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On the uniqueness of solutions of a nonlinear elliptic problem arising in the confinement of a plasma in a Stellarator device.

We obtain the uniqueness of solutions of a nonlocal elliptic problem when the nonlinear terms at the right hand side are assumed to be prescribed. The problem arises in the study of the magnetic confinement of a plasma in a Stellarator device.

1. Introduction.

The main goal of this communication is to prove the uniqueness of the solution of a two dimensional free boundary problem modeling the magnetic confinement of a plasma in a Stellarator device. The model consists of a second order partial differential equation of elliptic type, obtained from the 3-D ideal MHD system by Hender and Carreras [4] by using toroidal averaging arguments and a suitable system of coordinates. This problem has recently been studied by Díaz [1] who introduced the following formulation in the form of a free boundary problem. Let \( \Omega \) be an open, bounded, regular and connected set contained in \( \mathbb{R}^2 \), and let

\[
\lambda > 0, \quad F\gamma > 0, \quad a, b \in L^\infty(\Omega), \quad a \geq 0, \quad b > 0 \quad \text{a.e. in} \ \Omega.
\]

Given \( \gamma \in \mathbb{R}: = \{ t \in \mathbb{R}: t < 0 \} \), the problem is to find

\[
u : \Omega \rightarrow \mathbb{R} \quad \text{and} \quad F : \mathbb{R} \rightarrow \mathbb{R}_+
\]

such that \( F(s) = F_\gamma \) for any \( s \leq 0 \) and the following conditions hold:

\[
(\mathcal{P}) \begin{cases}
-\Delta u & = aF(u) + F(u)F'(u) + \lambda b u_+ & \text{in} \ \Omega \\
\quad u & = \gamma & \text{on} \ \partial \Omega \\
\quad 0 & = \int_{\{u \leq \gamma\}} \{F(u)F'(u) + \lambda u_+ b\} & \forall t \in [\text{essinf} u, \text{esssup} u]
\end{cases}
\]

In the sequel we will refer to the family of integral identities stated in \( (\mathcal{P}) \) as the Stellarator Condition.

In order to characterize the unknown function \( F \), the above problem was reformulated in Díaz [1] using the notion of relative rearrangement. There, he proved that if \( (u, F) \) is a solution of \( (\mathcal{P}) \) such that \( u \in \mathcal{U} \) where

\[
\mathcal{U} = \{ u \in W^{2,p}(\Omega), \ \text{for any} \ 1 \leq p < \infty : \text{meas} \{ x \in \Omega : \nabla u(x) = 0 \} = 0 \}
\]

then \( u \) satisfies the following uncoupled non local equation

\[
-\Delta u = a \left[ F_\gamma^2 - 2\lambda \int_0^{u_+(x)} \sigma b u_+ (|u > \sigma|) d\sigma \right]^{1/2} + \lambda u_+ (b - b_+ (|u > u(x)|)) \quad \text{in} \ \Omega
\]

where we denote \( \text{meas} \{ x \in \Omega : u(x) > t \} \) by \( |u > t| \), \( u_+ \) represents the decreasing rearrangement of \( u \) and \( b_+ \) is the relative rearrangement of \( b \) with respect to \( u \) (the definition of these functions and some of their properties can be found, for instance, in [6] and in its references). Moreover, if \( u \) satisfies (1) then the function \( F = F^u \) is given by

\[
F^u(t) = \left[ F_\gamma^2 - 2\lambda \int_0^{t^+} \sigma b u_+ (|u > \sigma|) d\sigma \right]^{1/2} \quad \text{for any} \ t \in [\text{essinf} u, \text{esssup} u].
\]

The existence of \( u \), solution of \( (\mathcal{P}) \) in the class of functions \( \mathcal{U} \) was proved by Díaz and Rakotoson [2, 3] under some additional assumptions.

Here we give a partial result to the uniqueness question. Notice that the equation of \( (\mathcal{P}) \) involves nonlinear terms which do not need to be convex neither concave functions. Our proof uses some a priori estimates, some properties of the relative rearrangement and the study of a suitable weighted eigenvalue problem. The idea of using an auxiliary linear eigenvalue problem is inspired by the technique used in Puel [5] to establish the uniqueness of solution of a different free boundary problem arising in the study of the plasma confinement in Tokamak devices.
1 The main result.

To state the uniqueness result we shall need to refer to the weighted eigenvalue linear problem

\[
(P^\mu) \left\{ \begin{array}{ll}
-\Delta w &= \mu g(x)w \text{ in } \Omega \\
\frac{\partial w}{\partial \nu} &= 0 \text{ on } \partial \Omega
\end{array} \right.
\]

as well as to a suitable positive constant \( \lambda_0 + \lambda_0(a, b, F_n, [\Omega], \mu_2) \) which depends on the data of the problem \( a, b, F_n \), on the constants of Poincaré and of a Sobolev’s Imbedding and on \( \mu_2 \), the second eigenvalue of \((P^\mu)\).

**Theorem.** Let \((u, F)\) with \( u \in U \) be a solution of \((P)\). Suppose that \( FF' \) is Lipschitz on \( \mathbb{R} \), i.e.,

\[|F(t)F'(t) - F(\hat{t})F'(\hat{t})| \leq \lambda K|t - \hat{t}|\]

for every \( t, \hat{t} \in \mathbb{R} \) and for some positive constant \( K \). Suppose also that the parameter \( \lambda > 0 \) is such that

\[\lambda < \lambda_0,\]

where \( \lambda_0 \) is the above mentioned constant and \( g \) is defined by

\[g(x) := C_2||b||_{L^\infty(\Omega)}a(x) + b(x) + K\]

for some known constant \( C_2 > 0 \). Then, if \((v, F)\) is another solution of \((P)\), then, necessarily, \( v \equiv u \).

**Proof.** Suppose that there exist two solutions \( u, v \) of \((P)\). The proof consists in two main steps:

- To verify that for small values of the parameter \( \lambda \) (\( \lambda < \lambda_0 \)) the solutions are necessarily ordered, for instance, \( u \leq v \). To do this we adopt a technique used by Puel [5] as well as some technical results on the regularity and positivity of the function \( F \).
- To derive a contradiction: if \( u \geq v \) in \( \Omega \) then, as \( u = v \) on \( \partial \Omega \) we obtain \( \nabla u \cdot n \leq \nabla v \cdot n \). But integrating the equation of \( P \) in \( \Omega \) and using the the Stellarator Condition, the Divergence theorem and the strictly decreasing character of \( F \) we arrive to

\[\int_{\partial \Omega} \nabla u \cdot n > \int_{\partial \Omega} \nabla v \cdot n\]

leading then to a contradiction.

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2 References

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