Nonlinear parabolic equations: qualitative properties of solutions

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Qualitative properties of free boundaries for some nonlinear degenerate parabolic equations

1. Introduction

This paper is a short survey of some recent work by the authors, concerning qualitative properties of nonlinear degenerate parabolic equations. The associated stationary problem was considered by the authors in [7] by using a local comparison technique involving some kind of local radial supersolutions, which was previously introduced by the first author in [5]. There the main interest was the study of the dead core, namely the subset where the (positive) solutions vanish identically; some necessary and/or sufficient conditions for the existence of a (non-empty) dead core, together with additional information about its size and location, were obtained (see [1] and [11]) for related work as well as the monograph [6]).

Here we apply the same kind of arguments to a rather large class of nonlinear (possibly) degenerate parabolic equations complemented with non-zero Dirichlet boundary conditions (see Problem (P) below). Some results for the case of pure powers, i.e., \( \phi(u) = u^p \) and \( f(u) = u^p \) were obtained in [8]. Here we extend this investigation to nonlinearities \( \phi \) and \( f \) which are not necessarily powers but have only a similar qualitative behaviour (see assumptions (H₁) and (H₂) below) near the origin. We refer the reader to [2] - [4] and [13] - [15] for other related work.

Very roughly speaking, a large part of our results seem to be new in this more general situation, and some of them extend to the case \( 0 < p < 1 \) theorems known for \( p > 1 \). More detailed information can be found below (see also [8][9]). An extended version of this survey, including also work in [8], with full proofs and many complementary results and applications will appear in [9]; in particular, we will give there applications to some reaction-diffusion systems arising in combustion theory (see [2][8]) and population dynamics with nonlinear diffusion ( [12]).

2. Main theorem

In this section we consider the following degenerate parabolic problem:
\[
\begin{aligned}
&\begin{cases}
    u_t - \Delta u + f(u) = 0 \quad \text{in } Q = \Omega \times (0, \omega), \\
    u(x, t) = h(x, t) \quad \text{on } \Sigma = \partial \Omega \times (0, \omega), \\
    u(x, 0) = u_0(x) \quad \text{in } \Omega,
\end{cases} \\
&h \in L^1(\Sigma), \quad h \geq 0 \quad \text{in } \Sigma; \quad u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \quad \text{on } \Omega.
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), under the following assumptions:

\( \psi \) is a continuous increasing function, \( \psi(0) = 0 \) and \( \psi' > 0, \psi'' > 0 \) in \( \Omega \times (0, \omega) \); 

\( f \) is continuous, \( f(0) = 0 \); there exists a continuous increasing (2.2) function \( f_0 \) such that \( 0 \leq f_0(s) \leq f(s) \) for every \( s \geq 0 \); 

\[ \phi \in C(\overline{\Omega}), \quad h \geq 0 \quad \text{in } \Sigma; \quad u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \quad \text{on } \Omega. \] 

Our main result in this section is the following theorem.

**Theorem 2.1.** Suppose that \( u \in C(\overline{\Omega}), \quad u \geq 0 \), is a solution of problem (P) with (2.1) - (2.3). Moreover assume that

\[ \int_0^1 \frac{1}{\phi'(\psi^{-1}(t))} \frac{ds}{f_0(t)} > + \infty \] 

(\( H_1 \))

and

\[ \int_0^1 f_0(s) \frac{ds}{\phi'(\psi^{-1}(s))} < + \infty \] 

(\( H_2 \))

are satisfied. Then there exists \( T_0 > 0 \) such that for every \( t \geq T_0 \) we have

\[ \eta = \phi^{-1}/N(\phi) \]

where \( \phi \) denotes the support of the corresponding function, and \( L \) is a constant depending on \( \psi, \psi', \psi'', \phi, f_0, u_0, \Omega \) and \( N \).

The main tool for the proof of Theorem 2.1 is the following Lemma, which generalizes Lemma 2.1 in [7]. Its proof can be found in [6].

**Lemma 2.1.** If we define \( \eta(s) = \psi^{-1}/N(\phi) \), where

\[ \hat{\psi}(r) = \int_0^r \frac{ds}{\phi'(\psi^{-1}(t))}^{1/2} \]

then for any \( x_0 \in \Omega \) we have

\[ -\Delta n(x-x_0) + \int_0^1 f_0(\psi^{-1}(\eta(x-x_0))) \geq 0 \quad \text{in } \Omega. \]

Moreover \( \eta(0) = \eta'(0) = 0 \) and \( \eta(s) > 0 \) if \( s \neq 0 \).

**Sketch of the proof of Theorem 2.1.** We define (this is an idea adapted from [10])

\[ \psi^{-1}(\eta(x-x_0) + \phi(U(t))) \]

where \( \eta(s) \) and \( \hat{\psi}(r) \) are as in Lemma 2.1 (we remark that by \( H_2 \)) we have

\[ \hat{\psi}(r) < + \infty \]

and \( U \) is a positive solution of the ordinary differential equation

\[ \frac{dU}{dt} + \frac{1}{2} f_0(U) = 0 \]

(2.5)

\[ V(0) = \|u_0\|_{C^0} \]

It is not difficult to see that, as a consequence of \( H_2 \), we have \( U(t) = 0 \) for any \( t \geq T_0 \)

From (2.1), (2.2) we obtain:

\[ \frac{d}{dt} (\phi^{-1}(\eta(x-x_0))) + \phi(U(t))) = -\Delta n(x-x_0) + f_0(\psi^{-1}(\eta(x-x_0))) + \phi(U(t))) \geq 0 \]

\[ \geq \frac{\psi'(U)}{\psi'(\phi^{-1}(\eta(x-x_0))))} \frac{dU}{dt} - \Delta n + \frac{1}{2} f_0(\psi^{-1}(\eta)) + \frac{1}{2} f_0(U) \geq 0 \]

\[ \geq \frac{dU}{dt} - \Delta n + \frac{1}{2} f_0(\psi^{-1}(\eta)) + \frac{1}{2} f_0(U) \geq 0 \]
by (2.4) and (2.5), taking into account that

\[ n \cdot \psi(u) \geq \psi(u) \]

implies the inequality

\[ \psi^{-1}(n \cdot \psi(u)) \geq u, \]

hence

\[ \psi'(\psi^{-1}(n \cdot \psi(u))) \geq \psi'(u), \]

once again by (2.1).

Concerning the boundary condition, it is easy to show that if we have

\[ 0 \leq h(x, t) \leq \|h\|_{L^\infty} \leq \psi^{-1}(\eta(|x - x_0|)) \leq \bar{u}(x, t), \]

then the inequality \( h(x, t) \leq \bar{u}(x, t) \) holds at the boundary. Indeed, if \( x \notin S(h, t) \), \( h(x, t) = 0 \) and the inequality is automatically satisfied. If not, it is sufficient that

\[ \psi(\|h\|_{L^\infty}) \leq \eta(|x - x_0|) \]

for any \( x \in S(h) \);

this is equivalent to

\[ \psi^{-1}(\psi(\|h\|_{L^\infty})) \leq |x - x_0| \]

or, otherwise stated, be

\[ d(x_0, U S(h, t)) \geq L, \]

where \( L = \psi^{-1}(\psi(\|h\|_{L^\infty})) \).

As for the initial condition, it is easily seen that

\[ 0 \leq u_0(x) \leq \|u_0\|_{L^\infty} \leq \psi^{-1}(\eta(|x - x_0|)) + \psi(\|u_0\|_{L^\infty}), \]

Thus we obtain (recall (2.2)):

\[
\begin{cases}
  u_t - \Delta u + f(u) - f(u) & \leq 0 \leq \bar{u}_t - \Delta \bar{u} \\
  u(x, t) \leq \bar{u}(x, t) & \text{on } \Gamma \\
  u_0(x) \leq \bar{u}(x, 0) & \text{in } \Omega;
\end{cases}
\]

it follows from comparison results for problem (P) with \( f(u) \) that

\[ 0 \leq u(x, t) \leq \bar{u}(x, t). \]

The proof ends by recalling that \( u(x_0, t) = 0 \) if \( t \geq T_0 \) and \( x_0 \) satisfies the above inequality.

**Remark 2.1.** It is also possible to prove similar results when replacing \( f(u) \) by \( c(x, t) r(u) \), with \( c(x, t) \geq 0 \) (see [8][9]). This seems to be particularly interesting for applications to reaction-diffusion systems.

**Remark 2.2.** If \( \eta(s) = s^m \), \( f_0(s) = s^p \), then \( (H_1) \) is equivalent to \( p < m \) and \( (H_2) \) is equivalent to \( p < 1 \). Now, for \( m = 1 \), \( (H_1) \) and \( (H_2) \) coincide. But if \( \psi(s) \neq s \) and \( f_0 \) is not a power, then \( (H_1) \) implies \( (H_2) \) but the converse is not true (see [10]).

**Remark 2.3.** Our theorem extends some work by Keranen [14] for the case \( N = 1 \), and also, for \( m \equiv 0 \) and \( \Omega = \mathbb{R} \), results by Kalashnikov [13] and Véron [15] concerning extinction of solutions in finite time. On the other hand, for \( m = 1 \), \( h \equiv 1 \), \( u_0 \equiv 1 \), estimates for the dead core as \( N(u, t) \supset \{ x \in \Omega | d(x, \Omega) \geq L \} \) can be found in [2] (see also [8]).

**Remark 2.4.** If \( (H_2) \) is satisfied but \( (H_1) \) does not hold, it is still possible to get estimates of the kind

\[ 0 \leq u(x, t) \leq U(t) \]

extending in this way some work by Beresty, Nambu and Poletier [4], respectively Véron [15]. Similar arguments also allow us to prove the estimate

\[ N(u, t) \supset \{ x \in \Omega | d(x, S(u_0)) \geq L \} \]

for some constant \( L' \).
REMARK 2.5. The same technique of proof allows us also to obtain local
(namely depending on the point \(x_0 \in \Omega\) and on the norm \(\|u\|_{L^\infty(B(x_0, \delta))}\))
estimates for the extinction time \(T_0(\varepsilon > 0; \text{see [2],[8],[9]}).\)

THEOREM 2.2. Assume that \(u \in C(\overline{\Omega}), u \geq 0,\) is a solution of problem \(\mathcal{P}\)
with \((2.1) - (2.3)\) and \((H_4).\) If \(x_0 \in \Omega\) satisfies

\[
0 \leq u_0(x) \leq \varphi^{-1}(\eta(|x-x_0|), 1/N)
\]

(2.5)

for any \(x \in B(x_0, \varepsilon),\) where \(\varepsilon = \psi_1^1/(\varphi(M)), M = \|u\|_{L^\infty(\Omega)},\) \(\eta(r, \mu) = \varphi^{-1}(r),\)
\(\psi_1^1\) as above), then \(u(x_0, t) = 0\) for any \(t > 0.\)

Sketch of the proof. On the set \(B(x_0, \varepsilon) \times (0, +\infty)\) define the function

\[
\bar{u}(x) = \varphi^{-1}(\eta(|x-x_0|), 1/N).
\]

Now, reasoning as in [6] we obtain

\[
\begin{cases}
\Delta u - \Delta \varphi(u) + f_0(u) \leq 0 \leq \Delta \varphi(u) - \Delta \varphi(\bar{u}) + f_0(\bar{u}) \quad \text{in } B(x_0, \varepsilon) \times (0, +\infty) \\
u(x, 0) = u_0(x) \leq \bar{u}(x) = \varphi^{-1}(\eta(|x-x_0|)) \quad \text{in } B(x_0, \varepsilon) \\
u(x, t) \leq M \leq \bar{u}(x) \quad \text{on } \partial B(x_0, \varepsilon) \times (0, +\infty),
\end{cases}
\]

where \(\|u\|_{L^\infty(\Omega)} \leq M.\) Then a comparison argument gives \(0 \leq u(x, t) \leq \bar{u}(x).\)

REMARK 2.6. Theorem 2.2 improves on some results in [4] for \(h = 0;\) indeed, we
only need the local estimate (2.5). If \(\varphi(s) = s^h, f_0(s) = \lambda s^p,\) then

\[
u(x) = K_\lambda |x-x_0|^{\frac{2}{1-h}}
\]

for some \(K_\lambda > 0.\)

THEOREM 2.3. Assume \(u \in C(\overline{\Omega}), u \geq 0,\) is a solution of the problem

\[
u_t - \Delta u + f_0(u) = 0 \quad \text{in } Q
\]
\[
u = 0 \quad \text{on } \Sigma
\]
\[
u(x, 0) = u_0(x) \quad \text{on } \Omega,
\]

where (2.2), (2.3) and \((H_4)\) are satisfied. If, moreover, \(u_t \in L^\infty(Q),\) then we have

\[
S(u(\cdot, t)) \subseteq S(u_0) + B(0, \psi_1^1(\varepsilon_t))
\]

for any \(t > 0\) and some \(C > 0,\) where \(C\) depends on \(\|u_0\|_{L^\infty(\Omega)}.

Sketch of the proof. Let \(t_0 > 0\) and \(x_0 \in S(u(\cdot, t_0)) \cap S(u_0).\) We consider the
region

\[
R(t_0) = \{(x, t)|0 < t < t_0, u(x, t) > 0, x \notin S(u_0)\}
\]

and the function

\[
\bar{u}(x) = \eta(|x-x_0|), 1/N).
\]

The function \(z(x, t) = u(x, t) - \bar{u}(x)\) satisfies

\[
z_t - \Delta z + B(x, t)z \leq 0 \quad \text{on } Q
\]

for a suitable \(B(x, t);\) then the Strong Maximum Principle implies that \(z\)
takes its maximum on the parabolic boundary of \(R(t_0).\) But, on the other hand, \(0 = u(x, t) \leq \bar{u}(x)\) for \((x, t) \in \partial B(t_0) - S(u_0),\) and \(z(x_0, t_0) > 0.\) Hence there exists a point \((\bar{x}, \bar{t})\) in \(S(u_0) \times (0, t_0)\) satisfying \(\bar{u}(x) < \bar{u}(x, \bar{t}).\) This
in turn implies

\[
d(x_0, S(u(\cdot, t))) \leq |x-x_0| \leq \varphi_1^1(u(x, t)) \leq \psi_1^1(u(x, \bar{t}) - \bar{u}(x_0)) \leq
\]
\[
\leq \psi_1^1(\varepsilon_t) \leq \psi_1^1(\varepsilon_t),
\]

which gives the result.
REMARK 2.7. The proof follows an idea of Evans and Knerr [10]. If $u_0 \in L^\infty(\Omega)$, $v_0 \in H^1_0(\Omega)$ and $h \in L^\infty(\Omega) \cap H^1(\Omega)$, then, following a theorem by Bénilan-Ha, $u_t \in L^\infty(\Omega)$.

REMARK 2.8. If $f_0(x) = x^p$, $0 < p < 1$, then $\psi_1(\gamma \alpha) = C t^{1-p/2}$.

REFERENCES


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