On a stochastic parabolic PDE arising in Climatology

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Abstract. We study the existence and uniqueness of solutions of a nonlinear stochastic pde proposed by R. North and R. F. Cahalan in 1982 for the modeling of non-deterministic variability (as, for instance, the volcano actions) in the framework of energy balance climate models. The more delicate point concerns the uniqueness of solutions due to the presence of a multivalued graph $\beta$ in the right hand side of the equation. In contrast with the deterministic case, it is possible to prove the uniqueness of a suitable weak solution associated to each given monotone (univalued and discontinuous) section $b$ of the maximal monotone graph $\beta$. We get some stability results when the white noise converges to zero.

Sobre una ecuación estocástica en derivadas parciales de tipo parabólico que surge en Climatología

Resumen. Estudiamos la existencia y unicidad de soluciones de una ecuación estocástica en derivadas parciales de tipo parabólico propuesta por R. North y R. F. Cahalan en 1982 para la modelización de variabilidad no determinista (como es el caso, por ejemplo, de la acción de volcanes) en el marco de los modelos de balance de energía. El punto más delicado se refiere a la unicidad de soluciones debido a la presencia de un gráfico multívoco $\beta$ en el término de la derecha de la ecuación. En contraste con el caso determinista, es posible mostrar la unicidad de una cierta noción de solución débil asociada a cada sección monótona (unívoca y discontinua) $b$ del gráfico máximo monótono $\beta$. En esta nota se dan unos resultados de estabilidad cuando el ruido blanco converge a cero.

1. Introduction

This note deals with the nonlinear stochastic pde

\[
\begin{aligned}
\mathbb{E}_{\beta, x} \left\{ \begin{array}{l}
\frac{du}{dt} + Du + f(x, t) = \varepsilon \mathcal{W}, \\
\varepsilon (x, t) \in (-1, 1) \times \mathbb{R}_+, \\
\varepsilon (x, 0) = \varepsilon_0(\varepsilon), \\
\varepsilon (x, 0) = 0,
\end{array} \right.
\end{aligned}
\]

where $D$ and $\varepsilon$ are positive constants,

\((H_0)\) $\beta$ is a bounded maximal monotone graph of $\mathbb{R}^2$, i.e., $m \leq z \leq M$, $\forall z \in \beta(s), \forall s \in \mathbb{R}$

\((H_x)\) $S : \mathcal{M} \to \mathbb{R}, S \in L^\infty(-1, 1), S_1 \geq S(x) \geq S_0 > 0$ a.e. $x \in (-1, 1)$

\(\varepsilon_0 \in C([-1, 1]), f \in L^\infty((-1, 1) \times \mathbb{R}_+)$ and the term $\mathcal{W}$ denotes a space-time white noise.

This kind of problems where proposed by R. North and R. F. Cahalan in 1982 ([5]) for the modeling of non-deterministic variability (as, for instance, the volcano actions) in the context of energy balance climate

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models. We recall that the distribution of temperature \( u(x, t) \) is expressed pointwise after a standard average process, where the spatial variable \( x \) is given by \( x = \sin \theta \) and \( \theta \) is the latitude. Notice that, for simplicity, we are replacing the natural degenerate diffusion term \((1 - x^2)u_x\) by the usual 1d-Laplacian operator and that the absence of boundary conditions for the degenerate diffusion is corrected by adding Neumann type boundary conditions since in the degenerate model the meridional heat flux \((1 - x^2)u_x\) vanishes at the poles \( x = \pm 1 \). The balance of energies leads to the problem

\[
\begin{align*}
  u_t - u_{xx} &= R_a - R_e, & (x, t) \in (-1, 1) \times \mathbb{R}_+, \\
  u_x(-1, t) &= u_x(1, t) = 0, & t \in \mathbb{R}_+, \\
  u(x, 0) &= u_0(x), & x \in (-1, 1),
\end{align*}
\]

where the terms \( R_a \) and \( R_e \) must be specified by means of constitutive laws (see, e.g., [5], [1] and [2]). The absorbed energy \( R_a \) depends, in a fundamental way, on the planetary albedo \( \beta \) representing the fraction of the incoming radiation flux which is absorbed by the surface. In ice-covered zones, reflection is greater than over oceans, therefore, the albedo is smaller. So, there is a sharp transition between zones of high and low albedo. In the energy balance climate models, a main change of the albedo occurs in a neighborhood of a critical temperature for which ice become white, usually taken as \( u = -10^o \). In the so called Budyko model the different values of the albedo are modeled by means of a discontinuous function of the temperature. As usual in pde, this function can be understood in the more general context of the maximal monotone graphs of \( \mathbb{R}^2 \). In particular, we assume that

\[
\beta(u) = \begin{cases} 
  m, & \text{if } u < -10, \\
  [m, M], & \text{if } u = -10, \\
  M, & \text{if } u > -10,
\end{cases}
\]

where \( m = \beta_i \) and \( M = \beta_w \) represent the albedo in the ice-covered zone and the free-ice zone, respectively and \( 0 < \beta_i < \beta_w < 1 \) (the value of these constants has been estimated by observation from satellites). In contrast to the above assumption, in the so called Sellers model \( \beta \) is assumed to be a more regular function (at least, Lipschitz continuous) piecewise constant function far from a neighborhood of \( u = -10 \). In both models, the whole absorbed energy is given by \( R_a = Q S(x) \beta(u) \) where \( S(x) \) is the insolation function and \( Q \) is the so-called solar constant.

The Earth's surface and atmosphere, warmed by the Sun, emit part of the absorbed solar flux as an infrared long-wave radiation. This energy \( R_e \) is represented, following the proposal by Budyko, by \( R_e = Bu - f(x, t) \). Here, \( B \) and \( f \) are obtained, again, by observation and depend on the greenhouse effect.

The main goal of this note is to present the mathematical analysis of the model ([5] was limited to the application of the Fourier method to the linear case \( \beta = 0 \)). We recall that in the deterministic case \((\varepsilon = 0)\) the existence of solutions was given in [1] (see [2] for the generalization to bidimensional models). When \( \beta \) is as the Sellers albedo this solution is unique, nevertheless, if \( \beta \) is multivalued it was shown there that there is lack of uniqueness of solutions except in the class of the, so called, non degenerate solutions. As we shall specify later, a curious fact is produced for problem \((E_{\beta, \varepsilon})\): the presence of a stochastic perturbation produces the uniqueness of the solutions associated to any given monotone (univalued and discontinuous) section \( \beta \) of the maximal monotone graph \( \beta \), (i.e. a function such that \( b(r) \in \beta(r) \), for any \( r \in \mathbb{R} \)).

In this note we use some previous results due to I. Gyöngy and E. Pardoux ([3],[4]) in order study the stability of solutions when \( \varepsilon \to 0 \). We shall show that the associated solution \( u^{\varepsilon, \delta} \) converges to a solution \( u^\delta \) of the deterministic problem and we characterize the limit for the case of the two distinguished sections of \( \beta \).

2. The Seller colbedo case

In this section we consider the formulation corresponding to the Sellers colbedo function. In fact, it is useful to start with a bounded truncation of the complete deterministic source function

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\[ F(x, t, r) = QS(x)\beta(r) - BT_n(r) + f(x, t) \]

where \( n \in \mathbb{N} \) and \( T_n(r) = \min\{|r|, n\} \, \text{sign}(r) \). We rewrite the problem as

\[
\left\{ \begin{array}{l}
u_t = u_{xx} + F(x, t, u) + \varepsilon \mathcal{W}, & \text{in } [-1, 1] \times \mathbb{R}_+, \\
u_x(-1, t) = u_x(1, t) = 0, & t \in \mathbb{R}_+, \\
u(\cdot, 0) = u_0(\cdot), & \text{on } [-1, 1].
\end{array} \right.
\]

Notice that \( F \) is a bounded Lipschitz function and so, in particular,

\[ |F(x, t, r) - F(x, t, \bar{r})| \leq K|r - \bar{r}|, \quad (2) \]

for some positive constant \( K \) and for \( x, y \in [-1, 1] \), \( t \in \mathbb{R}_+, r, \bar{r} \in \mathbb{R} \).

We recall that the notion of weak solution corresponds to \( B([-1, 1]) \otimes \mathcal{P} \) measurable and continuous random field \( \{u^\varepsilon(x, t)\}_{(x, t) \in [-1, 1] \times \mathbb{R}_+} \) such that

\[ \int_{-1}^{1} u^\varepsilon(y, t)\varphi(y)dy = \int_{-1}^{1} u_0(y)\varphi(y)dy + \int_{0}^{t} \int_{-1}^{1} [u^\varepsilon(y, s)\varphi''(y) + F(u^\varepsilon)(y, s)\varphi(y)] dyds \\
+ \varepsilon \int_{0}^{t} \int_{-1}^{1} \varphi(y)\mathcal{W}(dy, ds), \quad t \geq 0, \mathbb{P} \text{-a.s.}, \quad (3) \]

for \( \varphi \in C^2([-1, 1]) \cap \mathcal{C}([-1, 1]) \), \( \varphi(-1) = \varphi(1) = 0 \). Here \( F(u)(y, s) = F(y, s, u(y, s)) \), \( \mathcal{W} \) is a space-time white noise on a filtered probability space \( (\mathcal{O}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \), \( B([-1, 1]) \) is the collection of all the Borel sets of \([-1, 1]\) and \( \mathcal{P} \) is the \( \sigma \)-algebra of the progressively measurable subsets of \( \mathcal{O} \times \mathbb{R}_+ \).

**Theorem 1** ([6], [3]) Under the above conditions there exists a unique weak solution of \( \mathcal{E}_{F, \varepsilon} \). Moreover, if \( \varepsilon \leq \varepsilon' \) the comparison \( u^\varepsilon \leq u^{\varepsilon'} \) \( \mathbb{P} \text{-a.s.} \) holds. \( \square \)

The following result supply some information on the stability of the solutions when \( \varepsilon \to 0 \)

**Theorem 2** Under the above assumptions, if \( u^\varepsilon \) and \( u^{\varepsilon'} \) are solutions of \( \mathcal{E}_{F, \varepsilon} \) and \( \mathcal{E}_{F, \varepsilon'} \), respectively, one has

\[ \sup_x \mathbb{E} \left[ \left( u^\varepsilon(x, t) - u^{\varepsilon'}(x, t) \right)^2 \right] \leq |\varepsilon - \varepsilon'|^2 \left[ \frac{t}{2\pi} + K^2 \int_{0}^{t} \frac{1}{\sqrt{2\pi(t-s)}} \exp \left( K^2 \sqrt{\frac{2s}{t}} \right) ds \right], \quad (4) \]

for any \( t \geq 0 \).

**Proof.** Let \( G(x, t, y) \) be the fundamental solution of heat equation on \([-1, 1] \times \mathbb{R}_+ \) with homogeneous Neumann's boundary conditions. A simple probabilistic interpretation of \( G(x, y, t) \) involving a suitable Brownian motion, \( \{B(t)\}_{t \geq 0} \), on the complete probability space \( (\mathcal{O}, \mathcal{F}, \mathbb{P}) \), shows

\[ \int_D G(x, t, y)dx \leq \mathbb{P} \left( \{B(t) \in D\} \right), \quad \text{for all } D \in B([-1, 1]). \]

Then it follows

\[ G(x, t, y) \leq \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{(x - y)^2}{4t} \right), \quad x, y \in [-1, 1], \quad t \in \mathbb{R}_+, \]

whence inequality

\[ \int_{0}^{t} \int_{-1}^{1} G^2(x, y, t, s)dyds < \int_{0}^{t} \frac{ds}{2\sqrt{2\pi(t-s)}} < \infty \quad (5) \]

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holds. It is a simple exercise to verify that
\[
\int_{-1}^1 u^\varepsilon(y, t)\psi(y, t)dy = \int_{-1}^1 u_0(y)\psi(y, 0)dy
\]
\[
+ \int_0^t \int_{-1}^1 [u^\varepsilon(y, s) (\psi_{yy}(y, s) + \psi_s(y, s)) + F(u^\varepsilon)(y, s)\psi(y, s)]dyds + \varepsilon \int_0^t \int_{-1}^1 \psi(y, s)\mathcal{W}(dy, ds),
\]
for every \( \psi \in C_c^{2,1}([-1, 1] \times \mathbb{R}_+) \cap C_c^{1,0}([-1, 1] \times \mathbb{R}_+) \cap C([-1, 1] \times \mathbb{R}_+), \) \( \psi_\varepsilon(-1, t) = \psi_\varepsilon(1, t) = 0, \) \( t \in \mathbb{R}_+. \) Now, for any fixed \( t \) we define
\[
\psi(y, s) = G(\varphi, y, t - s) = \int_{-1}^1 G(y, z, t - s)\varphi(z)dz,
\]
where \( \varphi \) is as in (3). Since by construction
\[
G(\varphi, y, t) = \varphi(y) + \int_0^t G(\varphi, y, s)\varphi''(y)ds
\]
we deduce
\[
\psi(y, t) = \varphi(y) \quad \text{and} \quad \psi_s + \psi_{yy} = 0.
\]
So, the solution \( u^\varepsilon \) satisfies
\[
\int_{-1}^1 u^\varepsilon(y, t)\psi(y, t)dy = \int_{-1}^1 u_0(y)G(\varphi, y, t)dy + \int_0^t \int_{-1}^1 F(u^\varepsilon)(y, s)G(\varphi, y, t - s)dyds
\]
\[
+ \varepsilon \int_0^t \int_{-1}^1 G(\varphi, y, t - s)\mathcal{W}(dy, ds).
\]
Since \( \mathbb{E}[(u^\varepsilon(x, t))^2] \) is bounded in \([-1, 1]\) (see [3, Proposition 3.1]), Fubini's Theorem implies that \((u^\varepsilon(x, t))^2\) is integrable with respect to \( x, \) a.e. \( \omega \in \Omega. \) Then, if \( \varphi \) approaches a delta function, as \( t \) goes to 0, it follows
\[
\begin{align*}
\int_{-1}^1 u^\varepsilon(x, t) = & \int_{-1}^1 G(x, y, t)u_0(y)dy + \int_0^t \int_{-1}^1 G(x, y, t - s)F(u^\varepsilon)(y, s)dyds \\
+ & \varepsilon \int_0^t \int_{-1}^1 G(x, y, t - s)\mathcal{W}(dy, ds), \quad \text{IP - a.s.} \quad (x, t).
\end{align*}
\]
We note that since \( \mathcal{W} : B([-1, 1] \times \mathbb{R}_+) \to \mathbb{H} \) and the Gaussian space \( \mathbb{H} \) is contained in \( L^2(\mathcal{O}, \mathcal{F}, \mathbb{P}), \)
the estimate (5) gives a sense to the stochastic integral
\[
\mathcal{W}(G(x, \cdot, t - \cdot)) = \int_0^t \int_{-1}^1 G(x, y, t - s)\mathcal{W}(dy, ds).
\]
Next, for \( v(x, t) = u^\varepsilon(x, t) - u^\varepsilon'(x, t) \) we define \( V(x, t) = \mathbb{E}[(v(x, t))^2] \) and \( V(t) = \sup_x V(x, t). \) Then, from (6) we get that
\[
V(x, t) = \int_0^t \int_{-1}^1 \mathbb{E} \left[ \left( F(u^\varepsilon)(y, s) - F(u^\varepsilon')(y, s) + \varepsilon - \varepsilon' \right)^2 \right] G^2(x, y, t - s)dyds
\]
\[
\leq K^2 \int_0^t \int_{-1}^1 V(y, s)G^2(x, y, t - s)dyds + |\varepsilon - \varepsilon'|^2 \int_0^t \int_{-1}^1 G^2(x, y, t - s)dyds,
\]
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(see (2)) and
\[
V(t) \leq K^2 \int_0^t V(s) \left( \int_{-1}^1 G^2(x, y, t - s) dy \right) ds + |\varepsilon - \varepsilon'|^2 \int_0^t \int_{-1}^1 G^2(\tilde{x}, y, t - s) dy ds
\]
\[
\leq \frac{K^2}{2\sqrt{2\pi}} \int_0^t V(s) \frac{ds}{\sqrt{t - s}} + \frac{|\varepsilon - \varepsilon'|^2}{2\sqrt{2\pi}} \int_0^t \frac{ds}{\sqrt{t - s}}
\]
due to (5). Then, applying Gronwall inequality we get the result. \[\blacksquare\]

**Remark 1** Notice that (4) implies the uniqueness of the solution of problem (E_{F,\varepsilon}). \[\blacksquare\]

Since we are dealing here with the case \(\beta\) Lipschitz continuous, using that the solution of (E_{F,0}) is bounded due to the assumptions on the data (see [11]) and the representation (6) we get

**Corollary 1** We have \(u^\varepsilon \searrow u^0\), as \(\varepsilon \to 0\), at least in \(C([0, \infty); L^2(-1, 1; L^2(O, F, \mathbb{P}))\), with the convergence rate given by (4) for \(\varepsilon' = 0\), where \(u^0\) is the unique solution of the (deterministic) limit problem (E_{F,0}). \[\blacksquare\]

### 3. The multivalued Budyko coalbedo case

Consider now a maximal monotone graph \(\beta\) satisfying (H_{\beta}) and multivalued at \(r = -10\). Given any (univalued and discontinuous) section \(b\) of \(\beta\), we rewrite the energy balance model in terms of (E_{F_{b,\varepsilon}}) with
\[
F_{b}(x, t, r) = QS(x)b(r) - Br + f(x, t)
\]
(notice that now there is no truncation in the above definition). We start by mentioning that the results of [3] still hold. Indeed, \(F_{b}\) is a locally bounded and \(B([-1, 1]) \otimes B(\mathbb{R}_+) \otimes B(\mathbb{R})\) measurable function verifying the one side linear growth condition \(rF_{b}(x, t, r) \leq c(1 + r^4)\) for a constant \(c\) independent of \((x, t, r)\). Then, by using suitable approximations of \(F_{b}\) it was proved in [3, Theorem 5.1 and Theorem 5.2] the existence of a unique continuous and \(B([-1, 1]) \otimes \mathcal{P}\) measurable solution of the relative problem (E_{F_{b,\varepsilon}}).

Concerning the stability of the solutions when \(\varepsilon \to 0\) we have:

**Theorem 3** There is a unique continuous and \(B([-1, 1]) \otimes \mathcal{P}\) measurable solution \(u^{\varepsilon,b}\) of (E_{F_{b,\varepsilon}}). It converges to a solution \(u^b\) of (E_{F_{b,0}}), as \(\varepsilon \to 0\) and \(u^b\) is given by
\[
u^b(x, t) = \int_{-1}^1 u_0(x)G(x, y, t)dy + \int_0^t \int_{-1}^1 F_{b}(u^b)(y, s)G(x, y, t - s) dy ds, \quad (x, t) \in [-1, 1] \times \mathbb{R}_+. \tag{7}
\]

**Proof.** We shall use the classical theory of constructible solutions introduced following ideas by N. V. Krylov. In short, if \(h(x, t, r)\) is a \(B([-1, 1]) \otimes B(\mathbb{R}_+) \otimes B(\mathbb{R})\) measurable and bounded function, we construct the smooth approximation
\[
h_n(x, t, r) = n \int_{\mathbb{R}} h(x, t, r) \rho(n(r - z)) dz
\]
where \(\rho \in C^\infty(\mathbb{R})\) is a nonnegative function with \(\int_{\mathbb{R}} \rho(z) dz = 1\), and
\[
h_n(x, t, r) = n \int_{\mathbb{R}} h(x, t, r) \rho(n(r - z)) dz
\]
where \(\rho \in C^\infty(\mathbb{R})\) is a nonnegative function with \(\int_{\mathbb{R}} \rho(z) dz = 1\), and
\[
\left\{ \begin{array}{cc}
\tilde{h}_{n,k} & \leq \inf_{j=n, \ldots, k} h_j, \quad n \leq k, \\
H_n & \leq \sup_{j=n, \ldots, \infty} h_j.
\end{array} \right.
\]

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Then $\tilde{h}_{n,k}$ is Lipschitz continuous in $r$, uniformly with respect to $(x, t)$, and

$$\tilde{h}_{n,k} \wedge H_n, \quad \text{as } k \to \infty, \quad \text{and} \quad H_n \nearrow h, \quad \text{as } n \to \infty.$$ 

Now, if we apply the above procedure to $h(x, t, r) = F_b(x, t, r)$, from [3, Corollary 3.4 and Corollary 3.5] the solution $u_{n,k}^{\varepsilon, b}$ of $(E_0, \varepsilon, k)$ goes to the solution $u_{n,k}^{\varepsilon, b}$ of $(E_{H_n, \varepsilon})$, as $k \to \infty$, with

$$u_{n,k}^{\varepsilon, b} \leq u_{n,k}^{\varepsilon, b} \leq u_{n,k}^{\varepsilon, b}, \quad \text{if } k \leq k'.$$

Moreover, similar comparison arguments show that

$$u_{n,k}^{\varepsilon, b} \geq u_{n,k}^{\varepsilon, b}, \quad n \leq m \leq k, \quad \text{and} \quad u_{n,k}^{\varepsilon, b} \nearrow u_{n,k}^{\varepsilon, b}, \quad \text{as } n \to \infty,$$

where $u^{\varepsilon, b}$ is the unique continuous and $B([-1, 1]) \otimes \mathcal{P}$ measurable solution of $(E_{F_b, \varepsilon})$. Finally, from the convergence $u_{n,k}^{\varepsilon, b} \wedge u_{n,k}^{\varepsilon, b}$, as $\varepsilon \to 0$ we conclude the result. 

If $b = b^+$ (resp. $b^-$) denotes the section of $\beta$ of maximum (resp. minimum) norm then by using the method of super and subsolutions, that $b^+(r) \geq z$, $z \in \beta(r)$, and [2, Lemme 3] we get

**Corollary 2** $u^{b^+}$ (resp. $u^{b^-}$) is the maximal (resp. minimal) solution of the set of solutions of the deterministic problem.

**Remark 2** Obviously, if there is a nondegenerate solution $u$ (in the sense of [2]) of the deterministic problem then every solution of the stochastic problem $u^{\varepsilon, b}$ converges to $u$. 

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