Mathematical treatment of the magnetic confinement in a current carrying stellarator

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. Introduction

The magnetic confinement in a Stellarator can be modeled with help of averaging methods and Boozer vacuum coordinates (see [1, 2]). This leads to a two-dimensional Grad–Shafranov type problem for the averaged poloidal flux function. So, we assume that \((\rho, \rho \theta, \phi)\) are the Boozer vacuum coordinates system: i.e. \(\rho = \rho(x, y, z)\) is a function which is constant on each nested toroidal and positive except for the magnetic axis where \(\rho = 0; \theta = \theta(x, y, z)\) is the poloidal angle (i.e. \(\theta\) is constant on any poloidal loop) and \(\phi = \phi(x, y, z)\) is the toroidal angle (i.e. constant on any poloidal circuit). By averaging in \(\phi\) and adding a free boundary formulation to the Grad–Shafranov type equation, the problem can be stated in the following terms (see [3]): Let \(\Omega = \{(\rho, \theta) : 0 < \rho < R, \theta \in ]0, 2\pi[\}\) and define \(\partial \Omega = \Gamma_R \cup \Gamma_p \cup \Gamma_0\) by means of \(\Gamma_R = \{(\rho, \theta) : \rho \in ]0, R[\}\) and \(\Gamma_p = \{(\rho, 0) ; (\rho, 2\pi) : \rho \in ]0, R[\}\) and \(\Gamma_0 = \{(0, \theta) : \theta \in ]0, 2\pi[\}\). Given \(u : \Omega \rightarrow \mathbb{R} \) and \(F : \mathbb{R} \rightarrow \mathbb{R}_+\).
such that \( u \in W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega), p \geq 1 \) and \( F \in W^{1,\infty}(\inf u, \sup u), F(s) = F_0 \) for all \( s \leq 0 \) and \((u, F)\) satisfy

\[
-\mathcal{L}u = a(\rho, \theta)F(u) + F(u)F'(u) + \lambda b(\rho, \theta) p'(u) \quad \text{in } \Omega \\
u_{|_{\Gamma_0}} = \gamma; \quad u(\rho, 0) = u(\rho, 2\pi) \quad \text{for } \rho \in [0, R].
\] (1.1) (1.2)

Here \(-\mathcal{L}\) is a suitable second order elliptic operator (see [4]). Coefficients \(a\) and \(b\) are explicitly known in terms of the components of the metric associated to the Boozer coordinate system. The unknown \(u = u(\rho, \theta)\) represents the averaged poloida flux, \(F(u)\) is the covariant toroidal coordinate of the magnetic field and the constant \(F_0\) is the covariant toroidal coordinate of the magnetic field in the vacuum region. Function \(p(u(x))\) represents the pressure and it cannot be obtained explicitly from the magnetohydrodynamic system. Thus, \(p\) is given as a constitutive law. Usually \(p(s) = \frac{\frac{\lambda}{2}(s_+)^2}{s_+} \) where \(s_+ = \max\{s, 0\}, s \in \mathbb{R}\) and \(\lambda > 0\). The plasma region is defined at \(\{(\rho, \theta) \in \Omega : u(\rho, \theta) > 0\} = \Omega^p\) and the vacuum region as \(\{(\rho, \theta) \in \Omega : u(\rho, \theta) < 0\} = \Omega^v\).

Thus, a free boundary arises as the boundary of the set \(\{(\rho, \theta) \in \Omega : u(\rho, \theta) = 0\}\) (see [3, 4]). Notice that if \((\rho, \theta) \in \Omega^v\) then \(F(u) \in F_0\).

It is necessary to add another condition typical of any ideal Stellarator to conditions (1.1) and (1.2). It expresses the zero net current within each flux magnetic surface and can be written as

\[
\int_{\{u > \tau\}} [F(u)F'(u) + \lambda b(\rho, \theta) p'(u)] \rho d\rho d\theta = 0 \quad \forall t \in \left[ \inf u, \sup u \right].
\] (1.3)

Problem (1.1), (1.2), (1.3) has been treated in [3]. In practice, this condition does not hold and some known current arises at the interior of each magnetic surface. Some studies on the stability of the equilibrium configuration for the fixed boundary formulation are already in the literature (Cooper et al. [5]). In those works the current is assumed to have the form

\[
J(s) = J(1)(4s^2 - 3s^4)
\] (1.4)

within the flux magnetic surface corresponding to the parameter \(s\) (i.e. \(\{(\rho, \theta) \in \Omega : (\rho, \theta) = s\}\)) and where \(2\pi J(1)\) is the total current (toroidal). In [5] it is also assumed that the constitutive law for the real pressure function \(P\) is

\[
P(s) = P(0)(1 - 3s^2 + 2s^3).
\] (1.5)

Notice that in such a formulation it is assumed that the magnetic axis corresponds to \(s = 0\) and that \(s = 1\) corresponds to the free boundary.

We point out that the presence of a positive total current is a typical phenomenon of Tokamak devices (see e.g. Temam [6, 7], Berestycki and Brezis [8], Blum [9], Friedman [10], Mossino and Temam [11] and their references). In such a case, the global condition is formulated as

\[
\int_{\Omega} [F(u)F'(u) + \lambda b(\rho, \theta) p'(u)] \rho d\rho d\theta = I
\]
here $I > 0$ is the total current. For an exposition of the important differences between
the mathematical formulations of the magnetic confinement in Tokamak and Stellarator
devices see the paper [3] (see also Remark 1).

The main purpose of this paper is to extend the results of [3] to the study of the
new type of Stellarators (the so-called current carrying Stellarators). To explain how
place (1.3) by a new condition, we need to translate assumptions (1.4) and (1.5) in
piecewise of the formulation used in [3], where the boundary of the plasma region were
assumed corresponding to the level $u = 0$ and the magnetic axis to $\max_{\Omega} u = \|u_+\|_{L^\infty(\Omega)}$.

we define the following change of variable involving $\|u_+\|_{L^\infty(\Omega)}$ with $u$ verifying
(1.1) and (1.2): Let

$$s := \left(1 - \frac{t_+}{\|u_+\|_{L^\infty(\Omega)}}\right) \quad \forall t \in \left[\inf_{\Omega} u, \sup_{\Omega} u\right].$$ (1.6)

Thus,

$$\{s = 0\} \equiv \{u_+ = \|u_+\|_{L^\infty(\Omega)}\} \quad (\equiv \text{the magnetic axis}),$$

$$\{s = 1\} \equiv \partial\{u_+ = 0\} \quad (\equiv \text{the free boundary}).$$

Furthermore, if we fix a current $J(s)$ (as, for instance (1.4)), by the change of variable
in (1.6) we obtain a new expression

$$J(s) = j(t_+, \|u_+\|_{L^\infty(\Omega)})$$

terms of the new variable $t \in [\inf_{\Omega} u, \sup_{\Omega} u]$. Analogously, we will have $P(s) = p(t, \|u_+\|_{L^\infty(\Omega)})$. Here, and in what follows, we shall assume that

$$\begin{align*}
&j \in C(\mathbb{R} \times \mathbb{R}^+), \quad j(\sigma, 0) = 0 \quad \text{for all } \sigma \geq 0,
&j'(\sigma) \in C(\mathbb{R}^+ \times \mathbb{R}^+) \quad \text{and} \quad \eta := \sup \{j'(t, \sigma) : (t, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+\} < +\infty.
\end{align*}$$ (1.7)

e shall always use $j'$ to denote the derivative of $j$ with respect to the first component
et $j' := \frac{\partial j}{\partial t}$. The assumptions on $p$ are the following: $p(t, \sigma) \equiv p(t)$ with $p \in C^1(\mathbb{R})$
ch that

$$p(0) = 0, \quad 0 \leq p'(t) \leq \lambda t_+ \quad \text{and} \quad |p'(t) - p'(s)| \leq L|t - s|^\alpha$$ (1.8)

or some $\lambda > 0, L > 0$ and $\alpha \in ]0, 1[$. Notice that $p(t) = \frac{t}{2}(t_+)^2$ satisfies all the require-
ments. Finally, the new condition for current carrying Stellarator can be stated as

$$\int_{\{u > t\}} [F(u)F'(u) + p'(u)b(\rho, \theta)] \rho d\rho d\theta = j(t_+, \|u_+\|_{L^\infty(\Omega)}), \quad t \in \left[\inf_{\Omega} u, \sup_{\Omega} u\right].$$ (1.9)

For the sake of the exposition we shall assume that $\mathcal{L} = \Delta$ (the Laplace operator)
d we replace $(\rho, \theta)$ by the associated cartesian coordinate $x \in \Omega \subset \mathbb{R}^2$. Say the main
purpose of this paper is to prove the existence of a couple $(u, F)$, solution of the
llowing problem ($\mathcal{P}$)

$$-\Delta u = aF(u) + F(u)F'(u) + bp'(u) \quad \text{in } \Omega,$$ (1.10)
\[ u - \gamma \in H^1_0(\Omega), \quad (1.11) \]
\[ \int_{\{u > t\}} [F(u)F'(u) + b p'(u)] \, dx = j(t, \|u_+\|_{L^\infty(\Omega)}) \quad t \in \left[ \inf_{\Omega} u, \sup_{\Omega} u \right]. \quad (1.12) \]

Before stating our main result, we introduce the following useful convex cone:
\[ V(\Omega) = \{ v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v|_{\partial \Omega} \leq 0 \}. \]

**Theorem 1.** Suppose that \( \inf_{\Omega} |a| > 0 \) and \( \gamma \leq 0 \). Then there exists \( \Lambda > 0 \) such that
\[ \lambda \|b\|_{L^\infty(\Omega)} + \eta < \Lambda \]
there is a couple \((u, F), u \in V(\Omega) \) and \( F \in W^{1, \infty}(\inf_{\Omega} u, \sup_{\Omega} u) \) solution of (1.10), (1.11) and (1.12) satisfying also that \( \text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0 \) and that \( F \) is entirely determined by \( u \).

Inspired in [12] and [3] we will reformulate problem \((\mathcal{P})\) in terms of a new problem \((\mathcal{P}_\varepsilon)\), of nonlocal nature, where we eliminate the unknown \( F \) by a term involving the function \( u \), the decreasing rearrangement and the relative rearrangement of \( b \) with respect of \( u \). However, the lack of regularity of the derivative of the decreasing rearrangement makes it difficult to solve directly problem \((\mathcal{P}_\varepsilon)\). A family of problems \((\mathcal{P}_{\varepsilon, \gamma})\) are the introduced. Using a Galerkin method (of interest also for the numerical approach we will find a solution of \((\mathcal{P}_{\varepsilon, \gamma})\)). Finally, thanks to a result on the regularity of the derivative of the decreasing rearrangement (Lemma 4) we shall obtain a solution \((\mathcal{P}_\varepsilon)\) by making \( \varepsilon \to 0 \). The equivalence of the problems \((\mathcal{P}_\varepsilon)\) and \((\mathcal{P})\) (under a suitable condition) proves that this solution is also a solution of \((\mathcal{P})\). Finally, in Section 6 we give some qualitative properties on the solution.

### 2. Preliminary results

In this section we recall the notion of relative rearrangement and some useful properties of it. Let \( \Omega \) be a bounded measurable set of \( \mathbb{R}^N, N \geq 1 \). For any measurable subspace \( E \) of \( \Omega \), we denote by \(|E|\) its Lebesgue measure. Given a measurable function \( u : \Omega \to \mathbb{R} \) and any value \( t \in \mathbb{R} \), we denote by \( \{u = t\}, \{u > t\} \) and \( \{u \geq t\} \) the sets \( \{x \in \Omega : u(x) = t\} \), \( \{x \in \Omega : u(x) > t\} \) and \( \{x \in \Omega : u(x) \geq t\} \), respectively. Their measure will be indicated by \( |u = t|, |u > t| \) and \( |u \geq t| \), respectively. We will say that \( u \) has a flat region at the level \( t \) if \( |u = t| \) is strictly positive. It may exist, at most, a countable family of flat regions \( P_a(t_i) := \{u = t_i\} \). We denote by \( P(u) = \bigcup_{t \in \mathcal{D}} P_u(t) \) the union of all the flat regions of \( u \).

**Definition 1.** Let \( u : \Omega \to \mathbb{R} \) be the Lebesgue measurable function. The distribution function of \( u \) is defined by
\[ m_a(t) := |u > t| \quad \text{for any} \ t \in \mathbb{R}. \]
he generalized inverse of \( m_u \) is called the decreasing rearrangement of \( u \) and is denoted by \( u_* \). That is the function \( u_* : ]0, |\Omega|[ \to \mathbb{R} \) such that \( u_*(s) = \inf \{ t \in \mathbb{R} : |u - t| < s \} \) with \( u_*(0) = \text{ess sup}_\Omega u \) and \( u_*(|\Omega|) = \text{ess inf}_\Omega u \).

**Lemma 1.** Let \((v_n)_{n \geq 1}, v \in L^p(\Omega)\) such that \( \text{meas}\{x \in \Omega : v(x) = t\} = 0 \) for all \( t \in \mathbb{R} \) nd \( v_n \to v \) strongly in \( L^p(\Omega) \) and a.e. in \( \Omega \). Then

\[
|v_n > v_+(x)| \to |v > v_+(x)| \quad \text{a.e. } x \in \Omega.
\]

**Proof.** One has the following chain of inequalities

\[
|v > v_+(x)| \leq \lim \inf_{n \to \infty} |v_n > v_+(x)| \\
\leq \lim \sup_{n \to \infty} |v_n > v_+(x)| \\
\leq |v > v_+(x)| \quad \text{a.e. } x \in \Omega.
\]

ow, since by assumption \( v \) has no flat region, we have that \( |v = v_+(x)| = 0 \) and by e above inequality we get the conclusion. \( \square \)

We denote by \( \Omega_* \) the interval \( ]0, |\Omega|[ \). Later on we shall need to use some properties of the decreasing rearrangement. In particular, we shall use the following classical results.

**Lemma 2.** Let \( u \) and \( v \) be measurable functions in \( \Omega \). Then

For all \( s \in \Omega_* \), we have \( m_u(u_*(s)) \leq s \).

If \( u \) has no flat regions, then \( m_u \) is continuous and \( m_u(u_*(s)) = s \) \( \forall s \in \Omega_* \).

\( u \) and \( u_* \) are equimeasurable, i.e. \( |u_* > t| = |u > t| \) \( \forall t \in \mathbb{R} \).

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a Borel function such that \( \varphi(u) \in L^1(\Omega) \), then

\[
\int _\Omega \varphi(u(x)) \, dx = \int _{\Omega_*} \varphi(u_*(s)) \, ds;
\]

If \( u \leq v \) almost everywhere in \( \Omega \), then \( u_*(s) \leq v_*(s) \) \( \forall s \in \Omega_* \);

Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a nondecreasing function. Then \( \varphi(u_*(s)) = \varphi(u_*(s)) \) almost everywhere \( s \in \Omega_* \);

The mapping \( u \to u_* \) applies \( L^p(\Omega) \) into \( L^p(\Omega_*) \)

\( (1 \leq p \leq +\infty) \) and it is a contraction, i.e. \( \|u_* - v_*\|_{L^p(\Omega_*)} \leq \|u - v\|_{L^p(\Omega)} \)

\( (1 \leq p \leq +\infty) \). Moreover, \( \|u\|_{L^p(\Omega)} = \|u_*\|_{L^p(\Omega_*)}. \) \( \square \)

The proof of this lemma can be found, for instance, in [13–15].

We shall also need some regularity results on the first derivative of \( u_* \). More precisely we have:

**Lemma 3** ([16, 17]). Let \( \Omega \) be an open, bounded and connected regular set of \( \mathbb{R}^N \).

\( u \in W^{1,p}(\Omega) \) for some \( 1 \leq p \leq +\infty \), then:

i) \( u_* \in W^{1,p}_{\text{loc}}(\Omega_*) \),
(ii) There exists a constant \( Q = Q(\Omega) \) only dependent on \( \Omega \) such that if we define \( k(s) = \frac{1}{\Omega} \min\{s^{1-N}, (|\Omega| - s)^{1-N}\} \) \( \forall s \in \Omega_* \), then one has \( \|k \frac{du}{ds}\|_{L^p(\Omega_*)} \leq \|\nabla u\|_{L^p(\Omega)} \) for all \( 1 \leq p \leq +\infty \).

(iii) Finally, if \( N < p \leq +\infty \), then \( u_* \in W^{1,q}(\Omega_*) \) with \( 1 \leq q < \frac{Np}{(N-1)p+N} \) and

\[
\left\| \frac{du}{ds} \right\|_{L^p(\Omega_*)} \leq C \|\nabla u\|_{L^p(\Omega)}
\]

for some \( C = C(N, p, q, \Omega) \). \( \square \)

Making \( p \to +\infty \) in (iii) of Lemma 3, we obtain

**Corollary 1.** Let \( \Omega \) as above and \( u \in W^{1,\infty}(\Omega) \). Then \( u_* \in W^{1,q}(\Omega_*) \) for all \( 1 \leq q < \frac{N}{N-1} \) and \( u_* \in \mathcal{C}(\Omega_*) \). \( \square \)

The following lemma shows some additional regularity on \( \frac{du}{ds} \) under suitable as sumptions on \( u \) when \( N = 2 \).

**Lemma 4.** Let \( \Omega \subset \mathbb{R}^2 \). Then, for all \( w \in V(\Omega) \) one has

(i) \( \left\| \frac{dw}{ds} \right\|_{L^\infty(\Omega_*)} \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi} \),

(ii) \( \|w_+\|_{L^\infty(\Omega)} \leq \frac{\|\nabla w\|_{L^\infty(\Omega)}}{4\pi} \),

(iii) \( \left| \frac{d}{ds} w_+ (|w_+ > w_+(x)|) \right| \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi} \) a.e. \( x \in \Omega \).

**Proof.** Let \( w \in V(\Omega) \). Then

\[
\int_{\Omega} \Delta w (w_+ - t)_+ dx = \int_{\{w_+ > t\}} \Delta w (w_+ - t) dx \quad \text{for all } t > 0 \quad (2.1)
\]

since \( (w_+ - t)_+ \in H^1_0(\Omega) \), integrating by parts, we have

\[
\int_{\Omega} \Delta w (w_+ - t)_+ dx = -\int_{\{w_+ > t\}} |\nabla w_+|^2 dx. \quad (2.2)
\]

By classical arguments (see, for instance [13]), we have

\[
\frac{d}{dt} \int_{\Omega} \Delta w (w_+ - t)_+ dx = -\int_{\{w_+ > t\}} \Delta w dx. \quad (2.3)
\]

Combining (2.1), (2.2) and (2.3), one has

\[
-\frac{d}{dt} \int_{\{w_+ > t\}} |\nabla w_+|^2 dx = -\int_{\{w_+ > t\}} \Delta w dx \leq \|\Delta w\|_{L^\infty(\Omega)} |w_+ > t|. \quad (2.4)
\]
Arguing as in Talenti [14] and using the De Giorgi isoperimetric inequality, we get from (2.4)

\[
4\pi |w_+ > t| \leq \left( -\frac{d}{dt} |w_+ > t| \right) \left( -\frac{d}{dt} \int_{\{w_+ > t\}} |\nabla w_+|^2 \, dx \right)
\leq \left( -\frac{d}{dt} |w_+ > t| \right) |w_+ > t| \|\Delta w\|_{L^\infty(\Omega)}
\]

or a.e. \( t \in \inf_{\Omega} w_+, \sup_{\Omega} w_+ \). Thus \( 4\pi \leq (-\frac{d}{dt} |w_+ > t|) \|\Delta w\|_{L^\infty(\Omega)} \). Now, by standard arguments (see [13, 14]) we obtain (i), i.e.

\[
-\frac{d}{ds} w_{+, s}(s) \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi} \quad \text{a.e. in } \Omega_+.
\] (2.6)

ince we know already that \( w_{+, s} \in H^1_{\text{loc}}(\Omega_+) \), (2.6) infers that \( w_{+, s} \in W^{1, \infty}(\Omega_+) \) and an integration leads to \( \|w_+\|_{L^\infty(\Omega)} = w_{+, s}(0) \leq \frac{\|\Omega\|}{4\pi} \|\Delta w\|_{L^\infty(\Omega)} \) (since \( w_{+, s}(|\Omega|) = 0 \), i.e. (ii). From the equimeasurability, we obtain that

\[
\left[ \int_{\Omega \setminus P(w_+)} \left( \frac{d w_{+, s}}{ds} \right)^p \, dx \right]^{\frac{1}{p}} = \left[ \int_{\Omega \setminus P(w_+)} \left( \frac{d w_{+, s}}{ds} \right)^p \, ds \right]^{\frac{1}{p}}
\] (2.7)

or all \( p \in [1, +\infty[ \). Thus, letting \( p \rightarrow +\infty \), from (2.6), we deduce that

\[
-\frac{d^+ w_{+, s}}{ds} (|w_+ > w_+(x)|) \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi} \quad \text{a.e. in } \Omega \setminus P(w_+).
\]

esides, as \( w_{+, s} \) is right-continuous,

\[
-\frac{d^+ w_{+, s}}{ds} (|w_+ > w_+(x)|) = 0 \quad \text{for } x \in P(w_+)
\]

id (iii) holds. \( \Box \)

To pass to the limit in the iterative method that we shall use later we will need the wrong convergence of the first derivatives \( \frac{d}{ds} u_{n,s} \). To do that, we will use the notion of area regular function (see [18]):

**Definition 2.** Let \( u \in W^{1, 1}_{\text{loc}}(\Omega) \). For \( t \in \mathbb{R} \) we set \( m_{u, 0}(t) := |\{ x \in \Omega : u(x) > t \} \cap \partial u(x) = | \) and \( m_{u, 1}(t) := m_u(t) - m_{u, 0}(t) \). We will say that \( u \) is a coarea regular function if e Radon measure \( m_{u, 0}' \) is singular with respect to the Lebesgue measure on \( \mathbb{R} \).

Now, let us recall two conditions to get a coarea regular function obtained in [19] ee also [3])

**Lemma 5.** Let \( \Omega \) be an open set of \( \mathbb{R}^2 \) and \( u \in W^{2, p}_{\text{loc}}(\Omega) \) for some \( p > 1 \). Then \( u \) is coarea regular function. \( \Box \)

More in general, if \( \Omega \) is an arbitrary open set of \( \mathbb{R}^N \), we have a simpler statement.
Lemma 6. Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \). If \( \text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0 \) then \( u \) is a coarea regular function. \( \Box \)

The relation between the notion of coarea regular function and the convergence of the derivative sequence \( u_{n*} \) is obtained from an important result due to Almgren and Lieb [18]:

Lemma 7. Let \( u \) be a coarea regular function of \( W^{1,p}(\Omega) \), \( 1 \leq p \leq +\infty \) and let \( \Omega \) be a bounded regular open of \( \mathbb{R}^N \). If \( u_n \) is a bounded sequence of \( W^{1,p}(\Omega) \) converging to \( u \) in \( W^{1,1}(\Omega) \), then \( \frac{d}{ds} u_{n*} \) converges to \( \frac{d}{ds} u_* \) a.e. in \( \Omega_* \). Furthermore, if \( p \geq N \frac{d}{ds} u_n \) converges to \( \frac{d}{ds} u_* \) strongly in \( L^q(\Omega_*) \) for any \( 1 \leq q < \frac{p}{N-1} := \frac{1}{1+\frac{1}{p}-\frac{1}{N}} \). \( \Box \)

The proof of Lemma 7 has been presented in [19].

Now, we recall the notion of relative rearrangement of \( v \) with respect to \( u \). Let \( v \in L^1(\Omega) \), we define a function \( w \) in \( \tilde{\Omega}_* \) by:

\[
    w(s) = \begin{cases} 
        \int_{\{u > u_*(s)\}} v(x) \, dx & \text{if } |u = u_*(s)| = 0, \\
        \int_{\{u > u_*(s)\}} v(x) \, dx + \int_0^{s-|u_*(s)|} (v|_{P_n(u_*(s))})_*(\sigma) \, d\sigma & \text{if } |u = u_*(s)| \neq 0.
    \end{cases}
\]

Here \( v|_{P_n(u_*(s))} \) denotes the restriction of \( v \) to the set \( P_n(u_*(s)) \) and \( (v|_{P_n(u_*(s))})_* \) its decreasing rearrangement. The following lemma was proved in [11, 20].

Lemma 8. Let \( u \in L^1(\Omega) \) and \( v \in L^p(\Omega) \) for some \( 1 \leq p \leq +\infty \). Then \( w \in W^{1,p}(\Omega) \), and \( \frac{d}{ds} w \|_{L^p(\Omega_*)} \leq v \|_{L^p(\Omega)}. \) \( \Box \)

Definition 3. The function \( \frac{d}{ds} w \) is called the relative rearrangement of \( v \) with respect to the \( u \) and it is denoted by

\[
    v_{*u} = \frac{d}{ds}. \]

This function has many properties as we stated below (see, for instance [11, 15, 20, 21]):

Lemma 9. Consider \( u, v_1, v_2 \) three elements of \( L^1(\Omega) \). Then:

1. If \( v_1 \leq v_2 \) a.e. in \( \Omega \), then \( v_{1*} \leq v_{2*} \) a.e. in \( \Omega_* \).
2. If \( \varphi : \mathbb{R} \to \mathbb{R} \) is a Borel function such that \( \varphi(u) \in L^1(\Omega) \), then \((v_1 + \varphi(u))_{*u} = v_{1+*} \varphi(u_*)\). In particular, if we take \( \varphi \equiv k \) constant, then \((v_1 + k)_{*u} = v_{1+*} + k\).
3. The mapping \( v \to v_{*u} \) applies \( L^p(\Omega) \) into \( L^p(\Omega_*) \) for any \( 1 \leq p \leq +\infty \) and it is a contraction. In particular

\[
    v_{*u} \|_{L^p(\Omega_*)} \leq v \|_{L^p(\Omega)} \quad \text{for any } v \in L^p(\Omega) \text{ and } u \in L^1(\Omega). \quad \Box
\]

We shall be interested in expressing the function \([p'(u)b]_{*u}\) in terms of \( u_+ \) and \( b_{*u} \). To do that, we need the notion of mean operator introduced in [11].
Definition 4. (Second class of mean value operators). Let \( u, v \in L^1(\Omega) \), and \( g \in L^1(\Omega^*_e) \).

Let us define \( \mathcal{M}_{u,v}(g) \) as the function

\[
\mathcal{M}_{u,v}(g)(x) = \begin{cases} 
\frac{1}{|v|} \int_{|v| \geq v(x)} g(\sigma + |u > t_i|) \, d\sigma & \text{if } x \in \Omega - P(u) \\
\frac{1}{|v|_{e,\Omega}} \int_{|v|_{e,\Omega}} g(\sigma) \, d\sigma & \text{if } x \in P_{\Omega}(t_i)
\end{cases}
\]

i.e. \( x \in \Omega \) where \( v_i = v|_{P_{\Omega}(t_i)} \) the restriction of \( v \) to \( P_{\Omega}(t_i) \).

In particular, we will use the following lemmas (see [11, 19, 22]).

Lemma 10. The operator \( \mathcal{M}_{u,v}(g) \) is well defined and one has

i) if \( g \in L^1(\Omega^*_e) \), then \( \mathcal{M}_{u,v}(g) \in L^1(\Omega) \), besides

\[
\int_{\Omega^*_e} g(s) \, ds = \int_{\Omega} \mathcal{M}_{u,v}(g)(x) \, dx,
\]

ii) \( \mathcal{M}_{u,v} \) is a linear continuous map with norm one from \( L^p(\Omega^*_e) \) to \( L^p(\Omega) \) for any \( 1 \leq p \leq +\infty \).

We need also the following result which gives a relation between the relative rearrangement and the mean value operators:

Lemma 11. Let \( u \in L^1(\Omega) \) and \( v \in L^p(\Omega) \), \( 1 \leq p \leq +\infty \). For any \( g \in L^p(\Omega^*_e) \), \( \frac{1}{p} + \gamma = 1 \), we have

\[
\int_{\Omega^*_e} v_{\gamma u}(s) g(s) \, ds = \int_{\Omega} \mathcal{M}_{u,v}(g)(x) v(x) \, dx.
\]

i) if \( |P(u)| = 0 \) the last equality is reduced to

\[
\int_{\Omega^*_e} v_{\gamma u}(s) g(s) \, ds = \int_{\Omega} g(m_u(u(x))) v(s) \, dx.
\]

Using the last two lemmas, we have

Lemma 12 ([3]). Let \( u \in L^1(\Omega) \) and such that \( u_* \in \mathcal{C}(\Omega^*_e) \). Let \( F_0 : \mathbb{R} \to \mathbb{R} \) be a Borel function such that \( F_0(u) \in L^1(\Omega) \). Then, if \( b \in L^\infty(\Omega) \) we have

\[
[F_0(u)b]_{e,u} = F_0(u_*) b_{e,u}.
\]

Since we shall use later an approximate method, we shall need to know the behavior of the relative rearrangement of a fixed function \( b \) with respect to a sequence of actions \( u_n \) converging in \( L^1(\Omega) \). In that sense we have

Lemma 13 ([3]). Let \( u_n, u \) be in \( L^1(\Omega) \) and assume that \( u_n \) converges to \( u \) in \( L^1(\Omega) \). Then, for all \( v \in L^p(\Omega) \) (for a given \( p \), \( 1 < p \leq +\infty \)) we have

\[
(v_{\gamma \Omega \setminus P(u)})_{e,u} \to (v_{\gamma \Omega \setminus P(u)})_{e,u}
\]
weakly in $L^p(\Omega_+)$ if $p < +\infty$ and weakly* in $L^\infty(\Omega_+)$ if $p = +\infty$ (by $\chi_E$ we denote the characteristic function of the set $E$).

Under some additional regularity conditions it is possible to improve the above convergence:

**Lemma 14 ([23]).** Let $v \in L^p(\Omega)$ with $1 \leq p < +\infty$ and $u, u_n \in W^{1,\infty}(\Omega)$ such that:

\[ \text{meas}\{x \in \Omega : \nabla u(x) = 0\} = \text{meas}\{x \in \Omega : \nabla u_n(x) = 0\} = 0. \]

If $u_n$ converges to $u$ in $W^{1,q}(\Omega)$ for some $q > N$, then $v \ast u_n \rightharpoonup v \ast u$ strongly in $L^p(\Omega_+)$. Furthermore, $v \ast u_n(|u_n > u(\cdot)|)$ converges to $v \ast u(|u > u(\cdot)|)$ strongly in $L^p(\Omega)$. □

As a direct consequence of this lemma, we have the following result

**Lemma 15 ([23]).** Let $v \in L^p(\Omega)$ with $1 \leq p < +\infty$ and let $(\lambda_k, \varphi_k)^{+\infty}_{k=1}$ be the sequence of eigenvalues and eigenfunctions of $-\Delta$ on $\Omega$ with Dirichlet conditions; i.e. $-\Delta \varphi_k = \lambda_k \varphi_k$, $\varphi_k \in H^1_0(\Omega)$. Consider the finite dimensional vector space $V_m = \text{span}\{\varphi_1, \ldots, \varphi_m\}$ (the vector space spanned by $(\varphi_k)_{k=1}^m$). Then, the maps given by

\[ u \in V_m \setminus \{0\} \rightarrow v \ast u \in L^p(\Omega_+) \quad 1 \leq p < +\infty \]

and

\[ u \in V_m \setminus \{0\} \rightarrow v \ast u(|u > u(\cdot)|) \in L^p(\Omega) \quad 1 \leq p < +\infty \]

are strongly continuous over $L^p(\Omega_+)$ and $L^p(\Omega)$, respectively. □

As a complement of Lemma 7, we have the following useful result.

**Lemma 16.** Let $u_n, u \in W^{1,p}(\Omega)$, $1 \leq p \leq +\infty$ and $\Omega$ be a regular open subset of $\mathbb{R}^N$ such that

\[ \text{meas}\{x \in \Omega : \nabla u_n(x) = 0\} = 0 = \text{meas}\{x \in \Omega : \nabla u(x) = 0\}. \]

Then, if $u_n$ converges to $u$ in $W^{1,p}(\Omega)$ for some $p > N$;

\[ u' \ast u(|u_n > u(\cdot)|) \text{ converges strongly to } u' \ast u(|u > u(\cdot)|) \text{ in } L^q(\Omega) \]

for all $1 \leq q < \frac{N}{p} := \frac{1}{1 + \frac{1}{p} - \frac{1}{N}}$. □

The proof of this lemma is included in the proof of Lemma 14.

3. An equivalent formulation as a nonlocal problem

The main goal of this section is to show that if $u$ satisfies the family of conditions (1.12), i.e.

\[ \int_{\{u > t\}} \{F(u)F'(u) + p'(u)b\} \, dx = j(t_+, u_+(0)) \quad \forall t \in \left[\inf_\Omega u, \sup_\Omega u\right] \]

(3.1)
then it is possible to express $F$ in terms of $u$ decoupling in this way, the equation of such a family of conditions. This fact gives us the key to reformulate the problem (1.10), (1.11) and (1.12) like a nonlocal problem where we can eliminate the unknown function $F$. In this way, we reduce the original problem to the simpler problem $(\mathcal{P}_u)$

$$-\Delta u(x) = a(x)\mathcal{F}_u(x) + p'(u(x))[b(x) - b_{\ast u}(|u > u(x)|)]$$
$$+ j'_i(u_+(x), u_+(0))u'_+(|u > u(x)|) \quad \text{in } \Omega,$$

$$u - \gamma \in H^1_0(\Omega),$$

(3.2) (3.3)

where now $u$ is the unique unknown and function $\mathcal{F}_u$ is defined as follows:

$$\mathcal{F}_u(x) := \left[ F^2_v - 2 \int_{|u > u_+(x)|} [p(u_+)]' b_{\ast u}(s) \, ds ight. \left. + 2 \int_{|u > u_+(x)|} j'_i(u_+(s), u_+(0))u'_+(s)^2 \, ds \right]^{\frac{1}{2}}.$$  

(3.4)

In order to prove the equivalence between problems $(\mathcal{P})$ and $(\mathcal{P}_u)$, we will need some few lemmas. Their proofs are easy modifications of the corresponding lemmas in [3]. Here, we shall only show the significant differences. In all this section, we set $\hat{u} = \inf_{\Omega} u$ and $M = \sup_{\Omega} u$ which are justified since $u \in L^\infty(\Omega)$.

Given $u \in W^{1, \infty}(\Omega)$, we define the function $\mathcal{F}: \mathbb{R} \rightarrow \mathbb{R}_+$ by

$$\mathcal{F}(t) := \left[ F^2_v - 2 \int_0^{t_+} p'(s) b_{\ast u}(|u > s|) \, ds ight. \left. + 2 \int_0^{t_+} j'_i(s, u_+(0))u'_+(|u > s|) \, ds \right]^{\frac{1}{2}}.$$  

(3.5)

Then, we have

\textbf{Lemma 17.} Let $u \in W^{1, \infty}(\Omega)$ such that $\text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. Then $\mathcal{F}_u(x) = \mathcal{F}(u(x))$ for all $x \in \Omega$.

\textbf{Proof.} Similar to Proposition 1 of [3]. \qed

\textbf{Lemma 18.} Let $u \in V(\Omega)$ such that $\text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. Then

$$-\frac{d^+ u_+(|u > t|)}{ds} \leq \frac{||\Delta u||_{L^\infty(\Omega)}}{4\pi} \quad \text{for a.e. } t.$$  

(3.6)

\textbf{Proof.} It suffices to show (3.6) for $t \in [\inf_{\Omega} u, \sup_{\Omega} u]$. Let $p \in [1, +\infty[. \ Making \ the \ change \ of \ variables \ \tau = u_+(s)$ we have

$$\int_{\inf_{\Omega} u}^{\sup_{\Omega} u} |u_+'(|u > t|)|^p \, dt = \int_0^{\tau(\Omega)} |u_+'(s)|^{p+1} \, ds.$$
From Lemma 4, we then have for all $p \in [1, +\infty[$

$$
\left( \int_{\inf \Omega u}^{\sup \Omega u} |u_+^*(s) - |u > t| | d t \right)^{\frac{1}{p}} \leq \frac{\|\Delta u\|_{L^p(\Omega)}}{(4n)^{1 + \frac{1}{p}}}
$$

Letting $p \to +\infty$ we obtain (3.6). 

**Lemma 19.** Assume $u \in V(\Omega)$ such that $\mathrm{meas}\{x \in \Omega : \nabla (x) = 0\} = 0$ and $\min \{\mathcal{F}(t) : t \in [\tilde{m}, M]\} > 0$, $\tilde{m} := \inf \Omega u$, $M := \sup \Omega u$. Then (i) $\mathcal{F} \in W^{1, \infty}(\tilde{m}, M)$ and (ii) for almost every $t \in [\tilde{m}, M]$, we have

$$
\mathcal{F}(t)\mathcal{F}'(t) + p'(t)b_{su}(|u > t|) = j_1^*(t_+^*(u + u_0(0)))u_+^*(|u > t|).
$$

**Proof.** Since $b_{su} \in L^\infty(\Omega_+)$, $p \in C^1(\mathbb{R})$, then, the map

$$
\mathcal{F}(t)\mathcal{F}'(t) + p'(t)b_{su}(|u > \sigma|) d \sigma
$$

is in $W^{1, \infty}(\tilde{m}, M)$. From the assumptions on $j_1^*$ and Lemma 18, we deduce that the map

$$
\mathcal{F}(t)\mathcal{F}'(t) + p'(t)b_{su}(|u > \sigma|) d \sigma
$$

belongs to $W^{1, \infty}(\tilde{m}, M)$. Finally, proceeding as in [3, Lemma 12], we obtain the conclusion. 

**Lemma 20.** Let $u \in V(\Omega)$ such that $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$ and let $\mathcal{F}$ given by (3.5). Assume that $\min \{\mathcal{F}(t), t \in [\tilde{m}, M]\} > 0$. Then, for all $t \in [\tilde{m}, M]$,

$$
\int_{\{u > t\}}\{\mathcal{F}(u)\mathcal{F}'(u) + p'(u)b\} dx = j_1^*(u_+^*(u + u_0(0))).
$$

**Proof.** By Lemma 19, if we set $N = \{t \in [\tilde{m}, M] : \mathcal{F}(t)\mathcal{F}'(t) + p'(t)b_{su}(|u > t|) \neq j_1^*(u_+^*(u + u_0(0)))u_+^*(|u > t|)\}$, it has zero measure. Then $\{x \in \tilde{\Omega} : u(x) \in N\}$ and $\{s \in \tilde{\Omega}_s : u_s(s) \in N\}$ have zero measure (use equimeasurability, that $\{s \in \tilde{\Omega}_s : u_s(s) \in N\} \subset m_u(N)$ and that $m_u$ is absolutely continuous). Therefore,

$$
\mathcal{F}(u(x))\mathcal{F}'(u(x)) + p'(u(x))b_{su}(|u > u(x)|) = j_1^*(u_+^*(u + u_0(0)))u_+^*(|u > u(x)|)
$$

a.e. $x \in \tilde{\Omega}$. Integrating on $\{u > t\}$, one has

$$
\int_{\{u > t\}}\{\mathcal{F}(u(x))\mathcal{F}'(u(x)) + p'(u(x))b_{su}(|u > u(x)|)\} dx
$$

$$
= \int_{\{u > t\}} j_1^*(u_+^*(u + u_0(0)))u_+^*(|u > u(x)|) dx. \quad (3.7)
$$
By equimeasurability

\[
\int_{\{u > t\}} j_1'(u, u_{+*}(0)) u_{+*}'(u > u(x)) \, dx = \int_{\{u > t\}} j_1'(u, u_{+*}(s)) u_{+*}'(s) \, ds.
\]

If $t \geq 0$, one has $u_{+*}'(s) = u'(s)$ for almost every $s \in \{u > t\}$ since $\operatorname{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$ and $u \in V(\Omega)$. By the change of variable $\theta = u_{+*}(s)$

\[
\int_0^{u_{+*}(0)} j_1'(u_{+*}(s), u_{+*}(0)) u_{+*}'(s) \, ds = - \int_t^{\sup u_{+*}} j_1'(\theta, u_{+*}(0)) \, d\theta
\]

\[
= -j(0, u_{+*}(0)) - j(t, u_{+*}(0)). \quad (3.8)
\]

Moreover $j(0, u_{+*}(0)) = 0$ by Assumption (1.7). Thus, from (3.7) and (3.8), for all $t \geq 0$

\[
\int_{\{u > t\}} [F(u)F'(u) + p'(u)b_{+u}(u > u(x))] \, dx = j(t, u_{+*}(0)).
\]

Let $t < 0$. From relation (3.7)

\[
\int_{\{u > t\}} [F(u)F'(u) + p'(u)b_{+u}(u > u(x))] \, dx
\]

\[
= \int_{\{u \geq 0\}} j_1'(u, u_{+*}(0)) u_{+*}'(u > u(x)) \, dx
\]

\[
+ \int_{\{t < u < 0\}} j_1'(u, u_{+*}(0)) u_{+*}'(u > u(x)) \, dx \quad (3.9)
\]

notice that the last integral is zero). By the same change of variable as before

\[
\int_{\{u \geq 0\}} j_1'(u, u_{+*}(0)) u_{+*}'(u > u(x)) \, dx = - \int_0^{\sup u_{+*}} j_1'(\theta, u_{+*}(0)) \, d\theta
\]

\[
= j(0, u_{+*}(0)) - j(u_{+*}(0), u_{+*}(0))
\]

\[
= j(0, u_{+*}(0)). \quad (3.10)
\]

From (3.9) and (3.10), we have

\[
\int_{\{u > t\}} [F(u)F'(u) + p'(u)b_{+u}(u > u(x))] \, dx = j(0, u_{+*}(0)) = j(t, u_{+*}(0))
\]

for $t < 0$ and the conclusion holds. □

Lemma 20 shows that, given $u$, the function $F$ defined by (3.5) satisfies the original Stellarator condition (1.12), i.e. (3.1). Let us prove a result in the reciprocal way:

**Lemma 21.** Let $u \in V(\Omega)$ such that $\operatorname{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. If $F \in W^{1, \infty}([\tilde{m}, M])$ $s$ a function satisfying $F : [\tilde{m}, M] \rightarrow \mathbb{R}^+$, $F(t) = F_v$ for all $t \leq 0$ and for all $t \in [\tilde{m}, M]$
the relation (3.1) holds, then, necessarily \( F(t) = \mathcal{F}(t) \) for all \( t \in [\hat{m}, M] \) with \( \mathcal{F} \) given by (3.5).

**Proof.** We follow the same type of arguments as in [3]. From relation (3.1) we deduce that

\[
\int_{\{u > u_+(s)\}} \left\{ F(u) F'(u) + p'(u) b \right\} \, dx \quad \text{for all } s \in \Omega_+.
\]

Differentiating with respect to \( s \) this relation and using Lemma 12, for all \( s \in \Omega_+ \)

\[
j'_1(u_+(s), u_+(0)) u'_+(s) = F(u_+(s)) F'(u_+(s)) + p'(u_+(s)) b_*(s)
\]

since \( s = |u - u_+(s)| \) (because \( |P(u)| = 0 \)). The rest of the proof follows as in [3, Lemma 15]. \( \square \)

From the above lemmas, we can obtain the main result of this section:

**Theorem 2.** Let \( u \in V(\Omega) \) such that \( \text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0 \). Assume \( \hat{m} = \inf_{\Omega} u \leq 0 \) and \( \mathcal{F}_u(x) > 0 \) a.e. in \( \Omega \). Then, if \((u, F)\) is a solution of (\( \mathcal{P} \)) such that \( F : [\hat{m}, M] \to \mathbb{R}^+ \), \( F \in W^{1,\infty}([\hat{m}, M]) \) and \( F(t) = \mathcal{F}_F \) for all \( t \leq 0 \), then function \( u \) is also a solution of (\( \mathcal{P} \)) and necessarily \( F = \mathcal{F} \) with \( \mathcal{F} \) given by (3.5). Conversely, if \( u \) is a solution of (\( \mathcal{P} \)) and \( \mathcal{F} \) is given by (3.5) then, the couple \((u, \mathcal{F})\) is a solution of (\( \mathcal{P} \)) and \( \mathcal{F} \in W^{1,\infty}([\hat{m}, M]) \).

**Proof.** If \((u, F)\) is a solution of (\( \mathcal{P} \)), then thanks to the assumptions, \( F \) is defined necessarily by relation (3.5). Now, by Lemma 17, one has that \( \mathcal{F}_u(x) = F(u(x)) \) for all \( x \in \Omega \). The assumption \( \mathcal{F}_u(x) > 0 \) and the regularity \( u \in C(\bar{\Omega}) \) imply that \( \min\{F(t) : t \in [\hat{m}, M]\} > 0 \). Then, the conditions of Lemma 19 are fulfilled and so

\[
F(t) F'(t) + p'(t) b_{**}(|u > t|) = j'_1(t, u_+(0)) u'_+(|u > t|)
\]

for almost every \( t \in [\hat{m}, M] \). In particular, we deduce that \( u \) satisfies (3.2) and so \( u \) is a solution of (\( \mathcal{P} \)). Conversely, if \( u \) satisfies (\( \mathcal{P} \)) with \( \mathcal{F} \) given by (3.5), then \( \mathcal{F}(u(x)) = \mathcal{F}_u(x) > 0 \), \( x \in \bar{\Omega} \) by Lemma 17. Then Lemma 20 can be applied to get the relation (3.1) of (\( \mathcal{P} \)). Equation (1.10) is verified by \( u \) thanks to Lemma 19. \( \square \)

4. The approximate problem (\( \mathcal{P}_{\varepsilon} \))

Here and in what follows, we use the notation:

\[
I(v(x), \sigma) := \chi_{[|v| > v_+(x), |v| > 0]}(\sigma) \quad \sigma \in \Omega_+
\]

(the characteristics function of \([|v| > v_+(x), |v| > 0]\)),

\[
F_1(x, v, b_{**}) := -\int_{\Omega_+} I(v(x), s)[p(v_*)]'(s)b_{**}(s) \, ds,
\]

\[
F_{\varepsilon, 2}(x, v) := -\int_{\Omega_+} I(v(x), s) h_0(v_+(s)) j'_1(v_+(s), v_+(0)) \, ds,
\]

(4.1)

(4.2)

(4.3)
\[ F_\varepsilon(x, v, b_{**, v}) := \left[ F_2^2 - 2F_1(x, v, b_{**, v}) + 2F_{*2}(x, v) \right]^{1/2}, \]  
(4.4)

\[ H(v(x), b_{**, v}) := p'(v(x))[b(x) - b_{**, v}(v > v(x))], \]  
(4.5)

\[ J_\varepsilon(v(x)) := \xi[\varepsilon(v_{**}(v > v(x)))^{1/2}(v_{**}(v_{**}(x), v_{**}(0)))], \]  
(4.6)

Iways for a.e. \( x \in \Omega \) and for any function \( v \in H^1_0(\Omega) \). Here, we used the truncation functions

\[ h_\varepsilon(t) := \frac{t^2}{1 + \varepsilon t^2}, \quad \xi_\varepsilon(t) := \frac{t}{1 + \varepsilon |t|}. \]  
(4.7)

We shall adopt the notation \( F_2 := F_{0,2} \) and \( J := J_0 \) (for \( \varepsilon = 0 \)) concerning problem (3.2) and (3.3); that is

\[ F_2(x, v) := F_{0,2}(x, v) = \int_{[v > v_{**}(x)]} (v_{**}')^2(s)j''_{**}(v_{**}(s), v_{**}(0)) \, ds \]  
(4.8)

\[ J(v) := J_0(v) = v_{**}'(v > v_{**}(x))j''_{**}(v_{**}(x), v_{**}(0)). \]  
(4.9)

Now, let us consider the following approximate problem (\( \mathcal{P}_{\varepsilon} \)) for any fixed \( \varepsilon > 0 \); nd \( u_\varepsilon \) such that

\[ - \Delta u_\varepsilon = aF_\varepsilon(x, u_\varepsilon, b_{**, u_\varepsilon}) + H(u_\varepsilon, b_{**, u_\varepsilon}) + J_\varepsilon(u_\varepsilon) \quad \text{in} \ \Omega, \]  
(4.10)

\[ u_\varepsilon - \gamma \in H^1_0(\Omega) \cap W^{2,p}(\Omega); \ \forall p \geq 1. \]  
(4.11)

o simplify the boundary condition we define \( w_\varepsilon := u_\varepsilon - \gamma \). In order to prove the existence of \( w_\varepsilon \) we shall use a Galerkin method as in [24]. First, we shall find a solution \( w_{m\varepsilon} \) of some auxiliary problems (\( \mathcal{P}_{*, m} \)). We shall search \( w_{m\varepsilon} \in V_m \), where \( V_m \) : a finite dimensional space such that \( V_m \subset V_{m+1} \subset H^1_0(\Omega) \). Later, using appropriate estimates on the solutions \( w_{m\varepsilon} \) of (\( \mathcal{P}_{*, m} \)), we shall pass to the limit when \( m \) goes to infinity and so, we shall find a function \( w_\varepsilon \) such that \( w_\varepsilon + \gamma \) is a solution of (\( \mathcal{P}_{\varepsilon} \)).

.1. The Galerkin method. Existence of solution for a family of finite dimensional problems (\( \mathcal{P}_{*, \varepsilon} \))

Consider \( (\lambda_k, \varphi_k)_{k \geq 1} \) be the eigenvalues and eigenfunctions associated to \( -\Delta \) on \( \Omega \) with Dirichlet boundary conditions, i.e.

\[ -\Delta \varphi_k = \lambda_k \varphi_k, \ \varphi_k \in H^1_0(\Omega). \]  

et \( V_m = \text{span}\{\varphi_1, \ldots, \varphi_m\} \). On \( V_m \), we define the scalar product by \( [v, w] := \sum_{k=1}^m v^k w^k \) where \( v = \sum_{k=1}^m v^k \varphi_k \) and \( \omega = \sum_{k=1}^m w^k \varphi_k \). Let \( ||v||_{V_m} := [v, v]^{1/2} \) the associated norm.
Now, for $\gamma \leq 0$ fixed, we consider the operator $T^n_m : V_m \rightarrow V_m$ defined as

$$
[T^n_m v, \varphi] = \int_\Omega \nabla v \cdot \nabla \varphi \, dx - \int_\Omega aF_\ell(x, v + \gamma, b_{\ast(v+\gamma)}) \varphi \, dx - \int_\Omega H(v + \gamma, b_{\ast(v+\gamma)}) \varphi \, dx - \int_\Omega J_\ell(v + \gamma) \varphi \, dx \quad \forall v, \varphi \in V_m.
$$

(4.12)

We shall prove that this operator attains zero for some $w^n_m \in V_m \setminus \{0\}$. It is clear that $w^n_m$ satisfies $T^n_m w^n_m = 0$ in $V_m$ then $w^n_m$ satisfies the finite dimensional problem $(\mathcal{P}_{\ast, n}$ given by

$$
-\Delta(w^n_m + \gamma) = P_m[aF_\ell(x, w^n_m + \gamma, b_{\ast(w^n_m + \gamma)}) + H(w^n_m + \gamma, b_{\ast(w^n_m + \gamma)}) + J_\ell(w^n_m + \gamma)] \quad \text{in } \Omega,
$$

where $P_m$ is the orthogonal projection operator from $L^2(\Omega)$ onto $V_m$. To prove that $T^n_m$ has a zero in $V_m \setminus \{0\}$ we shall use Lemma 4.3 of [25]. We need to check that $T^n_m$ is a coercive and continuous map. We notice that $[T^n_m 0, \varphi] = F_\ell \int_\Omega a\varphi \, dx \neq 0$ for some $\varphi$ provided that $a \neq 0$.

Proposition 1. If

$$
\lambda_1 - \lambda_{\text{osc}} b > 0
$$

(4.13)

then

$$
[T^n_m v, v] \rightarrow +\infty \quad \text{as } ||v||_V \rightarrow \infty.
$$

In particular, $T^n_m$ is a coercive map.

Proof. Let $\gamma \leq 0$ fixed. We shall estimate

$$
[T^n_m v, v] = \int_\Omega \nabla v^2 \, dx - \int_\Omega aF_\ell(x, v + \gamma, b_{\ast(v+\gamma)}) v \, dx - \int_\Omega H(v + \gamma, b_{\ast(v+\gamma)}) v \, dx - \int_\Omega J_\ell(v + \gamma) v \, dx \quad \forall v \in V_m
$$

(4.14)

term by term. Let us start with $\int_\Omega H(v + \gamma, b_{\ast(v+\gamma)}) v \, dx$. Then,

$$
\left| \int_\Omega H(v + \gamma, b_{\ast(w^n_m + \gamma)}) v \, dx \right| \leq \lambda_{\text{osc}} b \int_\Omega v^2 \, dx
$$

(4.15)

where we used the assumptions $0 \leq p'(s) \leq \lambda s_+, \lambda > 0$ (see (1.8)) and that $\gamma \leq 0$. It is also clear that

$$
\left| \int_\Omega J_\ell(v + \gamma) v \, dx \right| \leq \frac{\alpha}{\beta} \int_\Omega |v| \, dx
$$

(4.16)
hanks to (1.7) and that function $\xi$ is bounded by $1/\varepsilon$. Let us estimate the term \( \int a \mathcal{F}_r(x,v + \gamma b_r(v + \gamma))v \, dx \). Dropping nonpositive terms and using the same argument as above,

$$
\int a \mathcal{F}_r(x,v + \gamma b_r(v + \gamma))v \, dx \leq \|a\|_{L^\infty(\Omega)} \left[ \frac{2}{\varepsilon} \|\mathcal{F}_r\|_{L^2(\Omega)} + \frac{2\eta}{\varepsilon} \right]^{1/2} \int |v| \, dx,
$$

(4.17)
in case $|\mathcal{F}_r| \leq 1/\varepsilon$. Using the above estimates, we have

$$
[T_m^e v, v] \geq \int \|\nabla v\|^2 \, dx - C_c \int |v| \, dx - \lambda \text{osc } b \int v^2 \, dx
$$

(4.18)
with $C_c = \|a\|_{L^\infty(\Omega)} \left[ \frac{2}{\varepsilon} \|\mathcal{F}_r\|_{L^2(\Omega)} + \frac{2\eta}{\varepsilon} \right]^{1/2}$ and for all $v \in V_m$. Applying Young's inequality $\int |v| \, dx$ we get

$$
\int |v| \, dx \leq \frac{K^2}{2} \int v^2 \, dx + \frac{\|\Omega\|}{2K^2} \quad \forall K > 0.
$$

Choosing $K^2 = \frac{\delta}{C_c}$ and $C_c \delta = \frac{C_c^2 \|\Omega\|}{4\delta}$, we obtain that

$$
[T_m^e v, v] \geq \lambda_1 \int v^2 \, dx - \delta \int v^2 \, dx - C_c \delta \int v^2 \, dx - \lambda \text{osc } b \int v^2 \, dx
$$

$$
= (\lambda_1 - \delta - \lambda \text{osc } b) \int v^2 \, dx - C_c \delta \quad \forall v \in V_m.
$$

(4.19)
from the Assumption (4.13) we obtain the coercivity of operator $T_m^e$ assumed we take small enough. □

**Proposition 2.** $T_m^e$ is a continuous map.

**Proof.** Since $(\varphi_k)_{k=1}^m$ is an orthogonal base of $(V_m, [\cdot, \cdot])$, $T_m^e v$ can be expressed in the following way

$$
T_m^e v = \sum_{k=1}^m [T_m^e v, \varphi_k] \varphi_k.
$$

(4.20)
o, the continuity of $T_m^e$ on $V_m$ is equivalent to the continuity of the application

$$
V_m \ni v \rightarrow [T_m^e v, \varphi]
$$

here $\varphi \in V_m$ is an arbitrary function. We shall prove the continuity of the different functions appearing in Definition (4.12) of $T_m^e$ once $\varphi$ is fixed. In the following, we shall take a sequence of functions $v_n \in V_m \setminus \{0\}$ and $v \in V_m$ such that

$$
v_n \rightarrow v \quad \text{in } V_m.
$$

We need a previous lemma.
Lemma 22. Let \( (v_n)_{n \geq 1} \) be a sequence of \( V_m \setminus \{0\} \) and let \( v \) be in \( V_m \setminus \{0\} \), such that
\[
v_n \rightharpoonup v \quad \text{in } V_m. \tag{4.21}
\]
Then one has
\[
v_n \rightharpoonup v \quad \text{strongly in } \mathcal{C}^k(\tilde{\Omega}) \forall k \in \mathbb{N} \cup \{0\}, \tag{4.22}
\]
\[
v_{n^*} \rightharpoonup v_* \quad \text{strongly in } \mathcal{C}(\tilde{\Omega}_*), \tag{4.23}
\]
\[
v'_{n^*} \rightharpoonup v'_* \quad \text{strongly in } L^q(\Omega_*) \forall 1 \leq q < \infty, \tag{4.24}
\]
\[
v'_{n^*}(\{v_n > v_n(\cdot)\}) \rightharpoonup v'_*(\{v > v(\cdot)\}) \quad \text{strongly in } L^q(\Omega) \forall 1 \leq q < \infty. \tag{4.25}
\]

Proof. On \( V_m \) all the norms are equivalent and since \( V_m \subset \mathcal{C}^\infty(\tilde{\Omega}) \), we have (4.22). As \( \Omega \) is connected and \( v_n, v \in \mathcal{C}^0(\tilde{\Omega}) \), then \( v_{n^*} \) and \( v_* \) are continuous in \( \tilde{\Omega}_* \). From the contraction property and (4.22) we get (4.23). On the other hand, since any element of \( V_m \setminus \{0\} \) is coarea regular (notice that it is analytic in \( \Omega \)), one has, from Lemma 7,
\[
\frac{dv_{n^*}}{ds} \rightarrow \frac{dv_*}{ds} \quad \text{in } L^q(\Omega_*), \quad \forall q \in \lbrack 1, 2 \rbrack. \tag{4.26}
\]
So,
\[
\frac{dv_{n^*}}{ds} \rightarrow \frac{dv_*}{ds} \quad \text{in } L^q(\Omega_*), \quad \forall q \in \lbrack 1, 2 \rbrack. \tag{4.27}
\]
Moreover, as the functions \( v_n \) are such that \( v_{n\Omega} = 0 \), they belong to \( V(\Omega) \). By (4.22) \( v \) is also in \( V(\Omega) \) and we know that \( |\Delta(v_n - v)|_{\mathcal{C}(\tilde{\Omega})} \rightarrow 0 \). Thus, from Lemma 4, we have
\[
\left| \frac{dv_{n^*}}{ds} \right|_{L^\infty(\Omega_*)} \leq \frac{\|\Delta v_n\|_{L^\infty(\Omega)}}{4\pi} \leq \text{constant independent of } n. \tag{4.28}
\]
We easily conclude from this estimate and (4.27) that
\[
\frac{dv_{n^*}}{ds} \rightarrow \frac{dv_*}{ds} \quad \text{in } L^p(\Omega_*), \quad \forall p \in \lbrack 1, +\infty \rbrack.
\]
Finally, the last statement comes from Lemma 16 and (iii) of Lemma 4. \( \square \)

Proof of Proposition 2. Continuity of the map
\[
V_m \ni v \rightarrow \int_\Omega H(v + \gamma, b_*(w_n + \gamma)) \varphi \, dx. \tag{4.29}
\]
It is equivalent to prove that the following equality holds
\[
\lim_{n \rightarrow \infty} \int_\Omega \varphi(x)p'((v_n + \gamma)(x))[b(x) - b_*(w_n + \gamma)((v_n + \gamma)(x))] \, dx
\]
\[
= \int_\Omega \varphi(x)p'((v + \gamma)(x))[b(x) - b_*(v + \gamma)((v + \gamma)(x))] \, dx \tag{4.30}
\]
for all $\varphi \in V_m$. We distinguish the case $v \neq 0$ from the case $v \equiv 0$. Since any element of $V_m \setminus \{0\}$ is an analytical function (recall that $\varphi_k$ are analytical functions), then $\text{meas}\{x \in \Omega : \nabla v(x) = 0\} = 0 = \text{meas}\{x \in \Omega : \nabla v_n(x) = 0\}$ and by Lemma 14, we get

$$b_{e(v_n+\gamma)}(v_n+\gamma > (v_n+\gamma)(\cdot)) \xrightarrow{n \to \infty} b_{e(v+\gamma)}(v + \gamma > (v + \gamma)(\cdot)) \text{ strongly in } L^q(\Omega) \tag{4.31}$$

for all $q \geq 1$ since $b$ is in $L^\infty(\Omega)$. By Assumption (1.8) we obtain, from the convergence Lemma 22, that

$$p'(v_n + \gamma) \xrightarrow{n \to \infty} p'(v + \gamma) \text{ strongly in } L^r(\Omega) \tag{4.32}$$

for all $r \geq 1$. Now, by the convergences (4.31) and (4.32) we obtain (4.30) when $v \neq 0$. Finally, if $v \equiv 0$, the $L^r(\Omega)$-strong convergence of $p'(v_n + \gamma)$ to $0$, the uniform boundedness in $L^\infty(\Omega)$ of $b_{e(v_n+\gamma)}(v_n+\gamma > (v_n+\gamma)(\cdot))$ and the weak convergence in $L^p(\Omega)$ of $b_{e(v_n+\gamma)}(v_n+\gamma > (v_n+\gamma)(\cdot))$ to $b_{e(v+\gamma)}(v + \gamma > (v + \gamma)(\cdot))$ (see Lemma 13) imply (4.30). Thus, the map defined by (4.29) is continuous.

**Continuity of the map**

$$V_m \ni v \mapsto \int_\Omega J_k(v + \gamma)\varphi \, dx. \tag{4.33}$$

Following the same steps as before we verify that

$$\lim_{n \to \infty} \int_\Omega \xi_\epsilon((v_n + \gamma)_+^r,(v_n + \gamma > (v_n + \gamma)(\cdot))) j'_i((v_n + \gamma)_+(x),(v_n + \gamma)_+(0))\varphi \, dx$$

$$= \int_\Omega \xi_\epsilon((v + \gamma)_+^r,(v + \gamma > (v + \gamma)(\cdot))) j'_i((v + \gamma)_+(x),(v + \gamma)_+(0))\varphi \, dx \tag{4.34}$$

for all $\varphi \in V_m$. Since by Lemma 22

$$v_n + \gamma \xrightarrow{n \to \infty} v + \gamma \text{ strongly in } C^k(\bar{\Omega}) \quad \forall k \in \mathbb{N} \cup \{0\},$$

we have

$$(v_n + \gamma)_+(x) \xrightarrow{n \to \infty} (v + \gamma)_+(x) \quad \forall x \in \Omega$$

and

$$(v_n + \gamma)_+(0) = \|(v_n + \gamma)_+\|_{L^\infty(\Omega)} \xrightarrow{n \to \infty} \|(v + \gamma)_+\|_{L^\infty(\Omega)} = (v + \gamma)_+(0).$$

The continuity of $j'_i$ in $\mathbb{R}^+ \times \mathbb{R}^+$ and the above convergences give

$$j'_i((v_n + \gamma)_+(x),(v_n + \gamma)_+(0)) \xrightarrow{n \to \infty} j'_i((v + \gamma)_+(x),(v + \gamma)_+(0)) \quad \forall x \in \Omega.$$ 

Applying again Lemma 22, the Lipschitz continuity and the boundedness of function $\xi_\epsilon$, one has

$$\xi_\epsilon((v_n + \gamma)_+^r,(v_n + \gamma > (v_n + \gamma)(\cdot))) \xrightarrow{n \to \infty} \xi_\epsilon((v + \gamma)_+^r,(v + \gamma > (v + \gamma)(\cdot)))$$
strongly in $L^q(\Omega)$ with $1 \leq q < \infty$. The above convergences and assumption (1.7), lead to the conclusion.

Continuity of the map

$$V_m \ni v \mapsto \int_\Omega aF_\varepsilon(x, v + \gamma, b_*(v + \gamma)) \varphi \, dx.$$  \quad (4.35)

As before, the first integral which appears in the definition of $F_\varepsilon$ (see (4.4)) converges to

$$2 \int_{|v + \gamma > (v + \gamma)_+(x)|} \left[ \rho((v + \gamma)_+(s)) b_*(v + \gamma)_+(s) \right] ds$$

from the strong convergence of Lemmas 22, 1, 14 and assumption (1.8). The convergence of the second integral to

$$2 \int_{|v + \gamma > (v + \gamma)_+(x)|} f_\varepsilon^i((v + \gamma)_+(s), (v + \gamma)_+(0)) b_\varepsilon((v + \gamma)_+(s)) ds$$

is obtained in the same way by using the continuity of $f_\varepsilon^i$.

The above steps show the continuity of the map $T^\varepsilon_m$. \square

**Theorem 3.** Assume (4.13). Then there exists at least $\omega_\varepsilon \in V_m$ solution of problem $(\mathcal{P}_{\varepsilon, m})$, i.e. satisfying $\forall \varphi \in V_m$

$$[T^\varepsilon_m w_\varepsilon, \varphi] = \int_\Omega \nabla w_\varepsilon \cdot \nabla \varphi \, dx - \int_\Omega aF_\varepsilon(x, w_\varepsilon + \gamma, b_*(w_\varepsilon + \gamma)) \varphi \, dx$$

$$- \int_\Omega H(w_\varepsilon + \gamma, b_*(w_\varepsilon + \gamma)) \varphi \, dx - \int_\Omega J_\varepsilon(w_\varepsilon + \gamma) \varphi \, dx = 0.$$  \quad (4.36)

**Proof.** From the coercivity of $T^\varepsilon_m$ and the continuity of $T^\varepsilon_m$ given by Propositions 1 and 2, we can apply the Brouwer Fixed Point Theorem (see e.g. [23, Lemma 4.3, p. 55]). \square

4.2. *A priori estimates on solutions of $(\mathcal{P}_{\varepsilon, m})$*

In this section, $w_\varepsilon$ is assumed to be a solution of $(\mathcal{P}_{\varepsilon, m})$. Taking $\varphi = w_\varepsilon$, from (4.18) and (4.36), we have

$$0 = [T^\varepsilon_m w_\varepsilon, w_\varepsilon] \geq \left( \lambda_1 - \lambda \text{osc } b - \delta \right) \int_\Omega |w_\varepsilon|^2 \, dx - C_{\lambda \delta}.$$  \quad (4.37)

Thus, choosing $\delta$ small enough, one has that

$$\|w_\varepsilon\|_{L^2(\Omega)} \leq C_{\varepsilon}$$

and from (4.17), we deduce

$$\int_\Omega |\nabla w_\varepsilon|^2 \, dx \leq C_{\varepsilon}$$  \quad (4.38)
for some positive constant $C_{e}$ only depending of $e$. From estimate (4.37) and using the same arguments used to obtain estimates (4.15), (4.16) and (4.17), we get

$$
\|\Delta w_{m}^{e}\|_{L^{2}(\Omega)} \leq \|aF_{e}(x, w_{m}^{e} + \gamma, b_{e}(w_{m}^{e} + \gamma))\|_{L^{2}(\Omega)} + \|H(w_{m}^{e} + \gamma, b_{e}(w_{m}^{e} + \gamma))\|_{L^{2}(\Omega)}
$$

$$
+ \|J_{e}(w_{m}^{e} + \gamma)\|_{L^{2}(\Omega)}
$$

$$
\leq \|a\|_{L^{\infty}(\Omega)} \left[ F_{e}^{2} + \frac{2\eta}{\epsilon} |\Omega| \right]^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \lambda \text{osc } b \|w_{m}^{e}\|_{L^{2}(\Omega)} + \frac{|\Omega|^{\frac{1}{2}}}{\epsilon} \leq C_{e}
$$

(4.39)

where $C_{e}$ also denotes a positive constant only depending on $e$. Thus, by standard regularity results, $(w_{m}^{e})_{m \geq 1}$ is uniformly bounded in $W^{2,2}(\Omega)$ with respect to $m$.

4.3. Passing to the limit $m \to \infty$: Existence of solution of $(S_{e})$

By the above estimates, there exists a subsequence of $\{w_{m}^{e}\}$, which we also denote by $\{w_{m}^{e}\}$, and $w^{e} \in W^{2,2}(\Omega)$ such that

$$
w_{m}^{e} \rightharpoonup w^{e} \quad \text{weakly in } W^{2,2}(\Omega),
$$

and so,

$$
w_{m}^{e} \longrightarrow w^{e} \quad \text{strongly in } W^{1,p}(\Omega), \quad \forall p \in [1, \infty[, \quad \text{and in } \mathscr{C}^{0}(\overline{\Omega}).
$$

Our next step is to verify that $T^{e}w^{e} = 0$. Here $T^{e}: H_{0}^{1}(\Omega) \to H_{0}^{1}(\Omega)$ is the operator defined by

$$
[T^{e} v, \phi] = \int_{\Omega} \nabla v \cdot \nabla \phi \, dx - \int_{\Omega} aF_{e}(x, v + \gamma, b_{e}(v + \gamma)) \phi \, dx
$$

$$
- \int_{\Omega} H(v + \gamma, b_{e}(v + \gamma)) \phi \, dx - \int_{\Omega} J_{e}(v + \gamma) \phi \, dx \tag{4.40}
$$

if $v, \phi \in H_{0}^{1}(\Omega)$. As before we have

$$
\|aF_{e}(x, w_{m}^{e} + \gamma, b_{e}(w_{m}^{e} + \gamma))\|_{L^{\infty}(\Omega)} \leq C_{e} \quad \forall m. \tag{4.41}
$$

Let us show that

$$
F_{1}(x, w_{m}^{e} + \gamma, b_{e}(w_{m}^{e} + \gamma)) \rightharpoonup F_{1}(x, w^{e} + \gamma, \tilde{b}^{e}) \quad \text{a.e. in } \Omega
$$

where

$$
b_{e}(w_{m}^{e} + \gamma) \rightharpoonup \tilde{b}^{e} \quad \text{weakly-star in } L^{\infty}(\Omega_{+})
$$

due to the uniform boundedness $\|b_{e}(w_{m}^{e} + \gamma)\|_{L^{\infty}(\Omega_{+})} \leq \|b\|_{L^{\infty}(\Omega)}$. Indeed, as in [3], one has

$$
\lim_{m \to +\infty} I(w_{m}^{e}(x) + \gamma, \sigma)[p((w_{m}^{e} + \gamma)_{+})]'(\sigma) = I(w^{e}(x) + \gamma, \sigma)[p((w^{e} + \gamma)_{+})]'(\sigma)
$$
a.e. in $\Omega_\ast$. Thus, by the strong convergence of $[p((w^e_m + \gamma)_\ast)]'$ to $[p((w^e + \gamma)_\ast)]'$ in $L^1(\Omega_\ast)$ thanks to Lemma 7 and the continuity of $p'$, we deduce that

$$I(w^e_m(x) + \gamma, \cdot)[p((w^e_m + \gamma)_\ast)]'(\cdot) \xrightarrow{m \to +\infty} I(w^e(x) + \gamma, \cdot)[p((w^e + \gamma)_\ast)]'(\cdot)$$

strongly in $L^1(\Omega_\ast)$ and since $b_\ast(w^e_m + \gamma) \xrightarrow{m \to +\infty} \tilde{b}^e$ we then obtain

$$F_1(x, w^e_m + \gamma, b_\ast(w^e_m)) \xrightarrow{m \to +\infty} F_1(x, w^e + \gamma, \tilde{b}^e) \quad \text{a.e. in } \Omega.$$  

By the same argument applied to $F_{n,2}(x, w^e_m)$, noting that, first,

$$\lim_{m \to +\infty} I(w^e_m(x) + \gamma, \cdot)h_\ast((w^e_m + \gamma)_\ast)(\cdot)) = I(w^e(x) + \gamma, \cdot)h_\ast((w^e + \gamma)_\ast)(\cdot)$$

a.e. in $\Omega_\ast$ and later, strongly in $L^1(\Omega_\ast)$, we obtain that

$$F_{n,2}(x, w^e_m + \gamma) \xrightarrow{m \to +\infty} F_{n,2}(x, w^e + \gamma) \quad \text{a.e. in } \Omega.$$  

Thus, we have (for some subsequence) that

$$F_\varepsilon(x, w^e_m + \gamma, b_\ast(w^e_m + \gamma)) \xrightarrow{m \to +\infty} F_\varepsilon(x, w^e + \gamma, \tilde{b}^e) \quad \text{weakly*- in } L^\infty(\Omega). \quad (4.42)$$

Analogously, one has that $\|b_\ast(w^e_m + \gamma)(w^e_m + \gamma) \|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\Omega)}$ and thus, for some subsequence, there exists a function $\tilde{b}^e \in L^\infty(\Omega)$ such that

$$b_\ast(w^e_m + \gamma)(w^e_m + \gamma) \xrightarrow{m \to +\infty} \tilde{b}^e$$

weakly-star in $L^\infty(\Omega).$

As before, $p'((w^e_m + \gamma)(\cdot))$ converges to $p'((w^e + \gamma)(\cdot))$ strongly in $L^1(\Omega)$ and thus,

$$H(x, w^e_m + \gamma, b_\ast(w^e_m + \gamma)(w^e_m + \gamma) \xrightarrow{m \to +\infty} H(x, w^e + \gamma, \tilde{b}^e(x))$$

weakly* in $L^\infty(\Omega).$

Finally, since $w^e_m$ converges in $L^1(\Omega_\ast)$ to $w^e_\ast$ (see Lemma 22) we have

$$J_\varepsilon(w^e_m + \gamma) \to J_\varepsilon(w^e + \gamma) \quad \text{strongly in } L^1(\Omega), m \to \infty.$$  

Then, $w^e$ verifies the weak formulation of the following problem

$$-\Delta w^e = aF_\varepsilon(x, w^e + \gamma, \tilde{b}^e) + p'(w^e + \gamma)[b - \tilde{b}_\varepsilon] + J_\varepsilon(w^e + \gamma) \quad \text{in } \Omega,$$

$$w^e \in H^1_0(\Omega) \cap W^{2,2}(\Omega)$$

(4.44)

for any $\varepsilon > 0$. To obtain a solution of $(\mathcal{P}_e)$, we only need to identify $F_\varepsilon(x, w^e + \gamma, \tilde{b}^e)$ as $F_\varepsilon(x, w^e + \gamma, b_\ast(w^e + \gamma))$ and $\tilde{b}^e$ as $b_\ast(w^e + \gamma)(|w^e + \gamma > (w^e + \gamma)(x))$. In this sense we have

**Proposition 3.** If $\max\{x \in \Omega : \nabla w^e(x) = 0\} = 0$ then $\tilde{b}^e = b_\ast(w^e + \gamma)$ in $\Omega_\ast$ and $\tilde{b}^e = b_\ast(w^e + \gamma)(|w^e + \gamma > (w^e + \gamma)(x))$. In particular, $w^e$ is a solution of $(\mathcal{P}_e)$.  

**Proof.** Use the analyticity of $w^e_m$ and Lemma 14. □
Next, we want to obtain a sufficient condition on the data in order to have the property meas \( \{ x \in \Omega : \nabla w^\epsilon(x) = 0 \} = 0 \). After that, we will let \( \epsilon \to 0 \) getting a solution \( u \) of (\( \mathcal{P}_\ast \)).

5. Condition on the data in order to get meas \( \{ x \in \Omega : \nabla w^\epsilon(x) = 0 \} = 0 \) and the existence of solution of (\( \mathcal{P}_\ast \))

From the above section, we already know the existence of \( w^\epsilon \in W^{2,2}(\Omega) \cap H_0^1(\Omega) \) verifying (4.43) and (4.44) for \( \gamma \) fixed. Then, setting \( u^\epsilon := w^\epsilon + \gamma \), function \( u^\epsilon \) verifies that

\[
-\Delta u^\epsilon = a[F_v^2 - 2F_1(x, u^\epsilon, \tilde{b}^\epsilon) + 2F_{\epsilon,2}(x, u^\epsilon)] \frac{1}{2} + H(u^\epsilon, \tilde{b}^\epsilon) + J_{\epsilon}(u^\epsilon) \quad \text{in } \Omega, \tag{5.1}
\]

\[
u^\epsilon = \gamma \in H_0^1(\Omega) \cap W^{2,2}(\Omega). \tag{5.2}
\]

We recall that \( F_1, F_{\epsilon,2}, H \) and \( J_{\epsilon} \) were defined in (4.2), (4.3), (4.5) and (4.6), and the truncation functions \( h_{\epsilon} \) and \( \tilde{z}_\epsilon \) were defined in (4.7).

Let us start by giving a condition on the data in order to have meas \( \{ x \in \Omega : \nabla w^\epsilon(x) = 0 \} = 0 \). We need the following technical result:

**Lemma 23.** Let \( \{ u^\epsilon \} \) verifying (5.1) and such that \( u^\epsilon - \gamma \in W_0^{1,\infty}(\Omega) \cap W^{2,2}(\Omega) \). If

\[
\nu := \frac{1}{4\pi} \left[ 2^{1/2} \eta \sqrt{\frac{1}{2}} \| a \|_{L^\infty(\Omega)} + \lambda |\Omega| \, \text{osc}_\Omega b + \eta \right] < 1 \tag{5.3}
\]

then

\[
\| \Delta u^\epsilon \|_{L^\infty(\Omega)} \leq \frac{\| a \|_{L^\infty(\Omega)} F_v}{1 - \nu}. \tag{5.4}
\]

In particular

\[
\| u^\epsilon_+ \|_{L^\infty(\Omega)} \leq \frac{\| a \|_{L^\infty(\Omega)} F_v |\Omega|}{4\pi(1 - \nu)} := S \tag{5.5}
\]

uniformly in \( \epsilon \).

**Proof.** We need some a priori estimates. Let \( \epsilon > 0 \) and let \( u^\epsilon \) be any solution of (\( \mathcal{P}_{\epsilon,v} \)). The function

\[
F_\epsilon(x, u^\epsilon, \tilde{b}^\epsilon) = [F_v^2 - 2F_1 + 2F_{\epsilon,2}] \frac{1}{2} \tag{5.6}
\]

with \( F_1 \) and \( F_{\epsilon,2} \) defined by (4.2) and (4.3) is bounded in \( \Omega \) because the integral \( F_1 \) is positive and \( F_{\epsilon,2} \) is bounded since \( \| \tilde{b}^\epsilon \| \leq 1/\epsilon \) and \( \| f \| \leq \eta \) (see (1.7)). In the same way, the term \( J_{\epsilon}(u^\epsilon) \), defined by (4.6), is bounded in \( \Omega \) and finally \( H(u^\epsilon, \tilde{b}^\epsilon) \), given by (4.5), is majorated by \( \lambda \| u^\epsilon_+ \|_{L^\infty(\Omega)} \, \text{osc}_\Omega b \). So, \( \Delta u^\epsilon \in L^\infty(\Omega) \) for any \( \epsilon \). Now, applying Lemma 4, we have that

\[
0 \leq -u^\epsilon_+ (s) \leq \frac{\| \Delta u^\epsilon \|_{L^\infty(\Omega)}}{4\pi}. \tag{5.6}
\]
Thus
\[\|u^e\|_{L^\infty(\Omega)} \leq \frac{\|\Omega\|}{4\pi} \|\Delta u^e\|_{L^\infty(\Omega)}. \tag{5.7}\]

Our following task is to prove that \(\Delta u^e\) is bounded in \(L^\infty(\Omega)\) uniformly with respect to the parameter \(e\). As before, using (5.7), we have the following estimates:
\[0 \leq F_i(x, u^e, \tilde{b}^e) \leq \lambda \|b\|_{L^\infty(\Omega)} \|u^e\|_{L^\infty(\Omega)} \leq \frac{\|\Omega\|}{4\pi} \|\tilde{b}\|_{L^\infty(\Omega)} \|\Delta u^e\|_{L^\infty(\Omega)}, \tag{5.8}\]
\[|F_{i,2}(x, u^e)| \leq \eta \|\Omega\| \frac{\|\Delta u^e\|^2_{L^\infty(\Omega)}}{(4\pi)^2}, \tag{5.9}\]
\[|H(u^e, \tilde{b}^e)| \leq \lambda \operatorname{osc} b \|u^e\|_{L^\infty(\Omega)} \leq \lambda \operatorname{osc} b \frac{\|\Omega\|}{4\pi} \|\Delta u^e\|_{L^\infty(\Omega)}, \tag{5.10}\]
\[|J_e(u^e)| \leq \frac{\eta}{4\pi} \|\Delta u^e\|_{L^\infty(\Omega)}. \tag{5.11}\]

Then
\[\|\Delta u^e\|_{L^\infty(\Omega)} \leq \|a\|_{L^\infty(\Omega)} F_e + \frac{1}{4\pi} \left[2^{\frac{3}{2}} \eta \|\Omega\|^{\frac{1}{2}} \|a\|_{L^\infty(\Omega)} + \lambda \|\Omega\| \operatorname{osc} b + \eta \right] \|\Delta u^e\|_{L^\infty(\Omega)}.\]

We obtain the conclusion from assumption (5.3) and the above estimate. \(\square\)

In order to get a solution of problem \((\mathcal{P}_{*,e})\), we need to verify the assumption \(\operatorname{meas}\{x \in \Omega : \nabla u^e(x) = 0\} = 0\). Indeed, by Proposition 3 we would have that
\[\tilde{b}^e = b_{*e} \quad \text{in } L^q(\Omega_*),\]
and
\[\tilde{b}^e(x) = b_{*e}(|u^e > u^e(x)|) \quad \text{in } L^\infty(\Omega)\]
and so \(u^e\) would satisfy \((\mathcal{P}_{*,e})\) thanks to (5.1). The following theorem gives a sufficient condition for this property.

**Theorem 4.** If \(\|b\|_{L^\infty(\Omega)}\) and \(\eta\) are small enough, that is
\[\left[\lambda \|b\|_{L^\infty(\Omega)} + \frac{\eta}{\|\Omega\|}\right] S < \inf_{\Omega} |a| \left[F_v^2 - 2 \lambda \|b\|_{L^\infty(\Omega)} S - \frac{2 \eta S^2}{\|\Omega\|} \right]^{\frac{1}{2}}, \tag{5.12}\]
then
\[\operatorname{meas}\{x \in \Omega : \nabla u^e(x) = 0\} = 0.\]
In particular, \(u^e\) satisfies problem \((\mathcal{P}_{*,e})\).

**Proof.** We argue by contradiction. Suppose that
\[\operatorname{meas}\{x \in \Omega : \nabla u^e(x) = 0\} \neq 0. \tag{5.13}\]
Then, from the equation $(\mathcal{P}_{\mu}^\varepsilon)$

$$0 = a[F_v^2 - 2F_t(x, u^\varepsilon, \tilde{b}^\varepsilon) + 2F_{x2}(x, u^\varepsilon)]_+ + H(u^\varepsilon, \tilde{b}^\varepsilon) + J_\varepsilon(u^\varepsilon)$$
a.e. on $\{x \in \Omega : \nabla u^\varepsilon(x) = 0\}$. Using the estimates (5.8), (5.9), (5.10) and (5.11), we get that

$$\lambda \text{osc } b \geq |H(u^\varepsilon, \tilde{b}^\varepsilon)| \geq \inf_{\Omega} a[F_v^2 - 2F_t(x, u^\varepsilon, \tilde{b}^\varepsilon) + 2F_{x2}(x, u^\varepsilon)]_+ - J_\varepsilon(u^\varepsilon),$$

$$\left[\lambda \text{osc } b + \frac{\eta}{|\Omega|}\right] S \geq \inf_{\Omega} a\left[F_v^2 - 2\lambda \|b\|_{L^\infty(\Omega)} S - \frac{2\eta S^2}{|\Omega|}\right]_+.$$

This contradicts assumption (5.12) proving the lemma. □

Now, we have

**Theorem 5.** Assume $\inf_{\Omega} a > 0$, $\gamma \in \mathbb{R}^-$ and that $\lambda \|b\|_{L^\infty(\Omega)} + \eta < \Lambda$ for a suitable $\Lambda > 0$. Then there is a solution of $(\mathcal{P}_\mu^\varepsilon)$. Moreover $u \in V(\Omega)$.

**Proof.** Our aim is to let $\varepsilon \to 0$. By the uniform estimate on $\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}$ given in Lemma 23, there exists some subsequence of $(u^\varepsilon)$ (which we will again denote by $u^\varepsilon$) and a function $\alpha \in L^\infty(\Omega)$ such that

$$\Delta u^\varepsilon \xrightarrow{\ast} \alpha \text{ weakly }^* \text{ in } L^\infty(\Omega).$$

By standard regularity, $u^\varepsilon$ belongs to a bounded set of $W^{2, p}(\Omega)$, for all $p \in [1, +\infty[$. Then, we have (for some subsequence) that

$$u^\varepsilon \rightharpoonup u \text{ weakly in } W^{2, p}(\Omega),$$

$$u^\varepsilon \to u \text{ strongly in } C^1(\bar{\Omega}).$$

In particular, $\alpha = \Delta u$, $\Delta u \in L^\infty(\Omega)$, $u \in V(\Omega)$ and the estimates (5.4) and (5.5) of Lemma 23 remain true replacing $u^\varepsilon$ by $u$. Then

$$\|\Delta u\|_{L^\infty(\Omega)} \leq \frac{\|a\|_{L^\infty(\Omega)} F_v}{1 - \nu}, \tag{5.14}$$

$$\|u_+\|_{L^\infty(\Omega)} \leq \frac{\|a\|_{L^\infty(\Omega)} F_v |\Omega|}{4\pi(1 - \nu)} := S. \tag{5.15}$$

Now, by Lemma 9, we have

$$\|\tilde{b}^\varepsilon\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\Omega)} \text{ and } \|\tilde{b}^\varepsilon\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\Omega)}$$

and so, $\tilde{b}^\varepsilon \xrightarrow{\varepsilon \to 0} \tilde{b}$ weakly* in $L^\infty(\Omega)$ and $\tilde{b}^\varepsilon \xrightarrow{\varepsilon \to 0} \tilde{b}$ weakly* in $L^\infty(\Omega)$. Furthermore, one has

$$u_+^{\varepsilon \to 0} \to u_+ \text{ in } L^p(\Omega) \forall p \in [1, +\infty[ \tag{5.16}$$
(Lemmas 4 and 7). From this convergence, we easily have:

\[ h_\varepsilon(u_{\varepsilon+}^\varepsilon) \xrightarrow{\varepsilon \to 0} (u_{\varepsilon+}^\varepsilon)^2 \quad \text{in } L^q(\Omega_+) \, \forall q \in [1, +\infty[. \]

Thus,

\[ I(u^\varepsilon(x), \cdot) h_\varepsilon(u_{\varepsilon+}^\varepsilon(\cdot)) \xrightarrow{\varepsilon \to +\infty} I(u(x), \cdot)(u_{\varepsilon+}^\varepsilon)^2. \]

In \( L^q(\Omega_+) \) for all \( q \in [1, +\infty[ \) and a.e. in \( x \). We get for all \( x \in \Omega \)

\[ F_{\varepsilon,2}(x, u^\varepsilon) \xrightarrow{\varepsilon \to 0} F_2(x, u) = \int_{|u_+ > u_+(x)|} |u_{\varepsilon+}^\varepsilon(\sigma)|^2 j'_i(u_{\varepsilon+}^\varepsilon(\sigma), u_{\varepsilon+}^\varepsilon(0)) \, d\sigma \]

(because \( j'_i \) is continuous on \( \mathbb{R}^+ \times \mathbb{R}^+ \)). Since \( |F_{\varepsilon,2}(x, u^\varepsilon)| \leq \eta \|\Delta u\|^2_{L^\infty(\Omega)} \), from Lebesgue’s theorem we have

\[ F_{\varepsilon,2}(\cdot, u^\varepsilon) \xrightarrow{\varepsilon \to 0} F_2(\cdot, u) \quad \text{in } L^p(\Omega) \quad \text{for all } p < +\infty. \]

With respect to \( J_\varepsilon(u^\varepsilon) \), we first obtain from Lemmas 4 and 23

\[ |u_{\varepsilon+}^\varepsilon(|u_+ > u_+(x)|)| \leq \frac{\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}}{4\pi} \leq \frac{S}{|\Omega|}. \]

Since \( \varepsilon|\xi(\eta u_{\varepsilon+}^\varepsilon(|u_+ > u_+(x)|))j'_i(u_{\varepsilon+}^\varepsilon(\sigma), u_{\varepsilon+}^\varepsilon(0))| \leq \eta \frac{\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}}{4\pi} \leq \eta \frac{S}{|\Omega|} \) by Assumption (1.7), and again by Lemma 23. Then, we may assume that

\[ \varepsilon|\xi(\eta u_{\varepsilon+}^\varepsilon(|u_+ > u_+(x)|))j'_i(u_{\varepsilon+}^\varepsilon(\sigma), u_{\varepsilon+}^\varepsilon(0))| \xrightarrow{\varepsilon \to 0} \ell_u j'_i(u_+(x), u_+(0)) \]

weakly* in \( L^\infty(\Omega) \) for some \( \ell_u \in L^\infty(\Omega) \). Thus \( u \) is a solution of

\[ -\Delta u = a[F_2^\varepsilon - 2F_i(x, u, \tilde{b}) + 2F_2(x, u)]_+ + p'(u)[b - \tilde{b}] + \ell_u j'_i(u_+(x), u_+(0)) \]

(5.17)

with \( \|\ell_u\|_{L^\infty(\Omega)} \leq \frac{\eta}{|\Omega|} S \) and \( |p'(u)[b - \tilde{b}]| \leq \lambda \text{osc}_\Omega b \|u_+\|_{L^\infty(\Omega)} \). Arguing as in Theorem 4, but now using equation (5.17), then if

\[ \left[ \lambda \|b\|_{L^\infty(\Omega)} + \frac{\eta}{|\Omega|} \right] S < \inf_{|\Omega|} |a| \left[ F_2^\varepsilon - 2\lambda \|b\|_{L^\infty(\Omega)} S - 2\eta S^2 \right]^{\frac{1}{2}} \]

we obtain that \( \text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0 \). Thus, we deduce \( \varepsilon|\xi(\eta u_{\varepsilon+}^\varepsilon(|u_+ > u_+(x)|))| \xrightarrow{\varepsilon \to 0} u_{\varepsilon+}^\varepsilon(|u_+ > u_+(x)|) \) a.e. \( x \in \Omega \) (see Lemma 1) and then \( \ell_u(\cdot) = u_{\varepsilon+}^\varepsilon(|u_+ > u_+(\cdot)|) \) a.e. in \( \Omega \) proving in that way that

\[ \varepsilon|\xi(\eta u_{\varepsilon+}^\varepsilon(|u_+ > u_+(\cdot)|))j'_i(u_{\varepsilon+}^\varepsilon(\cdot), u_{\varepsilon+}^\varepsilon(0))| \xrightarrow{\varepsilon \to 0} u_{\varepsilon+}^\varepsilon(|u_+ > u_+(\cdot)|)j'_i(u_+(\cdot), u_+(0)) \]

strongly in \( L^p(\Omega) \) \( \forall p \in [1, +\infty[. \) Applying Proposition 3 we can identify \( \tilde{b}(x) = b_{su}(|u_+ > u(x)|) \) in \( \Omega \), \( \tilde{b}(s) = b_{su}(s) \) in \( \Omega_+ \) and, in conclusion, \( u \) is a solution of (\( \mathcal{P}_\lambda \)).
The main result of the paper, stated in the Introduction, is now an easy consequence of the above result:

**Proof of Theorem 1.** By Theorem 5 there exists a solution of $(\mathcal{P}_*)$ such that $u \in V(\Omega)$. Moreover, the assumptions of Theorem 2 are fulfilled and so the couple $(u, \mathcal{F})$ is a solution of $(\mathcal{P})$. □

. Qualitative properties

In this section we shall give some qualitative properties on the founded solution $u$ of $(\mathcal{P})$. They are the following: a condition for the existence of the free boundary, an estimate on the measure of the plasma region and an estimate of the $L^1$-norm of $u_+$.

Let $\varphi_1$ be a normalized eigenfunction associated to the first eigenvalue $\lambda_1$ of the operator $-\Delta$ on $\Omega$ with Dirichlet boundary condition, i.e. $\varphi_1 \in H^1_0(\Omega)$ and $-\Delta \varphi_1 = \varphi_1$ on $\Omega$. We know that $\varphi_1 > 0$ on $\Omega$. Besides, we can renormalize it such that $\int_{\Omega} \varphi_1 \, dx = 1$.

**Theorem 6.** Assume that

$$-\gamma < F_\nu \int_{\Omega} a(x) \varphi_1(x) \, dx := -\gamma_0,$$

then any solution $u$ of $(\mathcal{P}_*)$ satisfies $u_+ \equiv 0$.

**Proof.** We argue as in [3, Theorem 8]. The proof relies on the identity

$$\lambda_1 \int_{\Omega} u \varphi_1 \, dx - \gamma = \int_{\Omega} a \mathcal{F}_u \varphi_1 \, dx + \int_{\Omega} H(u, b_{su}) \varphi_1 \, dx + \int_{\Omega} J(u) \varphi_1 \, dx$$

(6.1)

here $\mathcal{F}_u(x) := [F_\nu^2 - 2F_1(x, u, b_{su}) + 2F_2(x, u)]^\frac{1}{2}$ (see (4.2), (4.8) and (4.9)). Recall that $u$ satisfies the equation (3.2), i.e. $-\Delta u = a \mathcal{F}_u + H(u, b_{su}) + J(u)$ and then

$$\int_{\Omega} (u - \gamma) \nabla v \, dx = \int_{\Omega} a \mathcal{F}_u v \, dx + \int_{\Omega} H(u, b_{su}) v \, dx + \int_{\Omega} J(u) v \, dx \quad \forall v \in H^1_0(\Omega).$$

Choosing $v = \varphi_1$ and using that $\int_{\Omega} \nabla (u - \gamma) \nabla \varphi_1 \, dx = \int_{\Omega} (u - \gamma) (-\Delta \varphi_1) \, dx$ we get (6.1) from the above identity. To end the proof of Theorem 6, we argue by contradiction. Assume that $u_+ \equiv 0$. Thus, (6.1) reduces to

$$\lambda_1 \int_{\Omega} u \varphi_1 \, dx = \gamma + F_\nu \int_{\Omega} a \varphi_1 \, dx$$

(6.2)

(see $F_1(x, u, b_{su}) = H(u, b_{su}) = F_2(x, u) = J(u) = 0$ if $u_+ \equiv 0$ (see Definitions (4.2), (4.5), (4.8) and (4.9)). In that case, the first integral of (6.2) is nonpositive, and as relation (6.2) implies that $-\gamma \geq F_\nu \int_{\Omega} a \varphi_1 \, dx$. This relation contradicts the choice of $\nu$. □
Now, we shall estimate the measure of the plasma region

$$|u > 0| = \int_{\{u > 0\}} \text{d}x.$$  

This quantity can be estimated in terms of $\int_\Omega u_+ \text{d}x$ by using that

$$\int_\Omega u_+ \text{d}x \leq |u > 0| \max_{\Omega} u_+.$$  

We already know that $\max_{\Omega} u_+ \leq S$ (see (5.5) Lemma 23). So, we have

$$|u > 0| \geq \frac{1}{S} \int_\Omega u_+ \text{d}x.$$  

(6.3)

In order to estimate the $L^1$-norm of $u_+$ from below we use the identity (6.1). If we write $u = u_+ - u_-$ we obtain that

$$\int_\Omega [F_u + \lambda_1 u_-]\varphi_1 \text{d}x + \gamma = \int_\Omega \varphi_1 [\lambda_1 - \lambda(b - b_{\text{osc}}(u(x)))]u_+ \text{d}x$$

$$- \int_\Omega J(u)\varphi_1 \text{d}x.$$  

(6.4)

From (6.4), estimate (5.11) and the Lemma 23 we deduce that

$$\gamma - \gamma_0 + \int_\Omega a(x)[F_u(x) - F_\lambda(x)]\varphi_1(x) \text{d}x \leq [\lambda_1 + \lambda \text{osc } b]\|\varphi_1\|_{L^\infty(\Omega)}$$

$$\times \int_\Omega u_+(x) \text{d}x + \eta S \|\varphi_1\|_{L^\infty(\Omega)}.$$  

(6.5)

Now, we define $\mathcal{F}_u^1$ by

$$\mathcal{F}_u^1(x) := [F_u^2 - 2F_1(x, u, b_{\text{osc}})]^{1/2}.$$  

Then, it is clear that

$$\int_\Omega a[\mathcal{F}_u - F_\lambda]\varphi_1 \text{d}x = \int_\Omega a[\mathcal{F}_u - \mathcal{F}_u^1]\varphi_1 \text{d}x + \int_\Omega a[\mathcal{F}_u^1 - F_\lambda]\varphi_1 \text{d}x.$$  

We estimate the last two integrals:

$$\left| \int_\Omega a[\mathcal{F}_u^1 - F_\lambda]\varphi_1 \text{d}x \right| \leq \lambda^{1/2} \|b\|_{L^\infty(\Omega)}^{1/2} \|a\|_{L^\infty(\Omega)} \int_\Omega u_+ \text{d}x$$

and

$$\left| \int_\Omega a[\mathcal{F}_u - \mathcal{F}_u^1]\varphi_1 \text{d}x \right| \leq 2^{1/2} \eta^{1/2} |\Omega|^{3/2} \|a\|_{L^\infty(\Omega)} \|\varphi_1\|_{L^\infty(\Omega)}.$$
Thus, we get to
\[
\left| \int_{\Omega} a(\mathcal{T}_u - F) \varphi_1 \, dx \right| \leq \lambda^{1/2} \| b \|_{L^\infty(\Omega)}^{1/2} \| a \|_{L^\infty(\Omega)} \int_{\Omega} u_+ \, dx \\
+ 2^{1/2} \eta^{1/2} |\Omega|^{1/2} \| a \|_{L^\infty(\Omega)} \| \varphi_1 \|_{L^\infty(\Omega)}.
\]

he last inequality and (6.5) becomes
\[
\gamma - \gamma_0 \leq L(\lambda) \int_{\Omega} u_+ \, dx + \eta S \| \varphi_1 \|_{L^\infty(\Omega)} + 2^{1/2} \eta^{1/2} |\Omega|^{1/2} \| a \|_{L^\infty(\Omega)} \| \varphi_1 \|_{L^\infty(\Omega)} 
\]

ith \( L(\lambda) = [\lambda + \lambda \text{osc}_\Omega b + \lambda^{1/2} \| b \|_{L^\infty(\Omega)}] \| a \|_{L^\infty(\Omega)} \| \varphi_1 \|_{L^\infty(\Omega)}. \) From (6.6) we obtain
\[
\frac{\gamma - \gamma_0 - O(\eta^{1/2})}{L(\lambda)} \leq \int_{\Omega} u_+(x) \, dx.
\]

conclusion (combining this estimate and (6.3)) we have shown

**Theorem 7.** Assume the hypotheses of Theorems 1 and 6. Let \((u, F)\) be the solution of \((\mathcal{P})\) obtained in Theorem 1. Then
\[
|u_0| \geq \frac{\gamma - \gamma_0 - O(\eta^{1/2})}{SL(\lambda)} > 0
\]

\( \eta \) is small enough. □

**Remark 1.** The model that we studied here concerns the case of a Stellarator machine. Other models related on the magnetic confinement in Plasma Physics are rived for Tokamak machines. Most of these models have a different formulation. Nevertheless, the questions raised by those machines are just the same. For the sake of completeness, we provide here a few references on Tokamak machines that can guide the reader: Description of the Tokamak machine and derivation of the model [26–29]); Existence, uniqueness and Control theory for local models ([6–9,30–35,?]).

**References**


