{hk}(y) u_{y_h} \alpha^k H_{n-1}\, dx\, , \end{aligned}$$

where in the last step we used formula (2.4) (see Remark 2.7). On the other hand, by (2.16), (2.17), and (2.14) we have

$$\begin{aligned} &-\int_{\Gamma(y)} b_{hk}(y) u_{y_h} \alpha^h H_{n-1}\, dx \\ &= \int b_{hk}(y) \alpha_i^h u_{x_i} \alpha^k H_{n-1}\, dx + \int_{\Gamma(y)} b_{hk}(y) \phi^h \alpha^k H_{n-1}\, dx \\ &= \int_{\Gamma(y)} b_{hk}(y) \alpha^h \alpha^k |\nabla_x u| H_{n-1}\, dx + b_{hk}(y) \phi^h \int_{\Gamma(y)} \alpha^k H_{n-1}\, dx \\ &= \int_{\Gamma(y)} b_{hk}(y) \alpha^h \alpha^k |\nabla_x u| H_{n-1}\, dx\, , \end{aligned}$$

where in the last step we used the fact that ϕ^h is constant on $\Gamma(y)$ (see (2.16) and (2.18)); so we get

$$I_2 = -\frac{\partial}{\partial y_k} \left(b_{hk}(y) \frac{\partial}{\partial y_h} \int_{G(y)} u\, dx \right) + \int_{\Gamma(y)} b_{hk}(y) \alpha^h \alpha^k |\nabla_x u| H_{n-1}\, dx\, . \tag{3.3}$$

Furthermore, again by (2.25),

$$\begin{aligned} I_3 &= -\frac{\partial}{\partial y_h} \int_{G(y)} c_{ih}(y) u_{x_i}\, dx - \int_{\Gamma(y)} c_{ih}(y) u_{x_i} \alpha^h H_{n-1}\, dx \\ &= -\int_{\Gamma(y)} c_{ih}(y) u_{x_i} \alpha^h H_{n-1}\, dx \end{aligned} \tag{3.4}$$

since

$$\int_{G(y)} c_{ih}(y) u_{x_i}\, dx = c_{ih}(y) \int_{G(y)} u_{x_i}\, dx = 0\, .$$

Moreover, by applying (2.16), (2.17), and (2.14), we get

$$I_4 = \int_{\Gamma(y)} d_{hi}(y) u_{y_h} u_{x_i} \frac{1}{|\nabla_x u|} H_{n-1} \tag{3.5}$$

$$
\begin{aligned}
&= -\int_{\Gamma(y)} d_{hi}(y)\alpha^h_j u_{x_j} u_{x_i} \frac{1}{|\nabla_x u|} H_{n-1}\, dx \\
&\quad + d_{hi}(y)\phi^h \int_{\Gamma(y)} u_{x_i} \frac{1}{|\nabla_x u|} H_{n-1}\, dx \\
&= -\int_{\Gamma(y)} d_{hi}(y)\alpha^h u_{x_i} H_{n-1}\, dx\ ,
\end{aligned}
$$

as

$$\int_{\Gamma(y)} \frac{u_{x_i}}{|\nabla_x u|} H_{n-1}\, dx = 0\ .$$

Finally, the last integral in (3.1) can be estimated by using the classical Hardy's inequality for rearrangements (see also property (2.3))

$$\int_{G(y)} f(x,y)\, dx \leqq \int_0^s f^*(\sigma, y)\, d\sigma\ , \tag{3.6}$$

where $s = |G(y)|_n$.

Putting together (3.1) through (3.6), we obtain

$$
\begin{aligned}
&\int_{\Gamma(y)} a_{ij} \frac{u_{x_i}}{|\nabla_x u|} \frac{u_{x_j}}{|\nabla_x u|} |\nabla_x u|\, H_{n-1}\, dx + \int_{\Gamma(y)} b_{hk}(-\alpha^h)(-\alpha^k)|\nabla_x u| H_{n-1}\, dx \\
&\quad + \int_{\Gamma(y)} c_{ih} \frac{u_{x_i}}{|\nabla_x u|}(-\alpha^h)\, |\nabla_x uJ| H_{n-1}\, dx + \int_{\Gamma(y)} d_{hi}(-\alpha^h) \frac{u_{x_i}}{|\nabla_x u|} |\nabla_x u| H_{n-1}\, dx \\
&\quad - \frac{\partial}{\partial y_k}\left(b_{hk}(y) \frac{\partial}{\partial y_h} \int_{G(y)} u\, dx \right) \\
&\quad \leqq \int_0^s f^*(\sigma, y)\, d\sigma\ .
\end{aligned}
$$

Using ellipticity condition (1.10), we get

$$
\begin{aligned}
&\int_{\Gamma(y)} |\nabla_x u| H_{n-1}\, dx + \nu \int_{\Gamma(y)} \alpha^2 |\nabla_x u| H_{n-1}\, dx \\
&\quad - \frac{\partial}{\partial y_k}\left(b_{hk}(y) \frac{\partial}{\partial y_h} \int_{G(y)} u\, dx \right) \\
&\quad \leqq \int_0^s f^*(\sigma, y)\, d\sigma\ ,
\end{aligned}
$$

where $\alpha^2 = \sum_{h=1}^n (\alpha^h)^2$; hence

$$
\begin{aligned}
&\int_{\Gamma(y)} |\nabla_x u| H_{n-1}\, dx - \frac{\partial}{\partial y_k}\left(b_{hk}(y) \frac{\partial}{\partial y_h} \int_0^s u^*(\sigma, y) d\sigma \right) \\
&\quad \leqq \int_0^s f^*(\sigma, y) d\sigma\ .
\end{aligned} \tag{3.7}
$$

The first integral in (3.7) can be estimated from below via the classical isoperimetric inequality, the Schwarz inequality, and the co-area formula, so we have (see [7], [8], [18], [20])

$$\int_{\Gamma(y)} |\nabla_x u| H_{n-1}\, dx \geqq n^2 c_n^{2/n} \left[-\frac{\partial u^*}{\partial s}(s,y) \right] s^{2-2/n} . \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\begin{aligned} &-n^2 c_n^{2/n} s^{2-2/n} \frac{\partial u^*}{\partial s} - \frac{\partial}{\partial y_k}\left(b_{hk}(y) \frac{\partial}{\partial y_h} \int_0^s u^*(\sigma,y) d\sigma \right) \\ &\leqq \int_0^s f^*(\sigma,y)\, d\sigma . \end{aligned} \tag{3.9}$$

If

$$U(s,y) = \int_0^s u^*(\sigma,y)\, d\sigma ,$$

(3.9) can be written in the following way:

$$- n^2 c_n^{2/n} s^{2-2/n} \frac{\partial^2 U}{\partial s^2} - \frac{\partial}{\partial y_k}\left(b_{hk}(y) \frac{\partial U}{\partial y_h} \right) \leqq \int_0^s f^*(\sigma,y)\, dy , \tag{3.10}$$

with $(s,y) \in G^* = [0, |G'|_n] \times G''$. Beside differential inequality (3.10), function U satisfies the following boundary conditions:

$$\begin{cases} U(0,y) = 0 \quad U_s(|G'|_n, y) = 0, \ y \in G'' , \\ U(s,y) = 0 \quad y \in \partial G'', \ s \in (0, |G'|_n) . \end{cases} \tag{3.11}$$

Now we set

$$V(s,y) = \int_0^s v^*(\sigma,y)\, d\sigma ,$$

where $v(x,y)$ is the solution of (1.11). We handle problem (1.11) in the same way as (1.9), but we observe that in this case all the inequalities we used to get (3.10) become equalities. We obtain the differential equality

$$- n^2 c_n^{2/n} s^{2-2/n} \frac{\partial^2 V}{\partial s^2} - \frac{\partial}{\partial y_k}\left(b_{hk}(y) \frac{\partial V}{\partial y_h} \right) = \int_0^s f^*(\sigma,y)\, dy \quad \text{on } G^* ; \tag{3.12}$$

furthermore, V satisfies on ∂G^* the same boundary conditions (3.11) as U.

We point out that we do not need v to be analytic as the solution u of (1.9) is. Indeed, we used this regularity assumption essentially for deriving formula (2.25). Obviously this formula is trivial in the spherically symmetric case.

If we put

$$Z(s,y) = U(s,y) - V(s,y) = \int_0^s [u^*(\sigma,y) - v^*(\sigma,y)]\, d\sigma ,$$

from (3.10) and (3.12) we get

$$-n^2 c_n^{2/n} s^{2-2/n} \frac{\partial^2 Z}{\partial s^2} - \frac{\partial}{\partial y_k}\left(b_{hk}(y)\frac{\partial Z}{\partial y_h}\right) \leqq 0 \quad \text{on } G^* . \tag{3.13}$$

If we keep in mind the boundary conditions verified by Z on ∂G^*, we have $Z(s, y) \leqq 0$ on G^* by the maximum principle, that is, (1.12).

Remark 3.1. What can one say about the case when (1.12) becomes an equality? We observe that in each inequality we used to deduce (1.12), the equalities hold if and only if the problem (1.9) is the symmetrized problem (1.11). So, arguing as in [2], we can say that, when $u^* = v^*$, problem (1.9) must be symmetric with respect to the variable x, as (1.11) is.

Remark 3.2. The previous results can be used also to obtain sharp comparison results for nonhomogeneous Dirichlet problems. Indeed, let us consider the problem

$$-\Delta u = f \quad \text{in } \Omega , \qquad u = \varphi \quad \text{on } \partial\Omega$$

where Ω is a disk in $\mathbb{R}^2$, $\{x : |x| < r\}$ for example (but we can also consider an annulus or an exterior domain). Since

$$-\Delta u = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) - \frac{1}{\rho^2}\frac{\partial u^2}{\partial \theta^2} ,$$

where (ρ, θ) denote the polar coordinates in $\mathbb{R}^2$, we can apply the above arguments by using Steiner symmetrization with respect to the angular variable θ. We deduce

$$\int_0^s u^*(\theta, \rho) d\theta \leqq \int_0^s v^*(\theta, \rho) d\theta$$

for any $\rho \in (0, r)$ and $s \in [0, 2\pi]$, where v is the solution of the problem

$$-\Delta v = f^{\#} \quad \text{in } \Omega , \qquad v = \varphi^{\#} \quad \text{on } \partial\Omega .$$

Here $f^{\#}$ is the Steiner-symmetrized function of $f(\theta, \rho)$ with respect to θ, with $\theta \in [-\pi, \pi]$, $\rho \in (0, r)$, and $\varphi^{\#}(\theta)$ the Schwarz-symmetrized function of $\varphi(\theta)$, $\theta \in [-\pi, \pi]$.

Remark 3.3. By the same arguments used in proving Theorem 1.1, we can deduce an analogous result for operators of the type

$$Mu = -\sum_{i,j=1}^{n} (a_{ij}(x, y) u_{x_i})_{x_j} - \sum_{h,k=1}^{m} b_{hk}(y) u_{y_h y_k} .$$

In this case the proof is simpler because we can apply the weaker inequality (2.5) instead of (2.25). Indeed, proceeding as in the proof of Theorem 1.1, we meet the term

$$\sum_{h,k=1}^{m} b_{hk}(y) \int_{G(y)} u_{y_h y_k} dx \tag{3.14}$$

(see (3.1)). By (2.5), the integral (3.14) is estimated from below by the term

$$\sum_{h,k=1}^{m} b_{hk}(y) \frac{\partial^2}{\partial y_h \partial y_k} \int_0^s u^*(\sigma, y) d\sigma .$$

We go on as in the proof of Theorem 1.1, thus obtaining the comparison result (1.12), where u^* is the rearrangement of the solution u of the problem

$$Mu = f \quad \text{in } G, \qquad u = 0 \quad \text{on } \partial G ,$$

and v^* is the rearrangement of the solution v of

$$-\Delta_x v - \sum_{h,k=1}^{m} b_{hk}(y) v_{y_h y_k} = f^{\#} \quad \text{in } G^{\#} , \qquad v = 0 \quad \text{on } \partial G^{\#} .$$

Remark 3.4. The condition on G can be weakened, too. Let G be an open, bounded set of $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$, and let G'' be its projection on the linear manifold $\{x = 0\}$. Let

$$G^* = \{(s, y) \in R^{1+m} : y \in G'', \ s \in [0, |G_y|_n]\} .$$

We have $\partial G^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 &= \{(0, y) : y \in G''\} , \\ \Gamma_2 &= \{(s, y) : y \in \partial G'', \ s \in [0, |G_y|_n]\} \end{aligned}$$

and Γ_3 is the graph of the function

$$y \in G'' \to |G_y|_n , \tag{3.15}$$

that is,

$$\Gamma_3 = \{(|G_y|_n, y) : y \in G''\} .$$

If we proceed as in the proof of Theorem 1.1, we again obtain the differential inequality (3.13); in this case the function Z satisfies the following boundary conditions:

$$Z = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 , \qquad Z_s = 0 \quad \text{on } \Gamma_3 .$$

If we assume that function (3.15) is locally Lipschitz, we can apply a maximum principle from which we deduce the comparison result (1.12).

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