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Qualitative Study of Nonlinear Parabolic Equations: an Introduction

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1. INTRODUCTION: THE PROBLEM MODEL

Given \( \Omega \), open bounded regular set of \( \mathbb{R} \), \( N \geq 1 \), we consider the model problem

\[
(P) \quad \left\{ \begin{array}{l}
 b(u)_t - \text{div } A(x, u, \nabla u) + g(x, u) = f(t, x), \quad t > 0, \quad x \in \Omega, \\
 u = h, \quad t > 0, \quad x \in \partial \Omega, \\
 b(u(0, x)) = b(u_0(x)), \quad x \in \Omega.
\end{array} \right.
\]

Before making explicit the structural assumptions on the data \( b, A, f, h \) and \( u_0 \), we mention some important special examples. Perhaps the simpler example is the linear heat equation

\[ u_t - \Delta u = f. \tag{1} \]

\( b(s) = s, A(x, u, \xi) = \xi \) and \( g \equiv 0 \). This is a typical example of linear partial differential equation of parabolic type usually considered in undergraduate courses (see, e.g., John [31]). A modern treatment starts by introducing the notion of weak solution or by its reformulation as an abstract Cauchy problem in a Banach space

\[
\left\{ \begin{array}{l}
 \frac{du}{dt}(t) + Au(t) = f(t), \\
 u(0) = u_0,
\end{array} \right.
\]

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(see, e.g., Brezis [17]). It is well known, that one of the main results of the stabilization theory is that if

\[
\begin{align*}
  f(t, x) &\to f_\infty(x) \\
  h(t, x) &\to h_\infty(x)
\end{align*}
\]

as \( t \to \infty \)

in some suitable sense then the solution of the linear heat equation \( u(t, x) \) verifies that

\[
u(t, x) \to u_\infty(x) \quad \text{as} \quad t \to \infty
\]

in some functional space, with \( u_\infty \) satisfying the associated stationary problem

\[
\begin{align*}
  -\Delta u_\infty &= f_\infty(x) & \text{in} & \quad \Omega, \\
  u_\infty &= h_\infty & \text{on} & \quad \partial\Omega,
\end{align*}
\]

(\textit{the linear diffusion equation}). Notice that problem (2) is also included in the formulation \((P)\) by making \( b \equiv 0, A(x, u, \xi) = \xi, g \equiv 0, f = f_\infty \) and \( h = h_\infty \).

More in general, given a choice of \( b, A, g, f, h \) and \( u_0 \) leading to a specific formulation of \((P)\), the choice of choice of \( b \equiv 0, A \) and \( g \) as before leads to the formulation of the associated stationary problem. In this way \((P)\) includes also stationary problems. In order to present some nonlinear examples, it is useful to read \((P)\) as a balance of different phenomena

\[
\begin{align*}
  b(u)_t - \text{div} A + g(x, u) - f(x, t) &= 0. \\
  (I) \quad (II) \quad (III)
\end{align*}
\]

Let us make some comments on the accumulation term (I). It arises, for instance, in thermal processes when the heat capacity of the medium depends on the temperature. This is the case, e.g., when water and ice are simultaneously present and then \( b(u) \) is a strictly increasing function having a discontinuity at \( u = 0 \). This special case (called \textit{Stefan problem}) requires a delicate mathematical treatment.

In fact, as a general rule, the assumption \( b : \mathbb{R} \to \mathbb{R} \) nondecreasing is absolutely fundamental to formulate \((P)\) in the class of problems of parabolic type since otherwise the problem becomes \textit{ill posed} (as, for instance, \(-u_t - \Delta u = f; \text{the backward heat equation}\)).

This type of accumulation term (I) also arises in the theory of filtration of a fluid in a porous media. In that case

\[
b \in C^0(\mathbb{R}), \ b \text{ nondecreasing},
\]
see, e.g., Bear [10]). Now \( u(t, x) \) is not a temperature but the humidity of the soil. Different choices are possible: in the study of unsaturated soils \( b \) is assumed to be strictly increasing, as, for example, \( b(u) = |u|^{α-1}u \). In the case of partially saturated soils, \( b(u) \) is not strictly increasing but becomes constant at \( u > u^1 \), for some \( u^1 > 0 \). Notice that, in this physical framework, \( u \geq 0 \) and so the values of \( b \) on \( \mathbb{R}^- \) are not relevant. The, so called dam problem, corresponds to a limit case in which \( b \) is the Heaviside function. This choice \( b \) also arises in problems of a different physical context, as, for instance, the ele-Shaw problem or some problems arising in lubrication theory (see, e.g., Ayada and Chambat [9] where many other references can be found).

Let us refer now to the diffusion and convection terms involved in (II). The dependence of \( A(x, u, \nabla u) \) with respect to \( \nabla u \) (resp. \( u \)) leads to diffusion terms (resp. convection terms). Some examples of relevance in the applications are commented in the following. The, so called, nonlinear heat equation arises when the Fourier law fails and the thermal conductivity depends on the temperature (case of many gases, lubricating fluids, etc.). Then the diffusion heat leads to the expression

\[
\text{div} (k(u)\nabla u) = \Delta \beta(u) \quad \text{with} \quad \beta(s) := \int_0^s k(\sigma) d\sigma.
\]

most of the cases \( \beta(u) \) grows like a power

\[
\beta(u) = |u|^{m-1}u \quad \text{with} \quad m > 0.
\]

e above second order operator (sometimes written as \( -\Delta u^m \)) also arises the study of filtration in porous media (D’Arcy law) with \( m > 1 \) and in small physics when \( 0 < m < 1 \).

A different class of examples of nonlinear terms \( A(x, u, \nabla u) \) arises in the study of non-Newtonian fluids. The study of one-directional flows of some special fluids (as, for instance, polymer melts, suspensions, paints, animal od, honey, shampoo, etc.) leads to nonlinear diffusion operators of the form

\[
\text{div} (|\nabla u|^{p-2} \nabla u), \quad \text{denoted by} \quad \Delta_p u, \quad \text{for some} \quad p > 1.
\]

Notice that if \( p = 2 \) then \( \Delta_2 = \Delta \) (the linear Laplacian operator, arising in study of Newtonian fluids). The case \( 1 < p < 2 \) corresponds to pseudo-static fluids (as, e.g., gasoline, lubricating oil, etc.) and \( p > 2 \) arises in the consideration of dilatant fluids (as, for instance, the polar ice and glaciers, cano lava, etc.).
The above two operators may become degenerate since
\[ \Delta u^m = \text{div} \left( m u^{m-1} \nabla u \right) = m u^{m-1} \Delta u + m(m - 1) u^{m-2} |\nabla u|^2. \]
So, if \( m > 1 \) the coefficient of \( \nabla u \) vanishes on the set \( \{(t, x) : u(t, x) = 0\} \).
Analogously,
\[ \Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = |\nabla u|^{p-2} \Delta u + \nabla u \cdot \nabla (|\nabla u|^{p-2}) \]
and when \( p > 2 \) the coefficient of \( \nabla u \) vanishes on the set \( \{(t, x) : \nabla u(t, x) = 0\} \). Due to this reason the qualitative behavior of solutions of \( (P) \) may be very different (according to the assumptions on the data \( b, \mathbf{A}(x, u, \nabla u) \) and \( g \)) to the one of the solution of the linear heat equation. In fact, to show such kind of differences is one of the main goals of these notes.

We also mention that another relevant choice of nonlinear terms \( \mathbf{A}(x, u, \nabla u) \) arises in the study of transient minimal surfaces, in which case the second order diffusion operator is given by
\[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \]
Concerning the transport or convection terms, we mention that they arise very often in Fluid Mechanics. Usually they appear formulated in terms of an additive term, as, for instance, in the case of the temperature in a fluid
\[ -\Delta \beta(u) + \mathbf{w} \cdot \nabla u. \]
Diffusion Convection

If the fluid is incompressible (case of liquids) then \( \text{div} \mathbf{w} = 0 \) and so we get
\[ -\text{div} \left( k(u) \nabla u - u \mathbf{w} \right), \text{ i.e., } \mathbf{A}(x, u, \xi) = k(u) \xi + u \mathbf{w}. \]
Nevertheless, sometimes the convection term is not an additive term but appears in a different form.
\[ \text{div} \left( \Phi(\nabla u + K(b(u) e)) \right) \]
where
\[ \Phi(\xi) = |\xi|^{p-2} \xi, e \in \mathbb{R}^N \text{ and } K \in C^1(\mathbb{R} : \mathbb{R}). \]
This situation arises, for instance, in the study of turbulent flow of a fluid through a porous medium (with \( e \) the vector indicating the main filtration direction). For a general exposition on different examples of diffusion-convective
erators, containing many other references see Díaz [20] and Díaz and de Melin [25].

The expression (III) represents the absorption/forcing term. The presence of the term \( g(x, u) - f(t, x) \) is very typical of many problems arising in reaction-fusion problems in Biology, Chemistry and other contexts. By writing

\[
g(x, u) = g_1(x, u) - g_2(x, u),
\]

with \( g_1 \) and \( g_2 \) nondecreasing functions, we can distinguish the term of absorption \( g_1(x, u) \) (which contributes to make \(|u|\) smaller than if \( g_1 = 0 \)) from the term of forcing \( g_2(x, u) \) (which contributes to make \(|u|\) bigger than if \( g_2 = 0 \)).

In most of the cases

\[
g_1(x, u) = \lambda |u|^{q-1} u, \quad \lambda > 0,
\]

in \( q > 0 \) (the order of the reaction). Notice that if \( 0 < q < 1 \), \( g_1 \) is not a Lipschitz function.

Returning to the structural assumptions on the data, in the rest of the position, we shall always assume that

\[
b : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing, } b(0) = 0, \tag{3}
\]

\[
\left\{ \begin{array}{l}
A : \Omega \times \mathbb{R} \times \mathbb{R}^N \text{ is a Caratheodory function} \\
\text{(i.e., measurable in } x \text{ and continuous in } (u, \xi)), \\
\exists p > 1 \text{ such that } |A(x, u, \xi)| \leq C(|u|^{p'} + |\xi|^{p-1}), \quad \forall u \in \mathbb{R}, \\
\forall \xi \in \mathbb{R}^N \text{ with } p' = \frac{p}{p-1}, \quad p^* = \frac{Np}{N-p} \text{ and} \\
(A(x, u, \xi) - A(x, u, \xi^*)) \cdot (\xi - \xi^*) > 0, \quad \forall \xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*,
\end{array} \right. \tag{4}
\]

\[
g \text{ is Caratheodory function and} \tag{5}
\]

\[
|g(x, u)| \leq \gamma(|u|)(1 + d(x)), \quad d \in L^1(\Omega) \text{ and } \gamma \text{ strictly increasing,}
\]

\[
\int_{= f_1 + f_2, \ f_1 \in L^p(0, T : W^{-1,p'}(\Omega)), \ f_2 \in L^1((0, T) \times \Omega), \forall T > 0,} \\
h \in L^p(0, T : W^{1,p}(\Omega)) \cap L^\infty((0, T) \times \Omega), \forall T > 0, \tag{6}
\]

\[
u_0 \in L^\infty(\Omega).
\]

For the sake of simplicity in the exposition, we shall deal merely with

\textit{nded (weak) solutions}
Definition 1. We say that \( u \) is a bounded weak solution of \((P)\) if \( u - h \in L^p(0, T : W^{1,p}(\Omega)) \cap L^\infty((0, T) \times \Omega), \forall T > 0 \), and we have:

\[
\begin{align*}
& b(u)_t \in L^p(0, T : W^{-1,p'}(\Omega)) \quad \text{and} \\
& \int_0^T \langle b(u)_t, v \rangle_{W^{-1,p'} \times W^{1,p}_0} dt + \int_0^T \int_\Omega (b(u) - b(u_0))v_0 dx dt = 0 \\
& \forall v \in L^p(0, T : W^{1,p}_0(\Omega)) \cap W^{1,1}(0, T : L^1(\Omega)) \quad \text{with} \quad v(T, \cdot) \equiv 0,
\end{align*}
\]

and

\[
\begin{align*}
& \int_0^T \langle b(u)_t, v \rangle dt + \int_0^T \int_\Omega A(x, u, \nabla u) \cdot \nabla v dx dt + \int_0^T \int_\Omega g(x, u)v dx dt \\
& = \int_0^T \langle f_1, v \rangle dt + \int_0^T \int_\Omega f_2 v dx dt, \\
& \forall v \in L^p(0, T : W^{1,p}_0(\Omega)) \cap L^\infty((0, T) \times \Omega), \quad \forall T > 0.
\end{align*}
\]

The above definition is adapted from Alt and Luckhaus [2].

In the rest of this exposition we shall consider different qualitative properties of solutions of \((P)\) arising according the nature of the nonlinear term \( b(u), A(x, u, \nabla u) \) and \( g(x, u) \). Our plan is the following: Section 2 will be devoted to two comparison principles which will be important tools in our study. Two qualitative properties are presented in the rest of the exposition: the infinite extinction time property (Section 3) and the finite speed of propagating property (Section 4). In both of the above sections we shall apply the two comparison principles as well as some energy methods.

It is clear that the above presentation is far to be exhaustive. Problem like \((P)\) have attracted the attention of many specialists in the last forty year (perhaps the earliest mathematical paper on this subject was [38]). In consequence, many other very interesting qualitative properties are today available in the literature. The present notes only pretend to be an elementary introduction.

2. TWO USEFUL TOOLS

2.1. INTRODUCTION. The study of several qualitative properties for solutions of model problem \((P)\) will be carried out thanks to some useful tool: the comparison principles.

The most popular comparison principle has a pointwise nature and usually holds for elliptic and parabolic second order equations (as well as for first order hyperbolic equations). A first statement of such a principle is the following:
Theorem 1. (Pointwise comparison principle) Let \((f, h, u_0)\) and \((\hat{f}, \hat{h}, \hat{u}_0)\) two set of ordered data, i.e., such that

\[ f \leq \hat{f}, \quad h \leq \hat{h} \quad \text{and} \quad u_0 \leq \hat{u}_0, \]

their respective domains of definition. Let \(u\) and \(\hat{u}\) be (any) solutions of corresponding to \((f, h, u_0)\) and \((\hat{f}, \hat{h}, \hat{u}_0)\) respectively. Then

\[ u(t, x) \leq \hat{u}(t, x), \quad \text{for any} \ t > 0 \ \text{and a.e.} \ x \in \Omega. \]

In the case of linear problems, this property is a trivial consequence of the minimum principle (in fact, it suffices to assume \((\hat{f}, \hat{h}, \hat{u}_0) = (0, 0, 0)\) and so \(i = 0\)). The first (general) result for linear equations seems to be due to Paraf 1892 (later generalizations where due to Picard, Lichtenstein and, finally, of (in 1927) (see details in the book Gilbarg and Trudinger [30]).

It is clear that for the nonlinear case some conditions on \(b\), \(A\) and \(g\) are ded (notice that the pointwise comparison principle implies the uniqueness solutions). This topic is still under investigation (see the series of works by Benilan, J. Carrillo and others). Here we shall recall a particular result a short proof) stated in terms of an estimate for a suitable expression.

The second tool refers to another comparison principle, but this time, of a different nature. We can call it as the symmetrized mass comparison principle. The process of symmetrization need to be carefully presented. We start by the metrization of the domain \(\Omega\): Given \(\Omega\), an open bounded set of \(\mathbb{R}^N\), the metrized version of \(\Omega\) is the ball centered at the origin having the same measure than \(\Omega\). Let us call \(\Omega^*\) to this ball. The condition \(m(\Omega) = m(\Omega^*)\) has the isoperimetric inequality

\[ L \geq N\omega_N^{\frac{1}{N}} A^{\frac{N-1}{N}} \]

where \(L\) is the length of \(\partial \Omega\) (or \(m(\partial \Omega)\)), \(A\) is the area of \(\Omega\) (or \(m(\Omega)\)) and \(\omega_N\) is the area of the unit ball of \(\mathbb{R}^N\) (i.e., \(\omega_N = m(S^{n-1})\)).

3) the equality holds if and only if \(\Omega\) is a ball. This was a first noted by de Cartago (850 B.C.) (in \(\mathbb{R}^2\) the circles are the domains with fixed area having a longer perimeter). Rigorous proofs of (8) are due to Steiner (1882), varz (1890) and Schmidt (1939).

The second step of the process of symmetrization consists in the metrization of data \(f\) and \(u_0\). We shall use the notion of the decreasing-symmetric rearrangement of a function introduced by H.A. Schwarz in
1890: Given a function \( h : \Omega \to \mathbb{R}, \ h \in L^1(\Omega), \) we define the decreasing symmetric rearrangement of \( h, \ h^* \), as the (unique) function \( h^* : \Omega^* \to \) such that \( h^* \) is symmetric (i.e., \( h^*(x) = h^*(\tilde{x}) \) if \( |x| = |\tilde{x}| \)), \( h^* \) decreases if \( |x| \) decreases and the level sets of \( h \) and \( h^* \) are equimeasurables (i.e., \( m(\{x \in \Omega : h(x) > \theta\}) = m(\{x \in \Omega^* : h^*(x) > \theta\}), \forall \theta \in \mathbb{R} \)). A more systematic definition of \( h^* \) can be introduced as follows: we first define the distribution function of \( h \) by

\[
\mu : \mathbb{R} \to \mathbb{R}, \ \mu(\theta) := m(\{x \in \Omega : h(x) > \theta\}).
\]

Then we define the scalar decreasing rearrangement of \( h \) by

\[
\tilde{h} : [0, m(\Omega)] \to \mathbb{R}, \ \tilde{h}(s) := \inf \{ \theta \in \mathbb{R} : \mu(\theta) \leq s \}
\]

(notice that \( \tilde{h}(s) \sim \mu^{-1}(s) \)). Finally, we define the symmetric decreasing rearrangement of \( h \), by

\[
h^* : \Omega^* \to \mathbb{R}, \ h^*(x) := \tilde{h}(\omega_N |x|^N).
\]

Notice that, since \( h^* \) is symmetric, we can write \( h^*(x) = H(|x|) \) with \( H : \mathbb{R} \to \mathbb{R} \). Nevertheless \( H \neq \tilde{h} \) since \( H(r) = \tilde{h}(\omega_N r^N) \). Notice, also, that assume \( h \geq 0 \), by construction, we have that

\[
h \in L^1(\Omega) \text{ implies that } h^* \in L^1(\Omega^*) \text{ and } \int_\Omega h(x) dx = \int_{\Omega^*} h^*(x) dx \text{ (the Cavalieri Principle)}
\]

and that

\[
h \in L^\infty(\Omega) \text{ implies that } h^* \in L^\infty(\Omega^*) \text{ and } \text{ess sup } x \in \Omega h(x) = \text{ess sup } x \in \Omega^* h^*(x).
\]

The third step of the process is the symmetrization of the second order operator. We must replace the diffusion operator \( \text{div} \ A(x, u, \nabla u) \) by another isotropic diffusion operator, i.e., with the same behavior in any direction. Several possibilities arise. Here we shall consider, merely, a special case. Assume that condition (4) holds and that, in addition,

\[
A(x, u, \xi) \cdot \xi \geq |\xi|^p \quad \forall \xi \in \mathbb{R}^N.
\]
en we shall define as *symmetrized operator of* $\text{div} \ A(x, u, \nabla u)$ the one given

$$\Delta_p u = \text{div} (|\nabla u|^{p-2} \nabla u)$$

notice than if we take $A^*(x, u, \xi) = |\xi|^{p-2}$ then condition (4) holds with the tility sign instead the inequality one).
We also must introduce an *isotropic absorption* by assuming (besides (5)) the condition

$$\left\{ \begin{array}{ll}
g(x, u)u \geq \hat{g}(u)u & \text{a.e. } x \in \Omega, \\
\text{for some continuous function } \hat{g} : \mathbb{R} \to \mathbb{R}. \\
\end{array} \right. \quad (9)$$

Summarizing, we say that the *symmetrized problem of* $(P)$ is the following:

**Problem.** $(P^*)$: Find $U : [0, \infty) \times \Omega^* \to \mathbb{R}$ such that

$$\left\{ \begin{array}{ll}
b(U) t - \Delta_p U + \hat{g}(U) = f^*(t, x), & t > 0, \quad x \in \Omega^*, \\
U = h^*, & t > 0, \quad x \in \partial \Omega^*, \\
b(U(0, x)) = b(u_0^*(x)), & x \in \Omega^*. \\
\end{array} \right. \quad (P^*)$$

e $f^*(t, \cdot)$ and $u_0^*(\cdot)$ are the decreasing symmetric rearrangements of $f(t, \cdot)$ $w_0$, respectively. For the sake of simplicity in the exposition we shall ule now that

$$h = h^* = 0. \quad (10)$$

Let us make some remarks on the statement of the *symmetrized mass com-ison principle*. The first one is that some pioneer authors finding different ions between $u$ and $U$ where Saint-Venant (1856), Poya and Szego (1951) Weimberger (1962). The inequality

$$u^*(x) \leq U(x), \quad x \in \Omega^*, \quad (11)$$

first proved by G. Talenti, in 1976, for the case of the stationary problem ort absorption term (i.e., $b \equiv 0$ and $g \equiv 0$). Unfortunately, this (point- comparison fails to be true for parabolic problems (i.e., $b \neq 0$) or/and problems in presence of absorption terms ($g \neq 0$). In those cases we only compare the *distribution of the mass* of $u$ and $U$

**Theorem 2.** (Symmetrized Mass Comparison Principle (SMCP))

$$\int_{B(0, r)} u^*(t, x) dx \leq \int_{B(0, r)} U(t, x) dx, \quad \forall t > 0, \forall r \in [0, R],$$

med that $\Omega^* = B(0, R)$. 

Notice that this comparison can be, equivalently, expressed in terms of scalar decreasing rearrangement as
\[
\int_0^s \bar{u}(t, \sigma) d\sigma \leq \int_0^s \bar{U}(t, \sigma) d\sigma, \quad \forall t > 0, \forall s \in [0, m(\Omega)].
\]
The SMCP has many applications (as we shall see in other sections). The main philosophy of the applications is that function \( U \) can be easily estimated in many cases and thus, thanks to the SMCP, properties for \( U \) can be extended similarly for \( u \). Some books dealing with the symmetrization procedure are the ones by Bandle [6], Mossino [35] and Kawohl [33]. The proof we shall present here follows the memoir Díaz [21] (see also Díaz [22]). A different (but very original) approach is due to Abourjail and Benilan [1]. The first result in the literature for degenerate parabolic problems was Vázquez [41].

2.2. Proof of the Two Comparison Principles.

On the Pointwise Comparison Principle. We present here a particular version of this principle (more general results will be indicated later for the special case of the diffusion-convection operator arising in the study of turbulent flow of a fluid through a porous medium. More precisely, consider the problem

\[
(P_{\phi,K}) \begin{cases}
  b(u)_t - \text{div} \left( \phi(\nabla u + eK(b(u))) \right) + g(x, u) = f(x, t), & t > 0, x \in \Omega, \\
  u = h, & t > 0, x \in \partial \Omega \\
  b(u(0, x)) = b(u_0(x)) & x \in \Omega,
\end{cases}
\]

where \( \phi(\xi) = |\xi|^{p-2}\xi, \quad p > 1, \quad e \in \mathbb{R}^N \) and \( K \in C^0(\mathbb{R}, \mathbb{R}) \). Besides the conditions made explicit in Section 1 we shall make some extra assumptions:

\[
(H_{g,b}) \begin{cases}
  \text{there exists } C^* \geq 0 \text{ such that } \\
  g(\cdot, \eta) - g(\cdot, \widehat{\eta}) \geq -C^* (b(\eta) - b(\widehat{\eta})), & \forall \eta > \widehat{\eta}, \eta, \widehat{\eta} \in \mathbb{R},
\end{cases}
\]

(notice that \((H_{g,b})\) trivially holds if, for instance, \( g(\cdot, \eta) \) is nondecreasing in \( \eta \) or if \( g(\cdot, \eta) := \widetilde{g}(\cdot, s) \) with \( \widetilde{g}(\cdot, s) \) Lipschitz continuous in \( s \),

\[
(H_K) \begin{cases}
  K(b(\eta)) \text{ is Hölder continuous in } \eta \text{ of exponent } \gamma \geq \frac{1}{p} \text{ if } 1 < p < 2 \\
  \text{and } \gamma \geq \frac{1}{p'} \left( \frac{1}{p} + \frac{1}{p'} = 1 \right) \text{ if } p \geq 2, \\
  |K(b(\eta)) - K(b(\widehat{\eta}))| \leq C|\eta - \widehat{\eta})|^\gamma, & \forall \eta, \widehat{\eta} \in \mathbb{R},
\end{cases}
\]

(notice that condition (4) is now trivially satisfied).
THEOREM 3. Let \((f, h, u_0), (\tilde{f}, \tilde{h}, \tilde{u}_0)\) be such that \(f \leq \tilde{f}, h \leq \tilde{h}\) and \(u_0 \leq \tilde{u}_0\) on their respective domains. Let \(u, \tilde{u}\) be two bounded weak solutions of the \((H_{g,b})\) associated to \((f, h, u_0)\) and \((\tilde{f}, \tilde{h}, \tilde{u}_0)\), respectively. Assume, in addition, \(u\) and \(\tilde{u}\) are strong solutions, i.e.,

\[
 b(u)_t, b(\tilde{u})_t \in L^1((0,T) \times \Omega), \quad \forall T > 0. \tag{12}
\]

en \(u \leq \tilde{u}\) on \((0,T) \times \Omega\). More in general, if we replace the ordered data \((u_0, \tilde{u}_0)\) by the simpler condition \(h \leq \tilde{h}\) and \(f_1 \leq \tilde{f}_1\) then

\[
 [b(u(t,\cdot)) - b(\tilde{u}(t,\cdot))] + ||L^1(\Omega)|| \leq e^{C^*T} ||[b(u_0) - b(\tilde{u}_0)] + ||L^1(\Omega)||
 + \int_0^T e^{C^*\tau} ||[f_2(\tau,\cdot) - \tilde{f}_2(\tau,\cdot)] + ||L^1(\Omega)|| d\tau \tag{13}
\]

any \(t > 0\) \((C^* \text{ given in } (H_{g,b}))\), where \(\varphi_+ = \max(\varphi, 0)\).

**Proof.** We take as test function the following approximation of the \(\text{sign}^+(u)\) function: we start by defining \(\Psi_\delta(\eta) := \min(1, \max(0, \frac{\eta}{\delta}))\), for \(\delta > 0\). First, we define \(v = \Psi_\delta(u - \tilde{u})\). Notice that \(v \in L^p(0,T : W^{1,p}_0(\Omega)) \cap C([0,T) \times \Omega), \forall T > 0\), and that

\[
 \nabla v = \begin{cases} 
 \nabla \frac{1}{\delta}(u - \tilde{u}) & \text{if } 0 < u - \tilde{u} < \delta, \\
 0 & \text{otherwise}. 
\end{cases}
\]

n, since \(f_1 \leq f_2\), defining the set \(A_\delta := \{(t,x) \in (0,T) \times \Omega : 0 < u(t,x) - \tilde{u}(t,x) < \delta\}\) get

\[
 \int_0^T \int_{\Omega} (b(u)_t - b(\tilde{u})_t) \Psi_\delta(u - \tilde{u}) dx dt + I_1(\delta) + I_2(\delta)
 + \int_0^T \int_{\Omega} (g(x,u) - g(x,\tilde{u})) \Psi_\delta(u - \tilde{u}) dx dt
 \leq \int_0^T \int_{\Omega} (f_2 - \tilde{f}_2) \Psi_\delta(u - \tilde{u}) dx dt,
\]

\[
 I_1(\delta) = \frac{1}{\delta} \int_0^T \int_{A_\delta} \{\phi(\nabla u + K(b(u))e) - \phi(\nabla \tilde{u} + K(b(\tilde{u}))e)\} \cdot (\nabla u + K(b(u))e - \nabla \tilde{u} - K(b(\tilde{u}))e) dx dt,
\]
\[ I_2(\delta) = \frac{1}{\delta} \int_0^T \int_{A_\delta} \{ \phi(\nabla u + K(b(u)) e) - \phi(\nabla \widehat{u} + K(b(\widehat{u})) e) \} \cdot \{ -K(b(u)) e + K(b(\widehat{u})) e \} \, dx \, dt \]

(here \( T \) is arbitrary but fixed, \( T > 0 \)). Applying the Young inequality, \( \alpha \beta \times C(e)p^{-1}\alpha^p + \beta p^{-1}\beta^p \), we see that

\[ |I_2(\delta)| \leq \frac{\varepsilon}{\delta p'} \int_0^T \int_{A_\delta} |\phi(\nabla u + K(b(u)) e) - \phi(\nabla \widehat{u} + K(b(\widehat{u})) e)|^p' \, dx \, dt \]
\[ + \frac{C(e)}{\delta p} \int_0^T \int_{A_\delta} |K(b(u)) - K(b(\widehat{u}))|^p \, dx \, dt = I_2^a + I_2^b. \]

We shall only consider the case of \( p \in (1, 2) \) (the case \( p > 2 \) is similar and even, easier). We need an algebraic inequality.

**Lemma 1.** (see, e.g., Díaz and de Thelin [25]) Let \( \phi(\xi) := |\xi|^{p-2}\xi \) w. \( p > 1 \). Then, there exists \( C > 0 \) such that

\[ C \left| \phi(\xi) - \phi(\widehat{\xi}) \right|^{p'} \leq \left\{ (\phi(\xi) - \phi(\widehat{\xi})) \cdot (\xi - \widehat{\xi}) \right\}^{\frac{p}{2}} \left\{ |\phi(\xi)|^{p'} + |\phi(\widehat{\xi})|^{p'} \right\}^{1-\frac{p}{2}} \]

with \( \alpha = 2 \) if \( 1 < p < 2 \) and \( \alpha = p' \) if \( p \geq 2 \).

Using Lemma 1 we obtain that

\[ |I_2^a| \leq \varepsilon \widetilde{C} I_1(\delta), \]

for some \( \widetilde{C} \) independent of \( \delta \). Moreover

\[ I_2^b \leq \frac{C(e)}{\delta p} \int_{A_\delta} (C|u - \widehat{u}|)^p \, dx \, dt \leq \widetilde{C}(e)m(A_\delta)|\delta|^{p-1} \]

for some \( \widetilde{C}(e) > 0 \) independent of \( \delta \). Then

\[ I_1(\delta) + I_2(\delta) \geq I_1(\delta) - |I_2(\delta)| \geq (1 - \varepsilon \widetilde{C}) I_1(\delta) - \widetilde{C}(e)m(A_\delta)|\delta|^{p-1}. \]

Taking \( \varepsilon \) small enough (so that \( 1 - \varepsilon \widetilde{C} > 0 \)) and using that \( I_1(\delta) \geq 0 \) we have that

\[ \lim_{\delta \downarrow 0} (I_1(\delta) + I_2(\delta)) \geq 0 \]

and so

\[ \int_{u > \widehat{u}} (b(u) - b(\widehat{u})) \, dx \, dt + \int_{u > \widehat{u}} (g(x, u) - g(x, \widehat{u})) \leq 0. \]
on assumption \((H_{g,b})\) we deduce that
\[
\int_{u > \widehat{u}} (b(u) - b(\widehat{u}))_t \, dx \, dt \leq \int_{u > \widehat{u}} (b(u) - b(\widehat{u})) \, dx \, dt,
\]
that
\[
\int_0^T \int_{\Omega} \max\{b(u) - b(\widehat{u}), 0\}_t \, dx \, dt \leq \int_0^T \int_{\Omega} \max\{(b(u) - b(\widehat{u})), 0\} \, dx \, dt,
\]
d, finally
\[
\int_0^T \int_{\Omega} \max\{b(u(T, x)) - b(\widehat{u}(T, x)), 0\} \, dx \, dt \leq \int_0^T \int_{\Omega} \max\{(b(u) - b(\widehat{u})), 0\} \, dx \, dt.
\]
en, by Gronwall inequality
\[
b(u) \leq b(\widehat{u}) \quad \text{a.e.} \quad (t, x) \in (0, T) \times \Omega.
\]
b is strictly increasing this implies that \(u \leq \widehat{u}\) and the proof of the first inclusion ends. In the general case (i.e., when \(b\) is merely nondecreasing) it gains the consideration of the case in which \(A_\delta \subset \{b(u) = b(\widehat{u})\}\), for any \(\delta\) all, (since otherwise the above arguments apply). In that case \(I_2(\delta) \equiv 0\) implies that \(I_1(\delta) \equiv 0\). But from Lemma 1
\[
I_1(\delta) \geq C_\delta \int_0^T \int_{\Omega} \frac{|\nabla \Psi(\delta)(u - \widehat{u})|^2}{(|\nabla u + K(b(u))e|^p + |\nabla \widehat{u} + K(b(\widehat{u}))e|^p)^{\frac{2-p}{p}}} \, dx \, dt \geq 0.
\]
\(\Psi(u - \widehat{u}) = 0\) a.e. on \((0, T) \times \Omega\) which implies that \(u \leq \widehat{u}\) on this set. The of of the case \(p > 2\) and inequality (13) follows the same type of arguments.

Remark 1. It can be proved (see Díaz and de Thelin [25]) that if \(b\) is a schitz function and \(u_0\) is regular enough then any bounded weak solution strong solution (i.e., \(b(u)_t \in L^1(Q_T), Q_T := (0, T) \times \Omega\)). The proof of the tence of strong solutions under more general conditions on \(b\) is a delicate (see the recent results by Benilan and Gariepy [13]).

Remark 2. The (pointwise) comparison principle can be obtained for ker solutions by using more complicated arguments and other selected ones of solutions (entropy solutions, renormalized solutions, good solu-s,...). See the works by Benilan and Touré, Benilan and Wittbold, Carrillo, ...
Remark 3. The quantitative inequality (13) is a typical consequence of the application of abstract results (the $T$-accretiveness of the operator). An illustration of how this theory can be applied to the concrete case of proble $\left(P_{\phi,K}\right)$ (when $h \equiv 0$) is due to Bouhsiss [16]

**ON THE SYMMETRIZED MASS COMPARISON PRINCIPLE.** We recall this time we assume the additional conditions

$$A(x,u,\xi) \cdot \xi \geq |\xi|^p,$$

(1)

$$g(x,u)u \geq \hat{g}(u)u \text{ for some } \hat{g} \in C(\mathbb{R} : \mathbb{R}),$$

(1)

and, for simplicity, (10). Here we also assume that

$$f = f_{2} \in L^1_{loc}(0,\infty : L^1(\Omega)).$$

We shall only consider (for simplicity) the case in which $u$ and $U$ are nonnegative functions.

**THEOREM 4.** Assume that $\hat{g}$ is nondecreasing or locally Lipschitz and that the function

$$\varphi(\eta) := \hat{g}(b^{-1}(\eta))$$

is well defined and can be decomposed as

$$\varphi = \varphi_1 + \varphi_2$$

(1)

with $\varphi_1$ convex and $\varphi_2$ concave. Then

$$\int_{0}^{s} b(\tilde{U}(t,\sigma))d\sigma \leq \int_{0}^{s} b(\tilde{V}(t,\sigma))d\sigma \forall s \in [0,m(\Omega)], \forall t \in [0,\infty).$$

(1)

**Idea of the proof.** First of all we point out that conclusion (17) is stat by approximations of the data $(f,u_{0},b$ and $A)$ leading to the convergence solutions in $L^1(0,T : L^1(\Omega))$. Due to that, we can assume the data regular enough (and, in particular, that $u$ and $U$ are strong solutions $b(u)_t \in L^1(Q_T)$ $b(U)_t \in L^1(Q^*_T), Q^*_T := (0,T) \times \Omega$ and that $b$ is strictly increasing.

**Step 1. The radially symmetric problem.** We define

$$K(t,s) = \int_{0}^{s} b(\tilde{U}(t,\sigma))d\sigma$$

here $\widetilde{U}(t, \cdot)$ is the scalar decreasing rearrangement of $U(t, \cdot)$. First of all, let us prove that $U(t, x)$ decreases when $|x|$ increases. By the symmetry of the set $\Omega$ (and the uniqueness of solutions, implicitly assumed) we deduce that $(t, x) = U(t, |x|)$. Moreover $\nu_r := \frac{\partial}{\partial r} U(t, r), \ r = |x|$ verifies that

$$
\begin{cases}
\frac{\partial}{\partial t} (b(U) U_r) - \frac{\partial^2}{\partial r^2} (|U_r|^{p-2} U_r) + \tilde{g}'(U) U_r = F_r & \text{in } (0, T) \times (0, R), \\
U_r(t, 0) = 0, \quad U_r(t, R) \leq 0, & t \in (0, T), \\
U_r(0, r) = U_{0,r}(r) & r \in (0, R),
\end{cases}
$$

where $\Omega^* = B(0, R), U_0(r) = \tilde{u}_0(\omega_N r^N)$ and $F(t, r) = \tilde{f}(t, \omega_N r^N)$. Then by the maximum principle (here is possible to apply classical results since $U_r$ can be assumed to be smooth), as $F_r(t, \cdot) \leq 0$ and $U_{0,r}(\cdot) \leq 0, \ (t, \cdot) \leq 0$, i.e., $U(t, r)$ decreases when $r$ increases. In consequence, $U(t, \cdot) = (t, \cdot)$ (the function coincides with its decreasing symmetric rearrangement), and so

$$U(t, x) = \widetilde{U}(t, \omega_N r^N), \quad r = |x|.$$ 

Taking

$$s = \omega_N r^N \quad (s \in (0, m(\Omega)))$$

we get that

$$\frac{\partial K}{\partial s} (t, s) = b(\widetilde{U}(t, s)), \quad \frac{\partial U}{\partial r} = N \omega_N^{\frac{1}{N}} s^{\frac{N-1}{N}} \frac{\partial \widetilde{U}}{\partial s}.$$ 

We deduce that $K$ satisfies the parabolic (fully non-linear) problem

$$
\begin{cases}
\frac{\partial K}{\partial t} - a(s) \left[ \frac{\partial}{\partial s} b^{-1} \left( \frac{\partial K}{\partial s} \right) \right]^{p-2} \frac{\partial}{\partial s} b^{-1} \left( \frac{\partial K}{\partial s} \right) = \int_0^s \tilde{g} \left( b^{-1} \left( \frac{\partial K}{\partial s} (t, \sigma) \right) \right) d\sigma = \int_0^s \tilde{f}(t, \sigma) d\sigma, & s \in (0, m(\Omega)), \\
K(t, 0) = 0, \quad K(t, m(\Omega)) = 0, & t \in (0, T), \\
K(0, s) = \int_0^s b(\tilde{u}_0(\sigma)) d\sigma & s \in (0, m(\Omega)),
\end{cases}
$$

where

$$a(s) := \left[ N \omega_N^{1/N} s^{(n-1)/n} \right]^p.$$ 

**Step 2. Study of the rearrangement of u.** Given $\tilde{u}(t, \cdot)$ (the scalar decreasing rearrangement of the solution $u$ of (P)), we define

$$k(t, s) = \int_0^s b(\tilde{u}(t, \sigma)) d\sigma.$$
The main goal of this second step is to prove that \( k(t, s) \) is a subsolution of \( (\mathcal{F}N^\ast) \) in the sense that it verifies all the conditions but replacing the full nonlinear equation by the inequality

\[
\frac{\partial k}{\partial t} - a(s) \left( \frac{\partial}{\partial s} b^{-1} \left( \frac{\partial k}{\partial s} \right) \right)^{p-2} \frac{\partial}{\partial s} b^{-1} \left( \frac{\partial k}{\partial s} \right) + \int_0^s \tilde{g} \left( b^{-1} \left( \frac{\partial k}{\partial s}(t, \sigma) \right) \right) d\sigma \leq \int_0^s \tilde{f}(t, \sigma) d\sigma,
\]

\( s \in (0, m(\Omega)), t \in (0, T) \). The proof of this inequality is quite long and technical. This process can be also divided in several steps:

(i) Define the function \( T_{\tau, h} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) given by

\[
T_{\tau, h}(s) = 0 \quad \text{if} \quad 0 \leq s \leq t,
\]
\[
T_{\tau, h}(s) = s - t \quad \text{if} \quad t < s \leq t + h,
\]
\[
T_{\tau, h}(s) = h \quad \text{if} \quad s > t + h.
\]

We take \( v = T_{\tau, h}(u) \), as test function. Passing to the limit, as \( h \downarrow 0 \), we deduce that

\[
- \frac{\partial}{\partial \theta} \int_{u > \theta} |\nabla u|^p dx \leq \int_0^{\mu(\theta)} \tilde{f}(t, s) ds - \int_0^{\mu(\theta)} \tilde{g}(\tilde{u}(t, s)) ds - \int_{u > \theta} \frac{\partial b(u)}{\partial t} dx
\]

where we used the assumptions (14) and (9) and where \( \mu(\theta) \) denotes the distribution function of \( u(t, \cdot) \).

(ii) We have that

\[
N \omega_N^{1/N} (\theta^{(N-1)/N} \leq (-\mu'(\theta))^{1/p'} \left( - \frac{\partial}{\partial \theta} \int_{u > \theta} |\nabla u|^p dx \right)^{1/p}
\]

(this is a classical result in the rearrangement theory: the proof uses the so-called, Fleming-Rishel formula, the isoperimetric inequality and the notion of perimeter in the Giorgi sense).

(iii) the following identity holds

\[
\int_{u > \theta} \frac{\partial b(u)}{\partial t} dx = \int_0^{\mu(\theta)} \frac{\partial b(\tilde{u}(t, \sigma))}{\partial t} d\sigma = \frac{\partial k}{\partial t}(t, \mu(\theta))
\]

(although a first proof of this formula already appears in the book by Band [6] a more general, and rigorous, proof is due to Mossino and Rakotoson [36]. An easy manipulation of (i), (ii), (iii) leads to the wanted inequality for \( k \).
Step 3. Comparison using the fully nonlinear equation. First of all, notice at the comparison
\[ k(t, s) \leq K(t, s) \quad \forall t \in [0, T], \quad \forall s \in (0, m(\Omega)), \]
incides with the conclusion of the theorem. The main difficulty now is not associated to the very complicated diffusion operator but with the nonlocal nature of the zero order perturbation term. The key idea to obtain the result that, by assumption (16),
\[ \varphi(r) - \varphi(\bar{r}) \leq (\varphi_1'(r) + \varphi_2'(\bar{r})) (r - \bar{r}) \quad \forall r, \bar{r} \in \mathbb{R} \]
see for instance, Taylor formula, the convexity of \( \varphi_1 \) and the concavity of \( \varphi_2 \). Then
\[
\int_0^s \left[ \bar{g}(\tilde{U}(t, \sigma) - \bar{g}(\tilde{w}(t, \sigma)) \right] d\sigma \leq \int_0^s \left[ \varphi_1'(b(\tilde{U}(t, \sigma))) + \varphi_2'(b(\tilde{w}(t, \sigma))) \right] \cdot \left[ b(\tilde{U}(t, \sigma)) + b(\tilde{w}(t, \sigma)) \right] d\sigma \\
\leq C_1 |k(t, s) - K(t, s)| \\
+ C_2 \max_{\tau \in [0, T], \sigma \in [0, s]} |k(\tau, \sigma) - K(\tau, \sigma)|,
\]
some positive constants \( C_1 \) and \( C_2 \). The comparison is now a consequence of classical pointwise comparison principle also related to the T-accretiveness he complicated operator, but this time in the space \( C^0(\Omega) \), (details can be found in Díaz [21]; see also other references indicated at the Introduction of this section).

Remark 4. Thanks to a result due to Hardy, Littlewood and Polya in 1929, e.g., [6]), the comparison
\[
\int_0^s b(\tilde{w}(t, \sigma)) d\sigma \leq \int_0^s b(\tilde{U}(t, \sigma)) d\sigma \quad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty),
\]
lies that
\[
\int_0^s \Phi((\tilde{w}(t, \sigma))) d\sigma \leq \int_0^s \Phi(b(\tilde{U}(t, \sigma))) d\sigma \quad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty)
\]
any convex nondecreasing function \( \Phi \). In particular, if
\[ b \] is a concave function.
we get
\[ \int_{0}^{s} \tilde{u}(t, \sigma) d\sigma \leq \int_{0}^{s} \tilde{U}(t, \sigma) d\sigma \quad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty), \]

which is the conclusion presented at the Introduction of this section. Notice that a different application of the above result by Hardy, Littlewood and Poly is that
\[ ||b(u(t, \cdot))||_{L^q(\Omega)} \leq ||b(U(t, \cdot))||_{L^p(\Omega^*)} \]
for any \( q \in [1, \infty) \). Indeed, it suffices to use \( \Phi(r) = |r|^q \) and that
\[ \int_{0}^{m(\Omega)} |b(\tilde{u}(t, \sigma))|^q d\sigma = \int_{\Omega^*} |b(u^*(t, x))|^q dx = \int_{\Omega} |b(u(t, x))|^q dx. \]

3. THE FINITE EXTINCTION TIME PROPERTY

3.1. INTRODUCTION. One of the most natural questions concerning problem \((P)\) is the stabilization of solutions: Assume that
\[ f(t, \cdot) \longrightarrow f_\infty(\cdot) \quad \text{and} \quad h(t, \cdot) \longrightarrow h_\infty(\cdot) \quad \text{as} \quad t \rightarrow +\infty \]
in suitable functional spaces then \( u(t, \cdot) \longrightarrow u_\infty(\cdot) \) as \( t \rightarrow +\infty \) (in some suitable sense) with \( u_\infty(\cdot) \) solution of the associated stationary problem
\[ (P_\infty) \left\{ \begin{array}{l}
-\text{div} A(x, u_\infty, \nabla u_\infty) + g(x, u_\infty) = f_\infty(x), \quad x \in \Omega, \\
u_\infty = h_\infty, \quad \text{on} \ \partial \Omega.
\end{array} \right. \]

A general result, stated in terms of the omega limit set
\[ \omega(u) := \{ u_\infty \in W^{1,p}(\Omega) : \exists t_n \rightarrow \infty \text{ such that} \}
\[ u(t_n, \cdot) \rightarrow u_\infty \text{ in } L^p(\Omega), \text{ as } n \rightarrow \infty \}
\]
jointly with stronger convergence results (but for different particular case can be found in Díaz and de Thelin [25]. For stronger convergence result for one-dimensional particular equations see Feireisel and Simonon [28] at their references.

Very often \( f_\infty \equiv 0, h_\infty \equiv 0 \) and \( A \) and \( g \) are such that \( u_\infty \equiv 0 \) is the unique solution to problem \((P_\infty)\). In several applications (case of models in plasma physics and also in some chemical reactions) it is observed that there is a very strong stabilization in the following sense: there exists a finite time \( T_0 > 0 \) such that \( u(t, x) \equiv 0, \forall t \geq T_0 \text{ and a.e. } x \in \Omega \). This property is called:
The finite extinction time property and has been considered by many authors in the literature. The main goal of this section is to illustrate the application of the above two comparison principles to the study of this property. A third method (using energy arguments and so applicable to higher order parabolic problems and systems) will be also presented.

3.2. The finite extinction time via the pointwise comparison principle. A first result proving the occurrence of this property for some special formulation of problem (P) is the following

**Theorem 5.** Let $u$ satisfying

$$(P_{a,p}) \begin{cases} 
\left( |u|^{p-1} u \right)_t - \Delta_p u = 0, & t \in (0, \infty), x \in \Omega, \\
u = 0, & t \in (0, \infty), x \in \partial \Omega, \\
u(0, x) = u_0(x) & x \in \Omega,
\end{cases}$$

Then

$$u_0 \in C_c(\Omega), \text{ i.e., with supp } u_0 \text{ a compact subset of } \Omega.$$  \hspace{1cm} (19)

such that

$$(p - 1) < \alpha.$$  \hspace{1cm} (20)

then the finite extinction time property holds.

**Proof.** We assume $u$ in the class of solutions in which the pointwise comparison principle holds (due to the special formulation of $(P_{a,p})$ it can be shown (Benilan [11]) that this is our case for any $\alpha > 0$ and $p > 1$). Then if $(\underline{u})$ is a supersolution of problem $(P_{a,p})$ (resp. subsolution) then

$$\underline{u} \leq u \leq \overline{u}.$$  \hspace{1cm} (21)

if we are able to construct $\overline{u}$ (resp. $\underline{u}$) vanishing after a finite time this property also holds for $u$. Inspired in a pioneering paper (Sabinina [39]) we will construct $\overline{u}$ as a separable supersolution, i.e., $\overline{u}(t, x) = \Phi(t) w(x)$. Since want to have $\Phi \geq 0$ and $w \geq 0$, we define

$$N\overline{u} := \left( |\overline{u}|^{\alpha - 1} \overline{u} \right)_t - \Delta_p \overline{u} = (\Phi^\alpha)_t w^\alpha - \Phi^{p-1} \Delta_p w.$$  \hspace{1cm} (22)

take $\Phi$ such that

$$\begin{cases} 
(\Phi^\alpha)_t = -\lambda \Phi^{p-1}, & t \in (0, \infty), \\
\Phi(0) = M,
\end{cases}$$  \hspace{1cm} (22)
with \( \lambda > 0 \) and \( M > 0 \) to be determined. Due to the crucial assumption (2) the solution of (22) vanishes after a finite \( T_\Psi > 0 \) (notice that \( \Psi := \Phi^\alpha \) verifies an ODE with a term which is not Lipschitz \( \Psi_t + \lambda \Psi^{\frac{p-1}{\alpha}} = 0 \)). Notice also that (22) is integrable since it is a first order ordinary equation of separable variables. Then
\[
N \bar{u} = \Phi^{(p-1)} \left( -\lambda w^\alpha - \Delta_p w \right).
\]

In consequence we choose, as \( w \), the solution of the first eigenvalue problem for the \( \Delta_p \) operator, i.e., \( \lambda = \lambda_1 > 0 \) and
\[
\begin{align*}
-\Delta w &= \lambda_1 w^{p-1} \quad \text{on} \quad \Omega, \\
w &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
(22)

(the existence of a unique function \( w \) satisfying that \( w > 0 \) on \( \Omega \) or \( ||w||_{L^\infty(\Omega)} = 1 \) was due to Anane [3] and Barles [8]). Then
\[
N \bar{u} = \Phi^{p-1} \left( -\lambda_1 w^\alpha + \lambda_1 w^{p-1} \right) = \lambda_1 \Phi^{p-1} w^{p-1} \left( 1 - w^{\alpha-(p-1)} \right) \geq 0
\]
since \( 0 \leq w \leq 1 \) and \( \alpha > (p-1) \).

The boundary condition holds
\[
\bar{u}(t,x)|_{(0,\infty) \times \partial \Omega} = \Phi(t)w|_{\partial \Omega} = 0.
\]

The comparison between the initial data
\[
u_0(x) \leq M w(x), \quad x \in \Omega
\]
trivially holds by taking \( M \) big enough (recall the assumption (19) on \( u_0 \)). The construction of \( u \leq 0 \) is similar. \( \Box \)

Remark 5. The above statement can be improved in many different directions (but with longer proofs). For instance, in the case of \( p = 2 \) the homogeneity assumed on \( b \) is not needed. More precisely, in G. Díaz and J. Díaz [18], the finite extinction time property was established for the problem:
\[
\begin{align*}
b(u)_t - \Delta u &= f(x,t), \quad x \in \Omega, t > 0, \\
u &= 0, \quad x \in \partial \Omega, t > 0, \\
u(0,x) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]
by assuming
\[
\int_{0^+} \frac{ds}{b^{-1}(s)} < +\infty
\]
d the existence of $T_f$ such that $f(t, x) \equiv 0$, for $t > T_f$ and $x \in \Omega$. Notice at now $p = 2$ and that if $b(s) = |s|^{\alpha-1}s$ then (25) if and only if $\alpha > 1$, .., the same condition than (20). In fact, in this paper it is also shown that ndition (25) is also necessary for the existence of a finite extinction time.

Remark 6. Notice that the finite extinction time can not be satisfied (in se of the general formulation of (P)) each time that the strong maximum inciple holds (see, e.g., Nirenberg [37]) or the unique continuation property verified (see, e.g., Ghidaglia [29] and its references).

When condition (20) holds, it is said that we have a fast diffusion (in t, this term is more appropriate when talking on the balance between the accumulation and the diffusion terms). It is very easy to see that if we assume l) then the conclusion of the above theorem remains true under the presence a nondecreasing absorption term as, for instance,

$$\left(\left|u\right|^\alpha u\right)_t - \Delta_p u + \left|u\right|^{q-1} u = 0$$

any $q > 0$. The finite extinction time property also occurs due to suitable ance between the accumulation and absorption terms. It is the so called absorption case.

**Theorem 6.** Let $u$ satisfying

$$\begin{cases}
\left(\left|u\right|^\alpha u\right)_t - \Delta_p u + \left|u\right|^{q-1} u = 0, & t \in (0, \infty), \, x \in \Omega, \\
u = 0, & t \in (0, \infty), \, x \in \partial \Omega, \\
u(0, x) = u_0(x), & x \in \Omega,
\end{cases}$$

$$u_0 \in L^\infty(\Omega).$$

ume

$$\mu > 0 \text{ and } 0 < q < \alpha \text{ with } p > 1 \text{ arbitrary.}$$

n the finite extinction time property holds.

**Proof.** It is easy to see that the function $\bar{u}(x, t) = \Phi(t)$, with $\Phi$ the que) solution of the ODE

$$\begin{cases}
\left(\Phi^\alpha\right)_t + \mu \Phi^q = 0, & t \in (0, \infty), \\
\Phi(0) = M
\end{cases}$$

pare it with (22)) is a supersolution once that $M \geq \|u_0\|_{L^\infty(\Omega)}$. The mption (27) implies that $\Phi$ vanishes after some finite time $T_{\Phi}$. \qed
Remark 7. A general survey containing many references on this proper
is due to Kalashnikov [32].

3.3. The finite extinction time via the mass symmetrized comparison principle. Thanks to the mass symmetrized comparison principle,

it is possible to extend the last two theorems to more general equations for
which the construction of super and subsolutions can be very difficult (sp
ecially in the case of the first of the theorems).

Theorem 7. Let \( u \) be the solution of (P) with \( f \equiv 0, h \equiv 0, u_0 \in C_c(\Omega) \)

\( u_0 \geq 0 \) and assume \( b(u) = |u|^{\alpha - 1}u \). (14) and (9). We also suppose that one
of the following conditions holds:

\[
\begin{align*}
(p - 1) &< \alpha \quad \text{and} \\
\varphi(\eta) := \hat{g}(\eta)^{\frac{1}{\alpha - 1}} &\eta = \varphi_1(\eta) + \varphi_2(\eta), \eta \in \mathbb{R} \\
\text{with } \varphi_1 (\text{resp. } \varphi_2) \text{ nondecreasing and convex} \\
&\text{(resp. nondecreasing and concave),}
\end{align*}
\]

or

\[
\begin{align*}
\hat{g}(\eta) &= \mu |\eta|^{q - 1} \eta \quad \text{with } \mu > 0 \text{ and} \\
q &< \alpha.
\end{align*}
\]

Then the finite extinction time property is verified. More precisely, if we define
as \( T_{0,\Omega} \) the first extinction time (in which \( \|u(T_{0,\cdot})\|_{L^1(\Omega)} \equiv 0 \)) then

\[
T_{0,\Omega} \leq T_{0,\Omega^*},
\]

where \( T_{0,\Omega^*} \) is the first extinction time for the symmetrized problem \( (P^*) \).

Proof. By the mass symmetrized comparison principle and the result by
Hardy, Littlewood and Polya mentioned in the above Section we have that

\[
\|b(u(t,\cdot))\|_{L^1(\Omega)} \leq \|b(U(t,\cdot))\|_{L^1(\Omega)}
\]

for any \( t > 0 \). Assumption (29) (resp (30)) allows to apply Theorem 5 (resp
Theorem 8) which proves the result. \( \blacksquare \)

Remark 8. Notice that the general structure of \( A(x, u, \xi) \) may be the orig
of very complicated behaviors of the solution of the associated eigenvalt
problem

\[
\begin{align*}
-\text{div} A(x, w, \nabla w) &= \lambda w^{p - 1} \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

So that the arguments of the proof of Theorem 5 do not apply directly to
problem \( (P) \).
3.4. THE FINITE EXTINCTION TIME VIA AN ENERGY METHOD. A method
dich do not use any comparison principle can be applied to the study of this
property. The following is merely a special version of the method:

THEOREM 8. Let $u$ be the solution of $(P)$ with $h \equiv 0$,

\[ \begin{aligned}
& f \in L^\infty((0, \infty) \times \Omega) \text{ such that } \exists T_f > 0 \text{ with } \\
& f(t, x) \equiv 0 \text{ a.e. } t \geq T_f \text{ and a.e. } x \in \Omega,
\end{aligned} \tag{31} \]

\[ b(u) = |u|^{\alpha-1} u, \alpha > 0, \quad A \text{ satisfying (14) and } \]

\[ g(x, \eta) \geq 0 \quad \forall \eta \in \mathbb{R}. \tag{32} \]

sume that (20) holds (i.e., $p - 1 < \alpha$). Then the finite extinction property
ds.

Proof. We take as test function $v = |u|^{k-1} u$ (which we shall write, for
plicity, as $v = u^k$) with $k > 0$ to be determined later. We also write $u^\alpha$
read of $|u|^{\alpha-1} u$ by simplicity in the notation (nevertheless, it is not required
$u \geq 0$). Integrating on the open (bounded) set $\Omega$ in each term of the
ation we get:

\[ \int_\Omega \frac{\partial u^\alpha}{\partial t} u^k \, dx = \int_\Omega \alpha u^{(\alpha-1)+k} u \, dx \]

\[ = \frac{\alpha}{(\alpha + k)} \frac{d}{dt} \left( \int_\Omega u^{\alpha+k} \, dx \right) \]

\[ \text{justification of the final formula for } u \text{ weak solution of } (P), \text{ i.e., without}
condition } (u^\alpha)_t \in L^1(\Omega) \text{, is due to Alt and Luckhaus [2]),}

\[ - \int_\Omega \text{div } A(x, u, \nabla u) u^k \, dx = k \int_\Omega A(x, u, \nabla u) \cdot \nabla u u^{k-1} \, dx \]

\[ \geq k \int_\Omega |\nabla u|^p u^{k-1} \, dx. \]

using (31) and (32) we get that, if $t > T_f$, then

\[ \frac{\alpha}{(\alpha + k)} \frac{d}{dt} \int_\Omega u^{\alpha+k}(x, t) \, dx + k \int_\Omega |\nabla u|^p u^{k-1} \, dx \leq 0. \]

need the following interpolation result
Lemma 2. Let $p \geq 1$ and $k \geq 1$. There exists a constant $C = C(m(\Omega), N, k)$ such that if $w \in W_0^{1,1}(\Omega)$ and $\int_{\Omega} |\nabla w|^p |w|^{k-1} dx < +\infty$ we have that

$$\left( \int_{\Omega} |w|^s dx \right)^{\frac{p+k-1}{s}} \leq C k^p \int_{\Omega} |\nabla w|^p |w|^{k-1} dx$$

with

$$1 \leq s \leq \frac{N(p+k-1)}{N-p} \quad \text{if} \quad p < N,$$
$$1 \leq s \leq \infty \quad \text{if} \quad p = N,$$
$$s = \infty \quad \text{if} \quad p > N.$$

Idea of the proof of the Lemma. Define $z(x) = |w(x)|^{\frac{p+k-1}{p}} \text{sign}(w(x))$.

Then

$$\int_{\Omega} |\nabla z|^p dx = \left( \frac{p+k-1}{p} \right)^p \int_{\Omega} |\nabla w|^p |w|^{k-1} dx$$

and the conclusion follows from the application of the Poincaré-Sobolev and Hölder inequalities. \[\square\]

Continuation of the proof of Theorem 8. By the above lemma we have

$$\frac{\alpha}{\alpha + k} \frac{d}{dt} \left( \int_{\Omega} u^{\alpha+k}(t, x) dx \right) + C \left( \int_{\Omega} u^{s}(t, x) dx \right)^{\frac{p+k-1}{s}} \leq 0$$

for $t > T_f$. Applying Hölder inequality we get

$$\left( \int_{\Omega} u^{\alpha+k}(t, x) dx \right)^{\frac{1}{\alpha+k}} \leq C(n(\Omega)) \left( \int_{\Omega} u^{s}(t, x) dx \right)^{\frac{1}{s}}$$

(take $k = 1$ if $p \geq N$ and $k = \frac{N}{\alpha - (p-1)} - \alpha$ if $p < N$). Then if we define

$$Y(t) := \int_{\Omega} u^{\alpha+k}(t, x) dx$$

we have that

$$\left\{ \begin{array}{l}
Y'(t) + CY(t)^\gamma \leq 0 \quad \text{on} \quad (T_f, \infty), \quad \gamma = \frac{p+k-1}{\alpha+k} \in (0, 1), \\
Y(T_f) = Y_f > 0.
\end{array} \right.$$  

So, again, $\exists T_0 > T_f$ such that $Y(t) \equiv 0$ if $t \geq T_0$ and the conclusion holds.

Remark 9. Some similar energy method can be applied to the case of strong absorption (see, e.g., Tsutsumi [40]).
Remark 10. Under some extra decay assumptions on \( f(t, \cdot) \), near \( T_f \), it possible to show something unexpected: \( T_0 = T_f \) (see Antontsev and Díaz).

Remark 11. Similar energy methods applied to higher order quasilinear parabolic equations can be found in Bernis [14], [15].

Remark 12. One of pioneering applications of this type of energy methods concerning the case \( p = 2 \) and \( \Omega = \mathbb{R}^N \). In that case the condition for existence of a finite extinction time is

\[
\alpha > \frac{N}{N - 2},
\]

neger than \( \alpha > 1 \) correspondent to bounded domains (see Benilan and Véron [12]).

As a final and global remark we point out that the three methods used in this section can be also applied to the study of other different qualitative properties, as for instance, the existence of a finite blow-up time \( T_\infty \) (such \( \| b(u(t, \cdot)) \|_{L^r(\Omega)} \to +\infty \) as \( t \to +\infty \), for some \( r \in [1, +\infty) \)). Obviously, this property requires completely different assumptions on \( A, b \) and \( g \). The existence of the finite extinction time and the finite blow-up time for a couple of different nonlinear equations has been considered in Ohl and Peletier [34].

4. THE FINITE SPEED OF PROPAGATION PROPERTY

1. INTRODUCTION. The formulation of problem \( (P) \) is very general. It does not only the linear heat equation

\[
\frac{\partial u}{\partial t} - \Delta u = 0
\]

many other cases in which the behavior of the correspondent solutions is different to the one of the solution of the linear heat equation (remember remarks concerning the finite extinction time property as peculiar of fast or strong absorption and opposite to properties as the strong maximum principle or the unique continuation property which holds for the linear case).

Another qualitative property typical of some suitable nonlinear models are the finite speed of propagation of disturbances: if the initial datum
\( u_0 \) vanishes on a positively measured set of \( \Omega \) (i.e., \( \text{supp}(u_0) \subset \Omega \)) then \( \text{supp} u(t, \cdot) \subset \Omega \), for any \( t \in (0, t^*) \), for some \( t^* > 0 \).

This behavior (typical of the linear wave equation) fails for the linear heat equation (this can be illustrated in many ways: the strong maximum principle, the explicit representation formula for \( \Omega = \mathbb{R}^N \), etc). It is said that the linear heat equation has an \textit{infinite} speed of propagation.

When the finite speed of propagation holds then

\[
\text{supp} (u(t, \cdot)) := \{ x \in \Omega : u(t, x) \neq 0 \} \subset \Omega
\]

(at least for some small times \( t \)) and so some hypersurfaces \((0, \infty) \times \mathbb{R}^N \)

\[
\mathcal{F} = \bigcup_{t > 0} \mathcal{F}(t), \quad \mathcal{F}(t) = \partial (\text{supp} u(t, \cdot)) - \partial \Omega
\]

are formed. Those hypersurfaces are called as \textit{free boundaries} (since they are not a priori determined) and play a very important role in the study of the model (usually is in those free boundaries where are located the singularities of the gradient and/or the second derivatives of the solutions).

The main goal of this section is to illustrate how the two comparison principles can be applied to the study of the occurrence of this property. In the previous section, a third method (involving different energy arguments) will be also presented.

### 4.2. The Finite Speed of Propagation via the Pointwise Comparison Principle

As in the Subsection 3.2, the main idea will be to construct suitable super and subsolutions (now vanishing locally in some subdomain). In fact, those functions use to be constructed by modifying \textit{special solutions} of the equation (so this task is closer to a quantitative study of pde's than the usual approach to pde's by methods of \textit{functional analysis}).

To start with, let us consider the nonlinear equation

\[
\left( |u|^{\alpha-1} u \right)_t - \Delta_p u = 0, \quad \alpha > 0, \quad p > 1.
\]

Although we remain interested in the Cauchy-Dirichlet problem \( (P_{\alpha,q}) \), it is useful to start by considering the pure Cauchy problem (i.e., \( \Omega = \mathbb{R}^N \)). The very important family of \textit{exact solutions} is the one given by

\[
U_M(t, x) = \frac{1}{t^{\lambda}} \left[ C - k \sum_{i=1}^{N} |x_i|^{\beta p'} \right]^{(p-1)/(p-1-\alpha)} + \]

(\cdot)
ch arises when

\[(p - 1) > \alpha \tag{36}\]

tice that the fast diffusion was \((p - 1) < \alpha\), where

\[
p' = \frac{p}{p - 1}, \quad \beta = \frac{\alpha}{\alpha + 1(p - N) + (N - 1)p'},
\]

\[
\lambda = \frac{N\beta}{\alpha} \quad \text{and} \quad k = \beta(p' - 1) \frac{p - 1 - \alpha}{p},
\]

> 0 arbitrary). Such solutions were obtained, by first time, by G. I. 
enblatt in 1952 for the case \(p = 2\) (also in the case, they were refunded by 
E. Pattle in 1959). The case \(p \neq 2\) was found by A. Bamberger in 1975. 
point out that when \(p \neq 2\) the solution \(U_M\) is not radially symmetric with 
ext to the usual Euclidean norm of \(\mathbb{R}^N\). Nevertheless, it is possible to find 
xact solutions with free boundaries and symmetry (although they are 
so explicit as \(U_M\)). Many references on this topic can be found in the 
eys by Kalashnikov [32] and [42]. We also point out that:

\[
\int_{\mathbb{R}^N} U_M(t, x) dx = M, \quad M = M(C, \alpha, p, N),
\]

\[
U(t, \cdot) \rightarrow M\delta_0(x),
\]

that the free boundary generated by \(U_M\) is explicitly given by the equation

\[
\sum_{i=1}^{N} \left| x_i \right|^{p'} = \frac{C}{k} t^p'.
\]

A simple result is the following.

**Theorem 9.** Let \(u\) satisfying

\[
\begin{cases}
\left( |u|^{\alpha-1} u \right)_t - \Delta_p u = 0, & t \in (0, \infty), x \in \Omega, \\
u = 0, & t \in (0, \infty), x \in \partial \Omega, \\
u(0, x) = u_0(x), & x \in \Omega,
\end{cases}
\]

\[
\begin{cases}
u_0 \in C_c(\Omega) \quad \text{such that} \\
supp u_0 \subset B(x_0, R_0) \subset \Omega.
\end{cases}
\]

me that

\[(p - 1) > \alpha. \tag{37}\]

the finite speed of propagation holds.
\textbf{Proof.} As in Theorem 5, we can apply the pointwise comparison principle thanks to the result by Benilan [11]. By choosing \( M \) big enough and that to the assumption (37) we have that
\[
u_0(x) \leq U_M(\tau, x - \widehat{x}_0) \quad \forall x \in \Omega,
\]
for some \( \tau > 0 \). Since the function \( \overline{u}(t, x) := U_M(t + \tau, x - \widehat{x}_0) \) satisfies the system
\[
\begin{cases}
    \left( |\overline{u}|^{\alpha-1} \overline{u} \right)_t - \Delta_p \overline{u} = 0, & t \in (0, \infty), x \in \Omega, \\
    \overline{u} \geq 0, & t \in (0, \infty), x \in \partial\Omega, \\
    \overline{u}(0, x) \geq u_0(x), & x \in \Omega,
\end{cases}
\]
we conclude that
\[
u(t, x) \leq \overline{u}(t, x) \quad t > 0, x \in \Omega.
\]
By taking (if needed) different values of \( M \) and \( \tau \) we get, similarly that
\[-U_M(t + \tau, x - \widehat{x}_0) \leq u(t, x) \quad x \in \Omega, t > 0.
\]
Thus, at least for \( t \in [0, t^*] \) with \( t^* \) small enough, we conclude that
\[
u(t, x) \equiv 0 \quad \text{a.e.} \quad x \in \Omega - B(\widehat{x}_0, R(t))
\]
for some function \( R(t) \) and the result follows. \( \blacksquare \)

\textbf{Remark 13.} Again, the above statement can be improved in many different directions. For instance, in the case \( p = 2 \) we can replace \( b(u) = |u|^{\alpha-1} u \) by a general nondecreasing function satisfying that
\[
\int_{0^+} \frac{ds}{b(s)} < +\infty
\]
and the finite speed of propagation holds (see Díaz [19]). Notice that if \( p = \) and \( b(u) = |u|^{\alpha-1} u \) then (39) holds if and only if \( \alpha < 1 \), i.e., same condition as (38). If \( N = 1 \) (and \( p = 2 \)) it was proved by A.S. Kalashnikov (independently by L. A. Peletier) in 1974, that condition (39) is also necessary.

\textbf{Remark 14.} Once that the free boundary exists it becomes interesting to study its dynamics: how fast it starts near \( t = 0 \) (in some cases this is a waiting time), how it behaves for \( t \to +\infty \), the regularity of the free boundary, etc. Many of those questions remain still open (see the survey by Kalashnikov [32]).
When assumption (38) holds it is said that we have a slow diffusion. It easy to see that if (38) holds then the finite speed of propagation remains under the presence of nondecreasing absorption term as, for instance,
\[
(|u|^{\alpha-1}u)_t - \Delta_p u + \mu |u|^{q-1}u = 0, \quad \mu > 0,
\]
for any \( q > 0 \). The finite speed of propagation also occurs when the balance between the diffusion and absorption is suitable (called again as the strong absorption case). We can consider, even, the case of nonhomogeneous boundary conditions.

**Theorem 10.** Let \( u \) satisfying

\[
(P_{\alpha,p,q}) \begin{cases} 
(|u|^{\alpha-1}u)_t - \Delta_p u + \mu |u|^{q-1} = 0, & t \in (0, \infty), x \in \Omega, \\
u = h, & t \in (0, \infty), x \in \partial\Omega, \\
u(0, x) = u_0(x), & x \in \Omega,
\end{cases}
\]

then

\[
u \in L^\infty((0, \infty) \times \Omega) \cap L^p_{loc}(0, \infty : W^{1,p}(\Omega)), \quad h \geq 0 \text{ on } (0, \infty) \times \partial\Omega, \quad (40)
\]

\[
u = L^\infty(\Omega), \quad u_0 \geq 0 \text{ on } \Omega. \quad (41)
\]

sume

\[
u > 0 \quad \text{and} \quad 0 < q < p - 1. \quad (42)
\]

en the finite speed of propagation holds. More precisely: a) There exists a positive constant \( L > 0 \) such that the null set of \( u(\cdot, \cdot) \) is not empty assumed it the set

\[
\Omega - \left( \text{supp}(u_0) \cup \bigcup_{\tau > 0} \text{supp}(h(\tau, \cdot)) \right)
\]

gig enough, i.e.,

\[
\mathcal{I}(u(\cdot, \cdot)) := \{ x \in \Omega : u(t, x) = 0 \} \supset
\]

\[
\left\{ x \in \Omega; d(x, \text{supp}(u_0) \cup \bigcup_{\tau > 0} \text{supp}(h(\tau, \cdot))) \geq L \right\}
\]

any \( t > 0 \). b) If we assume, in addition, that

\[
q < \alpha \leq 1 \quad (43)
\]

n there exists \( t_0 \geq 0 \) such that for every \( t \geq t_0 \)

\[
N(u(t, \cdot)) \supset \left\{ x \in \Omega : d(x, \cup_{\tau > 0} \text{supp}(h(\tau, \cdot))) \geq \tilde{L} \right\}
\]

some \( \tilde{L} > 0. \)
Proof. We recall a result of Díaz [20] proving that the function
\[ w_\lambda(x) = C_\lambda^* |x - x_0|^{\frac{p-1}{q-1}}, \]
\[ C_\lambda^* = \left[ \frac{\lambda(p-1-q)\lambda^p}{p^{(p-1)}(pq + N(p-1-q))} \right]^{\frac{1}{p-1-q}} \]
satisfies that
\[ -\Delta_p w_\lambda + \lambda|w_\lambda|^{q-1}w_\lambda = 0, \]
assumed that (42) holds, i.e., \( \lambda > 0 \) and \( q < p-1 \). Let us prove a). Let \( x_0 \in \Omega \) (\( \text{supp} u_0 \cup \bigcup_{\tau > 0} \text{supp} h(\tau, \cdot) \)), and let \( R = d(x, (\text{supp} u_0 \cup \bigcup_{\tau > 0} \text{supp} h(\tau, \cdot))) \). Consider \( \Omega(x_0) := B(x_0, R) \cap \Omega \). Then \( \overline{u}(t, x) := W_\mu(x) \) is a local supersolution, i.e., a supersolution on \( \Omega(x_0) \) since
\[
\begin{align*}
(\|\overline{u}\|^{q-1}\overline{u})_t - \Delta_p \overline{u} + \mu|\overline{u}|^{q-1}\overline{u} &= 0 & \text{on} & \ (0, \infty) \times \Omega(x_0), \\
\overline{u}(0, x) &\geq 0 = u_0(x), & \text{on} & \ \Omega(x_0), \\
\overline{u}(t, x) &\geq 0 = h(t, x) & \text{on} & \ (0, \infty) \times \Omega(x_0) \cap \partial \Omega,
\end{align*}
\]
and the condition
\[ \overline{u}(t, x) \geq u(t, x) \quad \text{on} \quad (0, \infty) \times \partial \Omega(x_0) - \partial \Omega, \]
is satisfied if, for instance,
\[ C_\mu R^{\frac{p}{p-1-q}} \geq \|\overline{u}\|_{L^\infty((0, \infty) \times \Omega)} \quad (\geq u(t, x) \quad \text{a.e.}(t, x)), \]
i.e., if
\[ R \geq \left[ \frac{\|\overline{u}\|_{L^\infty((0, \infty) \times \Omega)}}{C_\mu} \right]^{\frac{p-1-q}{p}} \]
(notice that \( \|\overline{u}\|_{L^\infty((0, \infty) \times \Omega)} < \infty \) thanks to the assumptions on \( h \) and \( u_0 \), we can prove in many ways: for instance by using a suitable global supersolution). Then by the pointwise comparison principle on \( (0, \infty) \times \Omega(x_0) \), we obtain that
\[ 0 \leq u(t, x) \leq C_\mu^* |x - x_0|^{\frac{p-1}{q-1}} \]
and so \( u(t, x_0) = 0 \) (even if \( u \) is not necessarily continuous).

To prove part b) we take as local supersolution the function
\[ \overline{u}(t, x) := w_{\mu/2} + V(t) \]
\[ \begin{align*}
\frac{d}{dt} \left( |V|^{\alpha-1} V \right) + \frac{\mu}{2} |V|^{q-1} V &= 0,
V(0) &= ||u_0||_{L^\infty(\Omega)},
\end{align*} \tag{44} \]

Then

\[ V(t) = \left[ ||u_0||^{\alpha-q}_{L^\infty(\Omega)} - \frac{\mu(\alpha-q)}{2\alpha} t \right]^{\frac{1}{\alpha-q}}. \tag{45} \]

Denote

\[ (|\bar{u}|^{\alpha-1} \bar{u})_t = \alpha (w_{\mu/2}(x) + V(t))^{\alpha-1} \dot{V} \geq \frac{d}{dt} (|V|^{\alpha-1} V), \]

\[ \Delta_p \bar{u} = \Delta_p w_{\mu/2}, \]

\[ \mu |\bar{u}|^{q-1} \bar{u} \geq \frac{\mu}{2} |w_{\mu/2}|^{q-1} w_{\mu/2} + \frac{\mu}{2} |V|^{q-1} V, \]

d so

\[ \left( |\bar{u}|^{\alpha-1} \bar{u} \right)_t - \Delta_p \bar{u} + \mu |\bar{u}|^{q-1} \bar{u} \geq 0. \]

recover

\[ \bar{u}(0, x) = w_{\mu/2} + V(0) \geq ||u_0||^{\alpha-q}_{L^\infty(\Omega)} \geq u_0(x). \]

Finally, taking

\[ t_0 = \frac{2\alpha}{\mu(\alpha-q)} ||u_0||^{\alpha-q}_{L^\infty(\Omega)} \]

get that \( V(t) \equiv 0 \quad \forall t \geq t_0 \) and the conclusion follows as in part a).

\begin{remark}
The above result is taken from Díaz and Hernández [23] where, and more general, results can be found.
\end{remark}

\begin{remark}
In the model of chemical reactions, the null set \( N(u(t, \cdot)) \) is called as dead core. In that model usually \( h(t, x) \equiv 1 \) and so \( N(u(t, \cdot)) \) only occurs at the interior of \( \Omega \).
\end{remark}

\begin{remark}
Notice that if \( h \equiv 0 \) part b) shows the extinction in finite time. Notice also that assumptions (42) (in addition to (43)) implies the notion of dead core for \( t \) large even for \( h \equiv 1 \) and \( u_0 > 0 \). This property is similar to the so-called instantaneous shrinking of the support established by Brezis and Friedman in 1976, or by Evans and Knerr in 1979, for the case of \( \Omega = \mathbb{R}^N \) and \( u_0 > 0 \) such that \( \lim_{|x| \to \infty} u_0(x) = 0 \) (see references in the survey Kalashnikov [32]).
\end{remark}
4.3. The finite speed of propagation via the mass symmetrize comparison principle. The above method requires the construction sophisticated supersolutions. This is possible only for simple nonlinear operators. The application of the mass symmetrized comparison principle shows how important is to have symmetry conditions on the partial differential equation in order to have solutions with small support.

Theorem 11. Let $u$ be the solution of $(P)$ with $f \equiv 0$, $h \equiv 0$, $u_0 \in C_c(\Omega)$, $u_0 \geq 0$ and assume $b(u) = |u|^{\alpha - 1}u$, $(14)$ and $(9)$. We also suppose the following conditions

\[
\begin{align*}
(p - 1) &> \alpha, \\
\phi(\eta) := \tilde{\varphi} \left( |\eta|^{\frac{1}{p - 1}} \eta \right) &= \varphi_1(\eta) + \varphi_2(\eta), \quad \eta \in \mathbb{R} \\
\text{with } \varphi_1 \text{ (resp. } \varphi_2) &\text{ nondecreasing convex} \\
\text{(resp. nondecreasing concave),}
\end{align*}
\]

and

\[
\int_\Omega b(u(t, x)) dx = \int_{\Omega^*} b(U(t, x)) dx, \quad \forall t \geq 0, \tag{4'}
\]

where $U$ denotes the solution of the symmetrized problem. Then the supports of $u(t, \cdot)$ satisfy

\[
m \left( \text{supp } u(t, \cdot) \right) \geq m \left( \text{supp } U(t, \cdot) \right) \tag{4'}
\]

for any $t > 0$.

Proof. By using the mass symmetrized comparison principle, $(46)$ and the

\[
\int_\Omega b(u(t, x)) dx = \int_0^{m(\Omega)} b(\tilde{u}(t, \sigma)) d\sigma
\]

we have

\[
\begin{align*}
\int_\mathbb{R} b(\tilde{u}(t, \sigma)) d\sigma &= \int_0^{m(\Omega)} b(\tilde{u}(t, \sigma)) d\sigma - \int_0^g b(\tilde{u}(t, \sigma)) d\sigma \\
&\geq \int_0^{m(\Omega)} b(\tilde{U}(t, \sigma)) d\sigma - \int_0^g b(\tilde{U}(t, \sigma)) d\sigma.
\end{align*}
\]

Let

- support of $\tilde{u} = [0, R_u(t)], \quad 0 < R_u(t) \leq m(\Omega)$
- support of $\tilde{U} = [0, R_U(t)], \quad 0 < R_U(t) \leq m(\Omega)$
Recall that \( \tilde{u} \) and \( \tilde{U} \) are nondecreasing functions. Then, necessarily \( R_u(t) \geq \gamma(t) \) since otherwise we would deduce that
\[
\int_{R_u(t)}^{m(\Omega)} b(\tilde{u}(t, \sigma)) d\sigma \geq \int_{R_u(t)}^{R_U(t)} b(\tilde{U}(t, \sigma)) d\sigma > 0
\]
which is a contradiction. Finally, it suffices to remark that
\[
\tilde{u}(t, \cdot) = [0, m(\text{supp } u(t, \cdot))]
\]
and analogously for \( U \) and the conclusion holds. \( \blacksquare \)

Remark 18. Notice that by (47) if \( \text{supp } U(t^*, \cdot) = \Omega \), for some \( t^* > 0 \), then \( \text{pp } u(t^*, \cdot) = \Omega \).

Remark 19. Assumption (46) is satisfied, for instance, when the conservation of the mass holds, i.e.,
\[
\int_{\Omega} b(u(t, x)) dx = \int_{\Omega} b(u_0(x)) dx, \quad \forall t > 0.
\]
that case \( \int_{\Omega} b(u_0(x)) dx = \int_{\Omega} b(U_0(x)) dx = \int_{\Omega} b(U(t, x)) dx \) and (46) is verified. The conservation of the mass is typical of pure diffusion processes \( g = \overline{g} \). It can be shown (see Díaz [21]) that assumption (46) is verified when, besides the Dirichlet condition \( u(t, x) = 0 \), \( t > 0 \), \( x \in \partial \Omega \), have the additional information that
\[
\frac{\partial u}{\partial n}(t, x) = 0 \quad \text{for } t \in (0, \widehat{T}), \quad x \in \partial \Omega,
\]
some \( \widehat{T} > 0 \) (in that case the conclusion (47) holds at least for \( t \in [0, \widehat{T}) \)).

In the case of strong absorption we can allow a nonzero Dirichlet condition

\textbf{Theorem 12.} Let \( u \) be the solution of (P) with \( f \equiv 0 \) and
\[
h(t, x) \equiv h, \quad \text{a positive constant.}
\]
\[
u_0 \in L^\infty(\Omega) \text{ with } 0 \leq u_0(x) \leq h, \quad \text{a.e. } x \in \Omega.
\]

\[
(48) \quad (49)
\]
Assume \( b(u) = |u|^{q-1}u, (14), (9) \) and
\[
\tilde{g}(\eta) = \mu|\eta|^{q-1}\eta \quad \text{with } \mu > 0 \text{ and } q < (p - 1).
\]

Then the supports of \( u(t, \cdot) \) and \( U(t, \cdot) \) satisfy that
\[
m(\text{supp } u(t, \cdot)) \geq m(\text{supp } U(t, \cdot)) \quad \text{for } t > 0.
\] (5)

Idea of the proof. By introducing the change of variables \( v(t, x) = h_u(t, x) \) and \( V(t, x) = h - U(t, x) \) we can apply the mass symmetrized comparison principle to \( v \) and \( V \). Finally, it suffices to apply the result by Hard Littlewood and Polya for an appropriate choice of convex function \( \Phi \) (see Díaz [21]). ☐

Remark 20. Estimates (47) and (50) allows to compare the waiting time (when arising) for \( u \) and \( U \).

Remark 21. Estimate (50) shows that the dead core has a bigger measure under radially symmetric conditions. That was first observed in Bandle and Stakgold [7].

4.4. THE FINITE SPEED OF PROPAGATION VIA AN ENERGY METHODE

The study of the finite speed of propagation (and other qualitative properties) can be carried out by using some energy arguments which, in contrast with the ones of Section 3, now have a local character.

Theorem 13. Let \( A \) satisfying (4) and
\[
|A(x, u, \xi)| \leq C|\xi|^{p-1}.
\]
Let \( g(x, u) \) such that
\[
g(x, \eta)\eta \geq 0 \quad \forall \eta \in \mathbb{R}.
\]
Assume
\[
\alpha < (p - 1)
\]
and let \( u \) be a local solution of the equation
\[
(|u|^{\alpha-1}u)_t - \text{div} A(x, u, \nabla u) + g(x, u) = 0 \quad \text{on } (0, \infty) \times B(x_0, R)
\]
\( x \) some \( x_0 \in \mathbb{R}^N, R > 0 \) such that
\[
 u(0, x) = 0 \quad \text{a.e.} \quad x \in B(x_0, \rho_0), \quad \rho_0 < R.
\]

Then there exists \( t^* > 0 \) and \( \rho : [0, t^*] \rightarrow [0, \rho_0] \) nondecreasing such that
\[
 u(t, x) = 0 \quad \text{a.e.} \quad x \in B(x_0, \rho(t)).
\]

Idea of the proof. By multiplying by \( u \) and integrating by parts we get
\[
 \frac{\alpha}{\alpha + 1} \int_{B_\rho} |u(t, x)|^{\alpha + 1} dx + \int_0^t \int_{B_\rho} A(x, u, \nabla u) \cdot \nabla u dx \, ds
\]
\[
 \leq \int_0^t \int_{\partial B_\rho} u A(x, u, \nabla u) \cdot n d\Gamma ds
\]

This can be rigorously justified from the notion of bounded weak local solution. Here \( B_\rho = B(x_0, \rho) \). We introduce the local energies
\[
 E(t, \rho) := \int_0^t \int_{B_\rho} A(x, u, \nabla u) \cdot \nabla u dx \, ds
\]

and
\[
 b(t, \rho) := \text{ess sup}_{s \in (0, t)} \left( \frac{\alpha}{\alpha + 1} \int_{B_\rho} |u(s, x)|^{\alpha + 1} dx \right).
\]

Using Hölder inequality we get that
\[
 b + E \leq \frac{1}{pc} \left( \int_0^t \int_{B_\rho} |u|^p dx \, ds \right) \left( \frac{\partial E}{\partial \rho} \right)^{\frac{p-1}{p}}
\]

where we used that
\[
 \frac{\partial E}{\partial \rho}(t, \rho) = \int_0^t \int_{\partial B_\rho} A(x, u, \nabla u) \cdot \nabla u d\Gamma ds.
\]

We need the following

**Lemma 3.** (Interpolation-trace inequality) For any \( \sigma \in [0, p - 1] \) there is \( C > 0 \) and \( \theta \in [0, 1] \) such that for any \( w \in W^{1, p}(G), G \) open bounded of \( \mathbb{R}^N \), we have
\[
 ||w||_{L^p(\partial \Omega)} \leq C \left( ||\nabla w||_{L^p(G)} + ||w||_{L^{p+1}(G)} \right)^\theta \left( ||w||_{L^{p+1}(G)} \right)^{1-\theta}.
\]
Applying the Lemma and Young inequality we obtain that

\[ E^\gamma \leq C t^{1-\gamma \over 1-\gamma} \left( \frac{\partial E}{\partial \rho} \right) \]

for some exponent \( \gamma \in (0, 1) \). This implies the result. \( \square \)

**Remark 22.** Notice that the result holds without making explicit the boundary conditions. It has a local nature.

**Remark 23.** The first local energy method was due to S.N. Antontsev in 1981. A rigorous justification of his arguments, containing also several improvements, was made in Díaz and Veron [27].

**Remark 24.** Other qualitative properties (as the formation of dead core, the instantaneous shrinking of the support, etc) can be proved by this type of local energy arguments. See, e.g., Antontsev, Díaz and Shmarev [5]. These authors are preparing a book containing many other applications.

**Remark 25.** For the application of this type of arguments to higher order equations see Bernis [14], [15] and their references.

As a global, and final, remark we mention that the finite speed of propagation, the finite extinction time and other qualitative properties can be analyze for hyperbolic first order equations of the type

\[ \frac{\partial u}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} \Phi_i(u) + g(x, u) = f(t, x) \]

see Díaz and Veron [26] and Díaz and Kružkov [24].

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NONLINEAR PARABOLIC EQUATIONS

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