COMPACTNESS OF THE GREEN OPERATOR ASSOCIATED TO
THE POROUS MEDIA EQUATION

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Let $\Omega$ be an open regular set of $\mathbb{R}^N$, $N \geq 1$, and let $T > 0$. Consider the problem
\begin{align}
  u_t - \Delta \varphi(u) &= f & \text{in } \Omega \times (0,T) \\
  \varphi(u) &= 0 & \text{on } \partial \Omega \times (0,T) \\
  u(0,\cdot) &= u_0(\cdot) & \text{on } \Omega
\end{align}

where
\begin{align}
  \varphi \text{ is continuous nondecreasing and } \varphi(0) &= 0 \\
  u_0 &\in \mathcal{L}^1(\Omega).
\end{align}

Problems as (1), (2), (3) appear in many different contexts as, for instance, unsaturated flows through a porous medium. Existence, uniqueness and regularity results are today well-known in the literature (see e.g. [1]). Here we are interested in showing some compactness properties of the associated Green operator $G$ defined by
\[ G : \mathcal{L}^1(0,T;\mathcal{L}^1(\Omega)) \to C([0,T];\mathcal{L}^1(\Omega)) \]

\[ f \mapsto u, \text{ } u \text{ solution of (1), (2), (3),} \]

for a given $u_0$ satisfying (5). The compactness of $G$ (in some weak sense) has many different applications. For instance, it allows to obtain easily existence results for functional-perturbed equations (see [2] and [8]) or for nonlinear systems
\[ u_{i,t} - \Delta \varphi_i(u_i) = f_i(x,t,u_1,u_2) \quad i = 1,2 \]

under very weak assumptions on $f_i$ and $\varphi_i$ (see [7]). Other applications of the compactness of $G$ are concerned with the asymptotic behaviour, as $t \to \infty$, of

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solutions (1),(2),(3) (see [6]).

Our main result is the following

THEOREM 1. Let \( u_0 \in L^1(\Omega) \) and assume that \( \varphi \) is strictly increasing \( \varphi \) is strictly increasing \( (6) \)

Let \( F \) a weakly compact subset of \( L^1(\Omega) \). Then \( G(F) \) is a relatively (strongly) compact set of \( C([0,T];L^1(\Omega)) \).

The proof of Theorem 1 is based in the following result of interest by itself

THEOREM 2. Assume (6) and let \( S(t) \) be the semigroup of contractions on \( L^1(\Omega) \) associated to the operator \( -\Delta \varphi(\cdot) \). Then, for any \( t \in (0,T) \), \( S(t) \) transforms any weakly compact set of \( L^1(\Omega) \) into a relatively compact set of \( L^1(\Omega) \). In particular, the restriction \( S(t):L^p(\Omega) \rightarrow L^q(\Omega) \) is compact if \( 1 \leq q < p \leq +\infty \).

Remark 1. The study of the compactness of \( G \) was previously carried out in the work [3] for the special case of \( \varphi(s) = |s|^{m-1}s \) and \( m > 0 \). It is shown there that if \( N > 2 \) and \( m > (N-2)/N \) then \( S(t) \) is a compact semigroup in \( L^1(\Omega) \).

Moreover, if \( S(t) \) is compact and \( F \) is a bounded set of \( L^1(\Omega) \) then \( G(F) \) is relatively compact in \( C([0,T];L^1(\Omega)) \). Later in [5] it was shown the optimality of that result i.e. if \( m = (N-2)/N \) then \( S(t) \) is not compact in \( L^1(\Omega) \). Generalizations of the strong compactness of \( S(t) \) for more general functions \( \varphi \) were given in [2] and specially in [4] where it was shown that \( S(t) \) is compact in \( L^1(\Omega) \) if and only if

\[
\int_{-\infty}^{+\infty} \frac{ds}{\varphi(s)^{N/(N-2)}} < +\infty.
\]

Remark 2. The assumption (6) is optimal in order to get the conclusion of Theorem 1. Indeed, take as function \( \varphi \) the one associated to the Stefan problem:

\( \varphi(s) = s+1 \) if \( s \leq -1 \), \( \varphi(s) = s-1 \) if \( s \geq 1 \), \( \varphi(s) = 0 \) if \( s \in [-1,1] \).

Take \( \Omega = (0,1) \), \( u_0 = 0 \) and \( F = \{ f_n \in N: f_n(t)(x) = \sin nx \} \). Then it is easy to see that although \( F \) is weakly compact in \( L^1(Q) \), the set of solutions \( G(F) \) is given by \( G(F) = \{ u_n, n \in \mathbb{N}: u_n(t)(x) = t \sin nx \} \) which is not relatively compact.
in $C([0,T]:L^1(\Omega))$.

Remark 3. The proof of both results was given in [6]. Theorem 2 is shown by reducing the problem to the relative compactness of the set $S(t)(B)$ where $B$ is a bounded set of $L^\infty(\Omega)$. It is shown there that the conclusion comes from the gradient estimate

$$\left\| \nabla \phi(S(t)u_0) \right\|_{L^2(\Omega)}^2 \leq \frac{1}{t} \left\| J(u_0) \right\|_{L^1(\Omega)}$$

where $j(r) = \int_0^r \phi(s)ds$. The proof of Theorem 1 uses some approximation arguments. After that the conclusion follows from Theorem 2 by using the formula

$$\left\| u_r(t+\lambda)-S(\lambda)u_r(t) \right\|_{L^1(\Omega)} \leq \int_t^{t+\lambda} \left\| f(s) \right\|_{L^1(\Omega)} ds,$$

where $\lambda > 0$ and $u_r$ denotes the solution of (1), (2), (3) for a given function $f \in L^1(0,T;L^1(\Omega))$.

REFERENCES


