ON THE BOUNDARY LAYER FOR DILATANT FLUIDS

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1 Introduction: the boundary layer and the von Mises transformation.

This paper deals with the boundary layer associated to a class of non-Newtonian fluids, i.e., fluids for which the stress tensor $T$, at given temperature and pressure, is not a linear function of the spatial variation of the velocity $L = \nabla v$. This class of fluids is relevant in many contexts: chemical engineering (polymer melts, polymer solutions, suspensions, lubricants, paints, etc.), liquid crystals, oriented media, fibrous media, animal blood etc. (see, e.g., Schowalter [28] and Narasimhan [17]). The above notion of non-Newtonian fluids fails to bound the subject. An important subclass is the so called purely viscous non-Newtonian fluids. To introduce this notion we start from the Reiner-Rivlin principle of material objectivity

$$T = -P\mathbf{I} + \phi_1(I_1, I_2)D + \phi_2(I_2, I_3)D^2,$$

where $P$ is the pressure, $\mathbf{I}$ is the identity tensor and $I_i$ ($i = 1, 2, 3$) are the principal invariants of $D = \frac{1}{2}(\nabla v + \nabla v^T)$, the symmetric part of $L$. The special case of $\phi_1$ identically constant and $\phi_2 = 0$ corresponds to the case of incompressible Newtonian fluids. The more general case of $\phi_2 = 0$ and non-constant $\phi_1$ defines
the class of purely viscous non Newtonian fluids (also called generalized Newtonian fluids). It is useful to introduce the shear stress function

$$\tau(\kappa) = \frac{1}{2} \phi(\kappa) \kappa$$

where $\kappa$ represents the shear rate. The Power-law or Ostwald-de Waele model is the one associated to the case of

$$\tau(\kappa) = K |\kappa|^{p-2} \kappa.$$

where $p > 1$ is given as a constitutive property of the fluid. If $p = 2$ we find again the class of Newtonian fluids. The case of $p > 2$ corresponds to the so called dilatant fluids and the case $1 < p < 2$ to the pseudoplastic fluids.

The Navier-Stokes system associated to a two-dimensional stationary flow of an incompressible dilatant fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial}{\partial x} (|D|^{p-2} \frac{\partial u}{\partial x}) + \frac{\nu}{2} \frac{\partial}{\partial y} (|D|^{p-2} \frac{\partial u}{\partial y}) + \frac{\partial}{\partial y} (|D|^{p-2} \frac{\partial v}{\partial x})$$

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where $\mathbf{v} = (u, v)$ is the velocity, $P$ the pressure,

$$D = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} \end{pmatrix}$$

and

$$|D|^2 = u_x^2 + \frac{1}{2} (u_y + v_x)^2 + v_y^2.$$

In 1904, L. Prandtl [22] studied the influence of viscosity in a Newtonian flow at high Reynolds number in the presence of an obstacle. If we assume that the flow is exterior to a body (here represented by the interval $(0, X)$ in the $x$-axis) and that a representative value of the modulus of the velocity is $V$, then the Reynolds number is $R = \frac{VX}{\nu}$ (we can assume, for simplicity, that $\rho \equiv 1$). The transition from zero velocity at the wall to the free stream velocity (velocity of the outer flow) $(U(x), 0)$ takes place in a very thin layer: the boundary layer. To study such a layer, Prandtl used some simplifications. For instance, it is natural to expect that

$$\frac{\delta}{X} \ll 1,$$

where $\delta$ is the boundary layer thickness. It is not difficult to see that this property is equivalent to the condition

$$\left| \frac{\partial u}{\partial y} \right| \gg \left| \frac{\partial u}{\partial x} \right|.$$
Using dimensional analysis it can be shown that under this condition

\[ \left| \frac{\partial P}{\partial y} \right| << 1. \]

So, following Prandtl, we can assume the Bernoulli equation for the outer flow to be

\[ U(x) \frac{dU}{dx}(x) = -\frac{dP}{dx}(x). \]

Neglecting smaller terms, the Navier-Stokes system leads to the following problem:

\[
\begin{cases}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = UU_x + \nu \frac{\partial^2 u}{\partial y^2} & \text{in } Q, \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 & \text{in } Q, \\
u(0, y) = u_0(y) & y > 0, \\
v(x, 0) = 0, v(x, 0) = v_0(x) & x \in (0, X), \\
v(x, y) \rightarrow U(x) \text{ as } y \rightarrow \infty & x \in (0, X).
\end{cases}
\]

where \( Q = \{(x, y) : 0 < x < X, 0 < y\} \). In most physical problems \( v_0(x) \equiv 0 \); nevertheless, the case \( v_0(x) \leq 0 \) is also relevant in the so called suction problems.

To study problem \((PS)\), several reformulations are proposed in the literature. The key point is to work with the stream function \( \psi \) given by

\[
\begin{cases}
u = \frac{\partial \phi}{\partial y} \\
v = -\frac{\partial \phi}{\partial x} + u_0, \psi(x, 0) = 0
\end{cases}
\]

Notice that the level lines of \( \psi \) coincide with the current lines of \( v = (u, v) \).

The first mathematical treatment of \((PS)\) is carried out by studying the third order ordinary differential equation satisfied by \( \psi \) (see Schlichting [27] for the case of Newtonian flows). The second possibility is to introduce the von Mises transformation [34]

\[
\begin{align*}
\psi &= \psi(x, y) & \psi \in (0, \infty), \\
w(x, \psi) &= u^2(x, y) & x \in (0, X).
\end{align*}
\]

In this way, we arrive at the scalar problem

\[
\begin{cases}
\frac{\partial u}{\partial x} - \nu \sqrt{w} \frac{\partial}{\partial \psi} \left( \frac{\partial w}{\partial \psi} \right)^{P-2} \frac{\partial w}{\partial \psi} + v_0 \frac{\partial w}{\partial \psi} - 2UU_x = 0 & x \in (0, X), \psi \in (0, \infty), \\
w(0, \psi) = w_0(\psi) & \psi \in (0, \infty), \\
w(x, 0) = 0 & x \in (0, X), \\
w(x, \psi) \rightarrow U^2(x) & as \; \psi \rightarrow \infty,
\end{cases}
\]

where \( w_0(\psi) \) is defined through \( u_0(y) \). The P.D.E. appearing in \((P_w)\) is a nonlinear degenerate parabolic equation in which the \( x \) variable plays the role of time and \( \psi \) stands for the spatial variable. Some existence and uniqueness results for this
problem are due to Oleinik [18], [19] (case of $p = 2$) and Samokhin [26] (case of $p > 2$). The assumptions of those papers are the following:

$$ U(x) > 0 \quad \text{for } x \in (0, X), $$
$$ u_0(0) = 0 \text{ and } u_0(y) > 0 \quad \text{for } y > 0, $$
$$ u'_0(0) = 0, \quad (u_0', u_0^2) \in L^\infty(0, \infty)^2 $$

$$ U(0)U_x(0) - u_0(0) \frac{du}{dy} + \nu \left( \frac{du}{dy} \right)^{p-2} \frac{d^2 u}{dy^2} = 0(y^2) \quad \text{(consistency condition)}. $$

We also mention the results by Oleinik [19], Serrin [29] and Peletier [21] on the asymptotic behavior when $X = +\infty$.

2 The results

The main goal of this work is to study the coincidence set

$$ \{(x, \psi) : w(x, \psi) = U^2(x)\} $$

for the case of dilatant fluids. The boundary of this region could be called the exact boundary layer.

Remark 1 By the weak maximum principle, it is well known that necessarily $w(x, \psi) \leq U^2(x)$ in $(0, X) \times (0, \infty)$. In fact, if $p = 2$, it can be shown (see Oleinik [19]) that the strong maximum principle also holds and thus $w(x, \psi) < U^2(x)$ in $(0, X) \times (0, \infty)$, i.e. the coincidence set is empty. We recall that there are several attempts to make the boundary layer concept more precise. For instance, in Schlichting [27] it is defined as the zone where $u=0$. Our main results are the following

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**Theorem 1** (Existence of the coincidence set).

Assume $p>2, v_0(x) \leq 0$, and there exists $\psi_0 \in (0, \infty)$ such that $w_0(\psi) = U^2(0)$ for any $\psi \geq \psi_0$; Then there exists $C > 0$ and $\mu \in (0, 1)$ such that

$$ w(x, \psi) = U^2(x) \text{ for any } (x, \psi) \text{ such that } \psi \geq \psi_0 + Cx^\mu. $$

**Theorem 2** (Waiting distance along a streamline)

Assume $p > 2, v_0(x) \leq 0$ and that there exists $\psi_0 \geq 0$, $C > 0$ and $\sigma \in (0, 1)$ such that,

$$ \int_0^\infty (U^2(0) - w_0(\tau))^2 d\tau \leq C(\psi_0 - \psi)^{\sigma/(1-\sigma)} $$

for any $\psi \in (\psi_0 - \varepsilon, +\infty)$. Then, there exists $x_0 \in (0, X)$ such that

$$ w(x, \psi_0) = U^2(x) \text{ for any } x \in [0, x_0]. $$
Sketch of the proof of Theorem 1. It is based on a general Energy Method first introduced by one of the authors [1] and later improved and developed in [2], [3], [4], [5], [6], [7], [8], [9], [11], [12], [13], [14], [15], [16], [20] and [30] (see also [10], [23], [24], [31] and [32]).

First step. We introduce the homogenized unknown

\[ z(x, \psi) = U^2(x) - w(x, \psi). \]

We remark that by the comparison principle \( z(x, \psi) \geq 0 \) on \((0, X) \times (0, \infty)\). We also point out that arguing as in [18], or [26], it is possible to obtain the a priori estimate

\[ 0 < C_1 \leq U^2(x) - z(x, \psi) \leq C_2 \text{ for any } x \in (0, X), \psi \in (0, \infty), \]

for some constants \( C_1 \leq C_2 \). On the other hand, it is easy to see that \( z \) satisfies

\[
(P_z) \left\{ \begin{array}{ll}
\frac{\partial z}{\partial x} - \nu \sqrt{U^2 - z} \frac{\partial}{\partial \psi} \left( \frac{\partial^2}{\partial \psi^2} + w_0 \frac{\partial}{\partial \psi} \right) z = 0 & \text{ in } (0, X), \psi \in (0, \infty), \\
(0, \psi) = U^2(0) - w_0(\psi) & \psi \in (0, \infty), \\
z(x, 0) = U^2(x) & x \in (0, X), \\
z(x, \psi) \to 0 & \text{ as } \psi \to \infty.
\end{array} \right.
\]

We remark that \( z(x, \psi) = 0 \) on the coincidence set.

Second step: Integration by parts formula. We introduce the one-parameter energy domain

\[ Q_\rho = (0, x) \times (\rho, \infty) \]

where \( \rho \geq \psi_0 \) is arbitrary. Multiplying by \( z \) and by integrating (formally) by parts we obtain that

\[ \frac{1}{2} \int_0^\infty z^2(x, \psi) d\psi + \int_0^\infty \theta(s, \psi) \left| \frac{\partial z}{\partial \psi} (s, \psi) \right|^p d\psi + \frac{1}{2} \int_0^\infty (U^2(s, \psi) z^2(\psi) d\psi - \int_0^\infty \sqrt{w^2 - \left| \frac{\partial z}{\partial \psi} \right|^2} \left. \right|_{\psi = \rho} ds \]

where

\[ \theta(x, \psi) = \frac{2U^2(x) - 3z(x, \psi)}{2\sqrt{U^2(x) - z(x, \psi)}}. \]

It is easy to see that

\[ 0 < C_3 \leq \theta(x, \psi) \leq C_4 \text{ for any } x \in (0, X), \psi \in (0, \infty). \]

Third step. We introduce the energy functions

\[ b(x, \rho) = \sup_{0 \leq \psi \leq x} \frac{1}{2} \int_0^\infty z^2(x, \psi) d\psi. \]
\[ E(x, \rho) \geq \int_0^x \int_0^\infty \theta(s, \psi) \left| \frac{\partial^2 z}{\partial \psi^2}(s, \psi) \right|^p \, ds \, d\psi. \]

Applying the Holder inequality, we get that
\[
\left( \int_0^x \sqrt{\omega z} \left| \frac{\partial^2 z}{\partial \psi^2} \right| |z|^{\sigma-2} \, ds \right)^{\frac{1}{\sigma}} \leq C_\delta \left( \int_0^x \left| \frac{\partial E}{\partial \rho}(x, \rho) \right| \, ds \right)^{\frac{1}{2}} \left( \int_0^x |z(s, \psi)|^p \, ds \right)^{\frac{1}{2}}.
\]

Now we need a technical result

Lemma 3 (Trace-interpolation inequality, [16]). Let \( \sigma \in (0, 1) \) be given by \( \sigma = (p + 2)/3p \). Then
\[
\left( \int_0^x |z(s, \psi)|^p \, ds \right)^{\frac{1}{p}} \leq C_0 x^{(1-\sigma)/p} (E^{1/p} + x^{1/2} b^{1/2} y^{1-\sigma/2}).
\]

End of the proof of Theorem 1. By using the above inequalities we can find an exponent \( \mu \in (0, 1) \) and a positive constant \( C_\mu \) such that
\[
E^\mu \leq (E + b)^\mu \leq C_\mu x^{(1-\mu)/(p-1)} \left( \frac{\partial E}{\partial \rho}(x, \rho) \right).
\]

This inequality implies the conclusion due to the following easy result

Lemma 4 ([1]) Let \( y \in C([0, t_1] \times [0, \rho_0 + \delta]) \), \( y \geq 0 \) such that
\[
\Phi(y(t, \rho)) \leq C \omega y(t, \rho)
\]
for a.e. \( \rho \in [0, \rho_0 + \delta] \) and for any \( t \in [0, t_1] \), where \( \omega \geq 0 \) and \( \Phi \) is a continuous nondecreasing function such that \( \Phi(0) = 0 \) and
\[
\int_{\partial \Omega} \frac{\partial \Phi}{\partial n}(s) < \infty.
\]

Then there exists \( t_0 \in (0, t_1] \) and a function \( \rho(t) \) with \( 0 \leq \rho(t) \leq \rho_0 + \delta \) such that \( y(t, \rho) = 0 \) for any \( t \in [0, t_0] \) and any \( \rho \in [0, \rho(t)] \).

Remark 2 A different proof of Theorem 1, based upon the comparison principle, and under additional conditions, is due to [26].

Idea of the proof of Theorem 2. Using the same type of arguments and the assumption at \( x = 0 \) we obtain the differential inequality
\[
E^\mu \leq C_\mu x^{(1-\mu)/(p-1)} \left( \frac{\partial E}{\partial \rho}(x, \rho) \right) + C_\delta (\psi_0 - \psi)^{\mu/(1-\mu)}
\]
for any \( \psi \in (\psi_0 - \epsilon, +\infty) \). The conclusion comes now from an analysis of this differential inequality.
Lemma 5 (2) Let \( y \in C([0,t_1] \times [0,\rho_0 + \delta]), y \geq 0 \) such that

\[
\Phi(y(t,s)) \leq Ct^\omega \frac{\partial y}{\partial \rho}(t,s) + G((\rho - \rho_0)_+),
\]

for a. e. \( \rho \in [0;\rho_0 + \delta] \) and for any \( t \in [0,t_1] \), where \( \omega \geq 0 \) and \( \Phi \) is as in Lemma 4 and

\[
\exists \mu > 0 \text{ and } \varepsilon > 0 \text{ such that } G(s) \leq \varepsilon \Phi(\eta_\mu(s)), \text{ a.e. } s \in (0,\rho)
\]

with

\[
\eta_\mu(s) = \Theta_\mu^{-1}(s), \quad \Theta_\mu(\tau) = \int_{0^+}^{\tau} \frac{ds}{\mu \Phi(s)}.
\]

Then there exists \( t^* \in (0,t_1] \) such that \( y(t,\rho) = 0 \) for any \( t \in [0,t^*] \) and any \( \rho \in [0,\rho_0] \).

Remark 3 In the case of pseudo-plastic fluids \((1 < p < 2)\), it is possible to apply another kind of energy method (now using a suitable global energy) which leads to a different estimate on the location of the exact boundary layer: if \( X \) is large enough and \( U(x) \equiv 0 \ \forall x \geq x_U \) for some \( x_U > 0 \), then there exists \( x_0 \geq x_U \) such that \( w(x,\psi) = U^2(x) = 0, \forall x \geq x_0, \forall \psi \in (0,\infty) \) (see also [25]).

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References


