OPTIMAL GRADIENT BOUNDS FOR SOME SECOND ORDER QUASILINEAR EQUATIONS.

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ABSTRACT: We give a gradient estimate for any solution of a quasilinear second order equation of the form \(-\text{div}(Q(|V_u|)V_u)+f(u)=0\) in \(\Omega\) with \(u=k\) on \(\partial\Omega\). This includes the p-Laplacian operator \(Q(p)=c^p\) as well as the equation of surfaces of prescribed mean curvature \(Q(q)=1/(1+q)^{2^*}\). Our gradient estimates are of the type \(|V_u|\leq F(u)\) for some suitable function \(F\). The inequality becomes an equality in the one-dimensional case. This result was already known for strongly elliptic operators and \(f\in C^1\). The generalization to eventual degenerate operators and \(f\in C^1\) is motivated for some free boundary problems in continuum mechanics. The associated evolution problem is also considered. Detailed proofs of this preliminary report will appear elsewhere.


1. ELLIPTIC EQUATIONS. This communication deals with some pointwise gradient estimates for non-negative solutions of the problem

\[
\begin{aligned}
&-\text{div}(Q(|V_u|)V_u)+f(u)=0 \quad \text{in} \ \Omega \\
&u=k \quad \text{on} \ \partial\Omega,
\end{aligned}
\]

where \(\Omega \subset \mathbb{R}^N\) is a regular open bounded set, \(k\) is a positive constant,

\(Q \in C^2(0,\infty)\cap C^1([0,\infty))\), \(Q(q)>0\) and \(qQ(q)'>0\) if \(q>0\),

\(f \in C^1([0,\infty))\) and \(f(t)>0\) if \(t>0\).

Problems of this type appear in many different contexts: chemical reactions \(Q(q)=1\), non-Newtonian fluids \(Q(q)=q^p\), surfaces of prescribed mean curvature or the semiscum problem in capillarity \((Q(q)=1/(1+q)^{2^*})\); see references in Díaz [2] for the two first problems and Payne-Phillippin [6] for the third one.

Our main result is the following

THEOREM 1. Let \(u\) be a non-negative weak solution of (1) such that \(u\in W^{1,\infty}(\Omega;|V_u|+\|u\|)\cap C^1(\Omega)\). Then, for every \(x\in \Omega\) we have

\[
\begin{aligned}
|V_u(x)|\leq L^{-1}(F(u(x))-2(u(x)-m)) \quad \text{on} \ \Omega,
\end{aligned}
\]

where \(m\) is the minimum of \(u\) on \(\Omega\),

(*) Partially supported by the project n° 3308/83 of the CAICYT.
(5) \( A(q) = \frac{\partial^2 Q(s)}{\partial s^2} \) ds
ds
(6) \( F(x) = \frac{\partial^2 F(s)}{\partial s^2} ds \)
and
(7) \( a = \min\{0, \sqrt{(n-1)H(x)Q(\frac{3u}{\delta n}, x)}\} \)
x \in \Omega
with \( H(x) \) being the mean curvature of \( \Omega \).

Proof: Let \( J : \Omega \to R \) be defined by
(8) \( J(x) = A(\|Vu(x)\|) - F(u(x)) + g(u(x)). \)

In order to prove (4), or equivalently \( J(x) \geq g \) for any \( x \in \Omega \), we introduce the notation \( D_x = x \in \Omega : |Vu(x)| > x \) for \( x > 0 \) and proceed in different steps.

**First step.** We shall prove that if we define \( q(x) = |Vu(x)| \) and
(9) \( T(x) = \delta j(x) + \frac{Q(q(x))}{q(x)} u_k(x) u_j(x) J_{kj}(x) \)
then \( T \leq L^2(\Omega) \) and \( J \geq 0 \) on \( \Omega \), for any \( x > 0 \) (in (9) and in the following we use the Einstein summation convention). Indeed, by differentiating \( J \) (in the sense of distributions), we obtain

(10) \( \delta j \cdot (Q \cdot q)^n u_j u_k u_l - j \cdot j \cdot j \cdot j \cdot c_j - j \cdot c_j + e_j(u) u_j u_l + u_j u_k u_l \)

Using the equation in (1) and differentiating there with respect to \( x \) we get (after, at least, five minutes of computations) that

\[ T = (Q \cdot q)^n u_j u_k u_l (\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial q^2} \cdot Q^2) + \frac{\partial}{\partial q} \frac{\partial^2}{\partial q^2} + \frac{\partial}{\partial q} \frac{\partial^2}{\partial q^2} + \frac{\partial}{\partial q} \frac{\partial^2}{\partial q^2} - \frac{\partial}{\partial q} \frac{\partial^2}{\partial q^2} \cdot Q^2 \]

\[ - u_j u_k u_l f(u) \frac{\partial^2}{\partial q^2} + \frac{\partial}{\partial q} \frac{\partial^2}{\partial q^2} \cdot Q^2 \]

from which we obtain a bounded function in \( D_x \). On the other hand, using Cauchy-Schwarz inequality \( u_k u_l u_j u_k \)

\( a u_k u_l u_j \), as well as the identities

\( u_k u_l u_j = -\frac{\partial^2}{\partial q^2} + \text{terms containing } J_1 \)

\( (u_k u_l u_j)^2 = \frac{\partial^2}{\partial q^2} + \text{terms containing } J_1 \)

\( u_k u_l u_j u_j = \frac{\partial^2}{\partial q^2} + \text{terms containing } J_1 \)

we conclude that \( T(x) \geq 0 \) for \( x \in D \). Second step. We claim that \( J \) cannot take its maximum value on \( \Omega \) unless \( q = 0 \) on \( \delta \Omega \). Indeed, since \( u_k \) on

\( \delta \Omega \) and \( u_k \) in \( \Omega \), it follows that \( Q(\|Vu\|)u_k/\delta n > 0 \) on \( \delta \Omega \). By the divergence theorem

\[ \int_{\delta \Omega} Q(u) u_k/\delta n = \int_{\Omega} f(u) > 0. \]

Then, there is \( p \in \Omega \) such that \( Q(u)u_k/\delta n > 0 \) and, in consequence \( u_k/\delta n > 0 \).

But

(11) \( \frac{\partial u}{\partial n} = \frac{\partial u}{\partial n} - Q'(q)Q(q) - Q''(q) \frac{\partial u}{\partial n} \)

Hence, from equation (1)

(12) \( Q(q) \frac{\partial^2 u}{\partial n^2} + (n-1)H(u) \frac{\partial^2 u}{\partial n^2} + Q'(q) \frac{\partial u}{\partial n} = f(u) \) on \( \Omega \)

(remember that \( H(u) = \frac{\partial^2 u}{\partial n^2} + (n-1)H(u) \frac{\partial^2 u}{\partial n^2} \))

Combining (11) and (12) we conclude that \( \frac{\partial^2 u}{\partial n^2} = 0 \), which is a contradiction with the Hopf's maximum principle (which can be applied because \( \Omega \) is bounded and \( p \in \Omega \) for \( x \) small enough). Third step. \( J(x) \) takes its maximum value in every \( p \in \Omega \), such that \( u(p) = 0 \). To prove that, let \( p \in \Omega \) such that \( J(p) = \max J(x) \). By the above step \( p \in \Omega \) and \( J(p) = 0 \). If \( u(p) = 0 \) there is no \( J \) for \( p \in \Omega \) such that \( u(p) = 0 \). If \( |Vu(p)| = 0 \) we take \( e \cdot s \) and add \( D_x \) to \( \Omega \), by the strong maximum principle we conclude that \( J(p) = \max J(x) \). Since this is true for all \( e \cdot s \) and \( J \) is continuous we get \( J(x) = J(p) \) in \( \Omega \). Now, let \( p \in \Omega \) such that \( u(p) = 0 \). If \( p \in \Omega \) then \( J(p) = \max J(x) \). If \( p \in \Omega \) then \( J(p) = \max J(x) \).

REMARKS. 1. It is not difficult to show that the estimate (4) is optimal in the sense that, in fact, the equality is true if \( n = 1 \).

2. Theorem 1 extends previous results due to Payne-Phillipin [6] for the case of strongly elliptic quasilinear equations and \( f \in C^1 \). Our proof is also inspired on the adaptation made by Massie [5] of Payne's method, for semilinear equations.

3. The regularity assumed on \( u \) is not restrictive. This is well-known in many important particular cases including the \( p \)-Laplacian and the minimal surfaces operators (see Di Benedetto [4]).

4. Optimal pointwise gradient estimates are of a great interest in the study of the free boundary given by the boundary of the support of \( u \). In particular, estimate (4) is used in Daz-Saa-Thiel [3] in order to obtain a necessary condition for the existence of the free boundary for the equation (1) (which generalizes results collected in Daz [2]).

5. PARABOLIC EQUATIONS. Pointwise spatial gradient estimates can also be obtained for nonnegative solutions of
Theorem 2. Assume that \(|u_0| \leq C_k\) as well as

\[ f \text{ is nondecreasing} \quad \text{or} \quad f \text{ is locally Lipschitz continuous}, \]

Then if \(Q > 0\),

\[ d = \frac{1}{|\mathbf{V}_u(z)|} A^{-1}(F(u_0(z))) - u_0(z) - m \]

on \(\Omega\),

we have

\[ \frac{d}{\partial t} \leq \frac{1}{|\mathbf{V}_u(z)|} A^{-1}(F(u(t,z))) - u(t,z) - m \quad \text{in} \quad [0, T] \times \Omega \]

Proof. Due to assumptions (14) and (16) it is not difficult to show that \(u(t, \cdot) \in W^{1, \infty}(\Omega) \cap C^1(\Omega)\). Define \(m = \min_u, A \text{ and } F \text{ given by} \) (5) and (6) and let

\[ \alpha = \min\{0, \min (N-1)(x)(\frac{\partial u}{\partial n}(x,t)) - \frac{\partial u}{\partial n}(x,t)\} \]

Then if

\[ |\mathbf{V}_u(z)| \geq A^{-1}(F(u_0(z)) - u_0(z) - m \geq 0 \quad \text{on} \quad \Omega, \]

we obtain

\[ J(t,x) = A^{-1}(F(u(t,z))) - u(t,z) - m \quad \text{in} \quad [0, T] \times \Omega \]

and

\[ D(t) = \{x \in \Omega : \mathbf{V}_u(z) > c, \} \subset \Omega \]

In order to prove (19) (or, equivalently, \(J(t,x) \leq m\) for any \((t,z) \in [0, T] \times \Omega\)), we see that

\[ \Delta J = (\frac{q}{q^2} + Q') u_{jk} u_{jk} + \frac{1}{N} (Q + Q') (u_{jk} u_{jk}) - f(u_{jk}) u_{jk} - (f(\alpha) - \alpha) u \]

and so

\[ \Delta J = \frac{q}{q^2} u_{jk} u_{jk} - \frac{Q}{Q'} (Q' + Q) u_{jk} u_{jk} + \frac{1}{Q'} \left[ \frac{Q}{Q'} - 3 (Q')^2 \right] (u_{jk} u_{jk})^2 + (Q + Q') u_{jk} u_{jk} + (\frac{Q}{Q'}) (u_{jk} u_{jk}) - f(u_{jk}) u_{jk} + \]

\[ \alpha (f(u) - \alpha) (1 + \frac{Q'}{Q}) \leq 0 \text{ in} \quad D(t) \]

where we have used similar arguments to the elliptic part as well as

(15) and \(u > 0\). The maximum of \(J\) in \([0, T] \times \Omega\) must be attained in the parabolic boundary. But this maximum is not attained in the spatial boundary \((0, T) \times \mathbb{R}\) (use that \(u(t, \cdot) \neq \infty\) on \((0, T) \times \mathbb{R}\), \(u < 0\) on \((0, T) \times \mathbb{R}\) and argue as in the elliptic case). On \(t = 0\) we have \(J(0, x, z) = \infty\) by (18). Finally, if the maximum of \(J\) is not at \(t = 0\), it must be at some \((t, x, z) \in (0, T] \times \mathbb{R}\) and, as in the elliptic case, \(\forall J(t, x, z) = 0\) and \(u(t, x, z) = \).