Discrete Comput Geom 12:451–464 (1994)

Discrete & Computational
Geometry

An Algorithm To Compute Odd Orders and Ramification Indices of Cyclic Actions on Compact Surfaces*

E. Bujalance,[1] A. F. Costa,[1] J. M. Gamboa,[2] and J. Lafuente[3]

[1] Departamento de Matemáticas Fundamentales, Fac. Ciencias,
U.N.E.D., 28040 Madrid, Spain
emilio.bujalance@uned.es
antonio.costa@uned.es

[2] Departamento de Algebra, Fac. C. Matemáticas,
Universidad Complutense, 28040 Madrid, Spain

[3] Departamento de Geometría y Topología, Fac. C. Matemáticas,
Universidad Complutense, 28040 Madrid, Spain

Abstract. In this paper we get an effective algorithm to compute all odd orders and ramification indices of homeomorphisms acting on compact surfaces, orientable or not.

Introduction

A classical problem is to determine the actions of finite groups on compact topological surfaces. The groups acting on surfaces of genus 2 (resp. genus 3) were obtained by Wiman [13], Maclachlan [10], and Scherk [12]. More recently, Broughton [2] classified all group actions for genera 2 and 3.

In this paper we get an effective algorithm to compute all odd orders and ramification indices of homeomorphisms acting on compact surfaces, orientable or not. The study of even orders is more involved; it uses the odd case and it will be treated by the authors in a forthcoming paper.

We remark that our problem is equivalent to determining all odd orders

* E. Bujalance and A. F. Costa were partially supported by DGICYT PB 89-201 and Science Plan N SC1-CT91-0716, J. M. Gamboa was partially supported by DGICYT PB 89-379 and Science Plan N SC1-CT91-0716, and J. Lafuente was partially supported by Science Plan N SC1-CT91-0716.

and ramification indices of automorphisms of compact Klein surfaces. The input of our algorithm is a triple (g, k, α) where g, k are nonnegative integers and $\alpha = 1$ or 2. The output is the set $O(g, k, \alpha)$ of all odd integers $N > 1$ such that an orientable, if $\alpha = 2$, or nonorientable, if $\alpha = 1$, compact Klein surface of genus g exists whose boundary has k connected components, admitting an automorphism of order N. Also, whenever N occurs in $O(g, k, \alpha)$, and if f is an automorphism of order N acting on the surface S, the algorithm provides the topological type of the quotient $S' = S/\langle f \rangle$ and the ramification indices of the natural projection $S \to S'$. Of course the functorial correspondence between compact Klein surfaces and projective, smooth, real or complex algebraic curves allows us to compute, using the algorithm, the orders of birational automorphisms on such curves.

The article is organized as follows: the theoretical background needed to understand the paper is contained in Section 1. The algorithm, in the case $g + \alpha > 3$, is explained and described in Section 2, while the case $g + \alpha < 3$ is treated in Section 3, where some alternative procedures are also briefly commented on. To finish, we give some concrete examples.

1. Preliminaries

For fixed data g, k, α, let us denote by $K(g, k, \alpha)$ the family of orientable, if $\alpha = 2$, or nonorientable, if $\alpha = 1$, compact Klein surfaces of genus g whose boundary has k connected components. Independently of the value of k, if $g + \alpha > 3$, then every surface $S \in K(g, k, \alpha)$ can be written, by the Riemann Uniformization Theorem (see [11]), as a quotient $S = H/\Gamma$ of the hyperbolic plane $H = \{z \in C\colon \mathrm{Im}(z) > 0\}$ under the action of a non-Euclidean crystallographic (NEC in short) group Γ, i.e., Γ is a discrete subgroup of the extended Möbius transformations consisting of the transformations

$$z \to \frac{az + b}{cz + d}, \quad ad - bc > 0 \quad \text{or} \quad z \to \frac{a\bar{z} + b}{c\bar{z} + d}, \quad ad - bc < 0, \qquad a, b, c, d \text{ real}.$$

The algebraic structure of the group Γ is completely determined by the following list of numbers,

$$\sigma(\Gamma) = (g; \alpha; [m_1, \ldots, m_r]; \{(n_{i1}, \ldots, n_{is_i})\colon i = 1, \ldots, k\}),$$

which is called the signature of Γ. The meaning of g, α, and k is evident, while the numbers $m_1, \ldots, m_r$ (resp. $n_{i1}, \ldots, n_{is_i}$) denote the ramification indices of the canonical projection $\pi\colon H \to S$ on the interior points of S (resp. on the i-boundary component of S). Moreover, a finite group G acts as a group of automorphisms on S if and only if another NEC group Λ containing Γ as a normal subgroup exists such that $G = \Lambda/\Gamma$. This way, the classical Riemann–Hurwitz formula can be read as: order of $G = \mu(\Gamma)/\mu(\Lambda)$, where μ denotes the hyperbolic area of a fundamental region of the corresponding group. For these basic results the reader can see, e.g., [4].

According to $\alpha = 1$ or 2 and $k = 0$ or $k \neq 0$, results concerning the set $O(g, k, \alpha)$ appear in [7, Theorem 4], [3, Theorem 3.7] and [4, Theorems 3.1.2 and 3.1.3]. We summarize them as follows:

Theorem 1.1. *Let g, k be nonnegative integers and let $\alpha = 1, 2$, with $g + \alpha > 3$. A positive odd integer $N \in O(g, k, \alpha)$ if and only if nonnegative integers r, g', k' with $\alpha + g' \geq 2$, and positive divisors $m_1, \dots, m_r, l_1, \dots, l_{k'}$, of N, each $m_i \geq 2$, exist such that:*

(a) $\alpha g + k - 2 = N(\alpha g' + k' - 2 + \sum_{i=1}^{r} (1 - 1/m_i))$.
(b) $k = \sum_{j=1}^{k'} N/l_j$.
(c) *If $\alpha + g' = 2$, then* $\text{l.c.m.}\{m_1, \dots, m_r, l_1, \dots, l_{k'}\} = N$.
(d) *If $\alpha = 2$, the l.c.m. of the set $\{m_1, \dots, m_r, l_1, \dots, l_{k'}\}$ equals the l.c.m. of the set obtained after deleting one of these elements.*

Remarks. 1. The numbers $m_1, \dots, m_r$ (resp. $l_1, \dots, l_{k'}$) do not necessarily need to be distinct.

2. By definition, the l.c.m. of the empty set equals 1.

3. Assume that $N \in O(g, k, \alpha)$ satisfy the conditions above and the integers g', k', $m_1, \dots, m_r$, $l_1, \dots, l_{k'}$ solve equations (a)–(d). Then a surface $S \in K(g, k, \alpha)$ and an automorphism f on S of order N exist and, from the proof of Theorem 1.1, the quotient surface

$$S' = S/\langle f \rangle \in K(g', k', \alpha),$$

the numbers $m_1, \dots, m_r$ are the ramification indices on the interior points of S' of the canonical projection $S \to S'$, and the numbers $l_1, \dots, l_{k'}$ are determined by the action of f on the set of boundary components of S. In particular, note that S' is orientable if and only if the same holds for S. This follows from [8] and, among other things, it makes the study of the odd case simpler.

2. Algorithm To Compute $O(g, k, \alpha)$ in the Case $g + \alpha > 3$.

As was said before, the input of our algorithm is the triple (g, k, α). Theorem 1.1 above shows that the existence of a surface $S \in K(g, k, \alpha)$ and an automorphism on S of odd order N is equivalent to a formula with parameters and existential quantifiers. On the other hand, it is known from [13], in the orientable case, and [3], in the nonorientable one, that if $g + \alpha > 3$, then the order of all automorphisms of surfaces without boundary of genus g is bounded above by $2(\alpha g + \alpha - 1)$. Also, if f is an automorphism of $S \in K(g, k, \alpha)$ with $k > 0$, it extends to a surface without boundary $S' \in K(g, 0, \alpha)$, see Theorem D in [6]. Hence

"If $g + \alpha > 3$ and $N \in O(g, k, \alpha)$, then $N \leq 2(\alpha g + \alpha - 1)$."

In particular the parameters $m_1, \dots, m_r, l_1, \dots, l_{k'}$ of Theorem 1.1 are also bounded above, since they are divisors of N. Also, since each $m_i \geq 3$ it follows from condition (a) in Theorem 1.1 that $\alpha g' + k' + 2r/3 \leq \alpha g + k$. Hence all "variables" r, g', k', $m_1, \dots, m_r, l_1, \dots, l_{k'}$ are bounded above *a priori*, i.e., in terms of the input (g, k, α).

Thus, there is a finite set of candidates occurring in $O(g, k, \alpha)$ and to decide if a given odd number $N \leq 2(\alpha g + \alpha - 1)$ occurs in $O(g, k, \alpha)$ requires only a finite number of elementary arithmetic computations. However, we do not apply this naive method directly. Obviously, if $N \in O(g, k, \alpha)$, then $O(g, k, \alpha)$ contains all divisors of N. This leads us to determine, first, all prime powers occurring in $O(g, k, \alpha)$. Afterwards, we check the products of prime powers that appear in the set $O(g, k, \alpha)$. Consequently, the first step of our algorithm is:

Step 1. Determine the set

$$X = X(g, k, \alpha) = \{p^e \colon p \text{ in an odd prime}, e \geq 1, p^e \leq 2(\alpha g + \alpha - 1)\}.$$

Why have we begun looking for the prime powers occurring in $O(g, k, \alpha)$? The reason is that we obtained in [5] a much better upper bound for the prime powers in $O(g, k, \alpha)$ than the general one $2(\alpha g + \alpha - 1)$ referred to before. To be precise let p be an odd prime, let $e \geq 1$ be an integer, and consider the "p-adic" expansion of k:

$$k = a_e p^e + \sum_{j=0}^{e-1} a_j p^j, \qquad 0 \leq a_j < p \quad \text{for} \quad 0 \leq j \leq e-1, \quad a_e \geq 0.$$

Then define $\rho(p, e, k) = \sum_{j=0}^{e} a_j$ and it follows from Theorem 6 in [7], Theorems 4.1 and 4.3 in [3] and Theorems 1 and 3 in [5] that if $p^e \in O(g, k, \alpha)$, then

$$\alpha g + k - 1 \geq \mu(p, e, k, \alpha) \qquad \text{if} \quad (\alpha, k) \neq (2, 0)$$

and

$$g \geq \mu(p, e, 0, 2) \qquad \text{if} \quad (\alpha, k) = (2, 0),$$

where the function $\mu(p, e, k, \alpha)$ is defined as

$$\mu(p, e, k, \alpha) = \begin{cases} 2 & \text{if} \quad (p, e, k, \alpha) = (3, 1, 0, 2), \\ \dfrac{p^{e-1}(p-1)}{\alpha + k - \alpha k} + (2 - \alpha)k & \text{if} \quad (p, e, k, \alpha) \neq (3, 1, 0, 2), \quad k \leq 1, \\ 1 + p^{e-1}(p + 1 - \alpha) & \text{if} \quad k = 2, \\ p^e(\rho(p, e, k) - \alpha) + 1 & \text{if} \quad k > 2, \quad k \not\equiv 0, 1 \bmod p, \\ p^e(\rho(p, e, k) - \varepsilon) + \varepsilon + 1 - \alpha & \text{if} \quad k > 2, \quad k \equiv \varepsilon \bmod p, \quad \varepsilon = 0, 1. \end{cases}$$

Note that in the quoted papers we obtained the so-called minimum algebraic genus of surfaces with k boundary components admitting an automorphism of order p^e, and we have just expressed the bound in terms of the data (g, k, α). Hence, the second step in our algorithm is:

Step 2. Determine the set

$$Y = Y(g, k, \alpha) = \{p^e \in X : \mu(p, e, k, \alpha) \leq \begin{cases} \alpha g + k - 1 & \text{if} \quad (\alpha, k) \neq (2, 0), \\ g & \text{if} \quad (\alpha, k) = (2, 0). \end{cases}$$

Now we must check what values in Y satisfy Theorem 1.1. Since the divisors of $N = p^e$ for prime p have the form p^i with $0 \leq i \leq e$ we can reformulate Theorem 1.1 in this case as follows: with the notation there, let

$$x_i = \#\{m \in \{m_1, \ldots, m_r\} : m = p^i\}; \; y_j = \#\{l \in \{l_1, \ldots, l_{k'}\} : l = p^j\}$$

for $1 \leq i \leq e$, $0 \leq j \leq e$. Then conditions (a) and (b) in Theorem 1.1 are nothing more than:

(a′) $\alpha g + k - 2 = p^e(\alpha g' + k' - 2) + \sum_{i=1}^{e} x_i(p^e - p^{e-i})$.
(b′) $k = \sum_{j=0}^{e} y_j p^{e-j}$, $k' = \sum_{j=0}^{e} y_j$.

Let us define $\Lambda = \Lambda(x_1, \ldots, x_e; y_0, \ldots, y_e) = \{1 \leq i \leq e : x_i + y_i \neq 0\}$ and

$$M = \begin{cases} \max \Lambda & \text{if } \Lambda \text{ is not empty,} \\ 0 & \text{otherwise.} \end{cases}$$

Then condition (c) is equivalent to:

(c′) If $\alpha + g' = 2$, then $M = e$.

Moreover, to say that the l.c.m. of $m_1, \ldots, m_r, l_1, \ldots, l_{k'}$ does not change if we delete one of these numbers is equivalent to saying that the maximum power of p occurring in $\{m_1, \ldots, m_r, l_1, \ldots, l_{k'}\}$, if distinct from one, occurs at least twice. In other words, (d) is equivalent to:

(d′) If $\alpha = 2$ and $M \neq 0$, then $x_M + y_M \geq 2$.

This way, the third step in our algorithm is:

Step 3. For every prime $p \in X(g, k, \alpha)$, let $e = e(p)$ be the positive integer such that $p^e \in Y(g, k, \alpha)$ but $p^{e+1} \notin Y(g, k, \alpha)$. Afterwards we decide if nonnegative integers g', k' with $\alpha + g' \geq 2$ and positive integers $x_1, \ldots, x_e$; $y_0, \ldots, y_e$ satisfying conditions (a′)–(d′) above exist. If this is so, we know that $1, p, \ldots, p^e \in O(g, \alpha, k)$. If not we repeat the process with p^{e-1}, and so on. The biggest p^d occurring in $O(g, k, \alpha)$ is called the p-primary solution.

How the computer handles the diophantine equations (a′)–(d′) is explained later in Section 4.

After this step we know all the odd prime powers occurring in $O(g, k, \alpha)$. To deal with the general case, let $\{p_1^{e_1}, \ldots, p_h^{e_h}\}$ be the set of all primary elements in $O(g, k, \alpha)$ and $C = C(g, k, \alpha) = p_1^{e_1} \cdots p_h^{e_h}$. The elements of $O(g, k, \alpha)$ divide C. Now, if in Theorem 1.1 we group the divisors m_i's (resp. l_j's) with the same value, the fourth step is:

Step 4. First determine the set

$$T = \{N \in \mathbf{N}\colon N \text{ divides } C, N \text{ is not a prime power}, N \leq 2(\alpha g + \alpha - 1)\}.$$

Each element $N \in T$ has the form $N = p_1^{v_1} \cdots p_h^{v_h}$ and the map

$$T \to \mathbf{N}^{h+1}\colon N \to (v_1 + \cdots + v_h, v_1, \ldots, v_h)$$

is injective. Hence we obtain a total ordering in T defined as

$$p_1^{v_1} \cdots p_h^{v_h} \leq p_1^{v_1'} \cdots p_h^{v_h'}$$

if $(v_1 + \cdots + v_h, v_1, \ldots, v_h)$ is smaller than $(v_1' + \cdots + v_h', v_1', \ldots, v_h')$ in the lexicographical order. Then, for the biggest element in T with respect to this order, we check if nonnegative integers g', k', with $g' + \alpha \geq 2$, two nonnegative integers s and t, distinct divisors $m_1, \ldots, m_t$ of N, distinct divisors $l_1, \ldots, l_s$ of N, and nonnegative integers $\mu_1, \ldots, \mu_t, \lambda_1, \ldots, \lambda_s$, $\mu_i \geq 1$, exist such that:

(a″) $\alpha N g' + N k' + \sum_{i=1}^{t} \mu_i (N - N/m_i) = 2N + \alpha g + k - 2$.
(b″) $k = \sum_{j=1}^{s} \lambda_j N/l_j$, $k' = \sum_{j=1}^{s} \lambda_j$.
(c″) If $\alpha + g' = 2$, then $\text{l.c.m.}\{m_1, \ldots, m_t, l_1, \ldots, l_s\} = N$.
(d″) If $\alpha = 2$ and F is the collection of all numbers $l_1, \ldots, l_s, m_1, \ldots, m_t$, each l_j counted λ_j times and each m_i counted μ_i times, then

$$\text{l.c.m. } F = \text{l.c.m.}(F - \{f\}) \qquad \text{for all} \quad f \in F.$$

Once this is done with the biggest element in T the process continues with the next one, and so on. This provides all the elements in $O(g, k, \alpha)$. Of course, whenever we check that $N = p_1^{v_1} \cdots p_h^{v_h}$ satisfies conditions (a″)–(d″) above, it follows that the same holds true for $N' = p_{1_1}^{v_1'} \cdots p_{h_h}^{v_h'}$ if each $v_q' \leq v_q$, $1 \leq q \leq h$, and the computer does not check these cases.

This finishes the algorithm to produce $O(g, k, \alpha)$ in the case $g + \alpha > 3$. Notice that to know that some odd $N \in O(g, k, \alpha)$ it suffices to know that equations (a″)–(d″) admit some solution. However, our procedure provides us with all the solutions of this system of equations. Hence, as remarked in Section 1, to get all possible topological types of surfaces of the form $S/\langle f \rangle$ where S runs over $K(g, k, \alpha)$ and f is an automorphism of order $N \in O(g, k, \alpha)$, and to know the ramification indices of the projection $\pi\colon S \to S/\langle f \rangle$ on the interior points of $S/\langle f \rangle$ and the action of f on the boundary components of S we go to:

Step 5. For every value $N \in O(g, k, \alpha)$ obtained in Step 4, find all possible solutions $g', k', l_1, \ldots, l_s, \lambda_1, \ldots, \lambda_s, m_1, \ldots, m_t, \mu_1, \ldots, \mu_t$ of system (a″)–(d″).

Then $S/\langle f \rangle \in K(g', k', \alpha)$, the projection π has a ramification set of the form $\{m_i, \overset{\mu_i}{\ldots}, m_i : i = 1, \ldots, t\}$, and f groups the boundary components of S in the following way: for every $j = 1, \ldots, s$ there are exactly λ_j blocks of N/l_j components of ∂S, each of them mapped onto the same connected component of the boundary of $S/\langle f \rangle$.

3. Algorithm To Compute $O(g, k, \alpha)$ in the Case $g + \alpha \leq 3$.

Let f be a homeomorphism of finite order odd N on the surface S. We say that the action of f on S is free if the set $\text{Fix}(f^l) = \{x \in S: f^l(x) = x\}$ of fixed points of f^l is empty, for all $0 < l < N$. Our first result in this section concerns the case $k = 0, \alpha = 2$ (and so $g = 0, 1$), i.e., the sphere and the torus. We abbreviate $2\mathbf{N} + 1$ for the set of odd positive integers.

Theorem 3.1.

(a) $O(0, 0, 2) = O(1, 0, 2) = 2\mathbf{N} + 1$.
(b) *Let f be a homeomorphism of odd order >1 on the sphere S. Then $Fix(f)$ has two points and the quotient $S/\langle f \rangle$ is a sphere.*
(c) *Let f be a homeomorphism of odd order $N \neq 3$ on the torus T. Then the action of f on T is free and the quotient $T/\langle f \rangle$ is a torus.*
(d) *Homeomorphisms f_1 and f_2 of order 3 on the torus T exist such that the action of f_1 on T is free,* Fix(f_2) *has three points, $T/\langle f_1 \rangle$ is a torus, and $T/\langle f_2 \rangle$ is a sphere.*

Proof. (a) Let N be an odd positive integer. Then the rotations of angle $2\pi/N$ shown in Fig. 1 are automorphisms of order N.

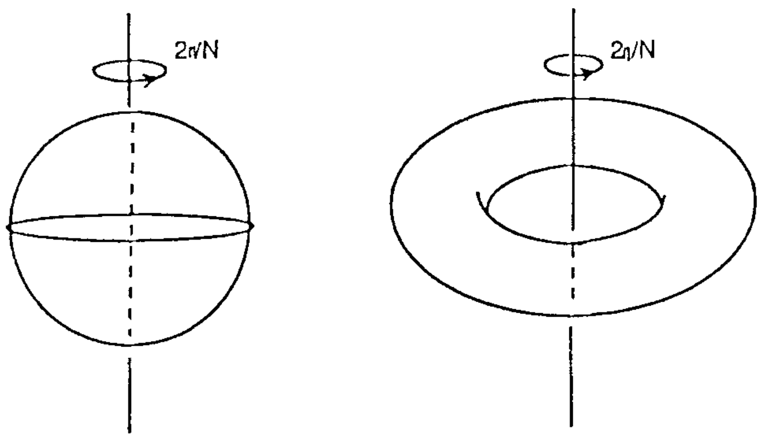

Fig. 1. Rotations of angle $2\pi/N$.

(b) Let N be the order of f. If χ denotes the Euler characteristic, then the Riemann–Hurwitz formula says

$$2 = \chi(S) = N_\chi(S/\langle f \rangle) - N \sum_{i=1}^{r} (1 - 1/m_i),$$

where $m_1, \ldots, m_r$ are the branching indices of the covering $\pi: S \to S/\langle f \rangle$. Then, necessarily, $\chi(S/\langle f \rangle) = 2$, i.e., $S/\langle f \rangle$ is a sphere, since otherwise the right-hand side of the formula above would be negative. Hence,

$$N < 2(N-1) = N \sum_{i=1}^{r} (1 - 1/m_i) < 2N$$

and, since each $m_i \geq 3$ because m_i divides N, we get $r = 2$. Thus

$$2 = N/m_1 + N/m_2, \quad \text{i.e.,} \quad m_1 = m_2 = N,$$

and therefore f fixes two points.

(c) and (d) Assume that f is an automorphism which acts nonfreely on T and let N be its odd order. We check that $N = 3$. In fact,

$$0 = \chi(T) = N_\chi(T/\langle f \rangle) - N \sum_{i=1}^{r} (1 - 1/m_i)$$

and since the action is not free, we can assume $m_1 > 1$. Thus $\chi(T/\langle f \rangle) > 0$, i.e., $\chi(T/\langle f \rangle) = 2$ and $T/\langle f \rangle$ is a sphere. Also

$$. \ 2 = \sum_{i=1}^{r} (1 - 1/m_i)$$

and, with each m_i being odd, the unique solution of the last equation is $r = 3$, $m_1 = m_2 = m_3 = 3$. Let p_1, p_2, p_3 be the branching points of the covering $\pi: T \to T/\langle f \rangle$. We get a surjective group homomorphism

$$\pi_*: \pi_1(T/\langle f \rangle - \{p_1, p_2, p_3\}) \to \mathbf{Z}_N,$$

where $\pi_1(T/\langle f \rangle - \{p_1, p_2, p_3\})$ is the fundamental group and is generated by the meridians x_1, x_2, x_3 around the points p_1, p_2, p_3. Since $\mathbf{Z}_N$ is generated by the images $\pi_*(x_i)$, each of them of order $m_i = 3$, we obtain $N = 3$. This proves part (c) since, of course, when f acts freely, the Riemann–Hurwitz formula gives $0 = \chi(T) = N\chi(T/\langle f \rangle)$, i.e., $\chi(T/\langle f \rangle) = 0$ and $T/\langle f \rangle$ is a torus. The existence of f_1 of part (d) was proved in (a) while the homeomorphism f_2 is given by the rotation of angle $2\pi/3$ which fixes the points O, P, and Q on the torus of Fig. 2. □

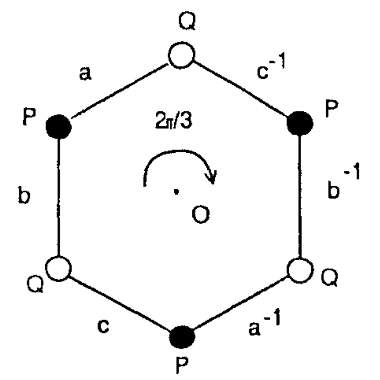

Fig. 2. The homeomorphism f_2.

We now study nonorientable surfaces with empty boundary, i.e., the projective plane and the Klein bottle. We first need the following:

Lemma 3.2. *Let S be a compact nonorientable surface without boundary and let $\hat{S}$ be its orientable double covering. Then every homeomorphism f of finite odd order N on S can be lifted to a homeomorphism $\hat{f}$ of $\hat{S}$ of order N.*

Proof. Let $\pi_1^+(S)$ be the subgroup of orientation-preserving elements of the fundamental group $\pi_1(S)$. Clearly, since f is a homeomorphism, the subgroup $\pi_1^+(S)$ is preserved under the action of f on $\pi_1(S)$, and this proves the lemma because $\hat{S}$ is given by $\pi_1^+(S)$. □

Let S denote either the projective plane or the Klein bottle. Its orientable double covering $\hat{S}$ is, respectively, the sphere or the torus. In both cases, let τ be the orientation-preserving involution on $\hat{S}$ such that $S = \hat{S}/\langle\tau\rangle$. Let us denote

$$H^\tau(\hat{S}) = \{f\colon \hat{S} \to \hat{S},\ f \text{ is homeomorphism on } \hat{S} \text{ and } \tau f = f\tau\}.$$

For every $f \in H^\tau(\hat{S})$ we get an odd order homeomorphism $F\colon S \to S$ defined by $F(0_x) = 0_{f(x)}$, where $0_x \in S$ is the orbit, under the action of τ, of the point $x \in \hat{S}$. From the lemma, the assignment $f \to F$ from $H^\tau(S)$ onto the group of homeomorphisms of S is surjective and preserves orders. Hence, as an immediate consequence of Theorem 3.1 we get:

Theorem 3.3.

(a) $O(1, 0, 1) = O(2, 0, 1) = 2\mathbf{N} + 1$.
(b) *Let f be a homeomorphism of odd order > 1 on the projective plane P. Then* $\text{Fix}(f)$ *is a unique point and $P/\langle f\rangle$ is a projective plane.*
(c) *Let f be a homeomorphism of odd order > 1 on the Klein bottle K. Then the action of f on K is free and $K/\langle f\rangle$ is a Klein bottle.*

Table 1

Surface	g	k	α	$O(g, k, \alpha)$	B.p.	R.i.	Quotient	Orb. dis.
Disk	0	1	2	$2N+1$	1	$2N+1$	Disk	(1, 1)
Annulus	0	2	2	$2N+1$	0		Annulus	(2, 1)
Möbius S.	1	1	1	$2N+1$	0		Möbius S.	(1, 1)

Remark. Notice that in the situation above, the unique homeomorphism on $\hat{S}$ which does not induce a homeomorphism on S is the homeomorphism of order 3 on the torus with three fixed points.

To finish this section we deal with the cases $\alpha + g \leq 3$, $k > 0$. The key point here, already used in [6], is that every automorphism f on the bordered surface S can be extended to an automorphism f^* of the unbordered surface S^* obtained from S by filling its holes with disks. From this and Theorems 3.1 and 3.3 we obtain:

Theorem 3.4. *Let α, g, k be three integers with $k > 0, g \geq 0, \alpha = 1, 2$, and $\alpha + g \leq 3$. Then, according to $\alpha g + k \leq 2$ or $\alpha g + k > 2$, the set $O(g, k, \alpha)$ and, for every $N \in O(g, k, \alpha)$, the topological features of the projection $\pi: S \to S\langle f \rangle$, where $S \in K(g, k, \alpha)$ and $f \in \mathrm{Aut}(S)$ has order N, are given in Tables 1 and 2.*

In these tables, given a positive integer l, we denote by $D(l)$ the set of odd positive divisors of l. Also the integers g', k', α' denote, respectively, the genus, number of boundary components, and orientability character of the quotient $S/\langle f \rangle$. The abbreivations B.p. and R.i. mean the number of branching points and ramification indices, respectively. Finally, the pairs (x, y) in the last column indicates "x orbits with y elements each."

Table 2

g	k	α	$O(g, k, \alpha)$	B.p.	R.i.	g'	k'	α'	Orb. dis.
0	>2	2	$D(k)$	2	N	0	k/N	2	$(k/N, N)$
0	>2	2	$D(k-1)$	1	N	0	$(k+N-2)/N$	2	$((k-1)/N, N)(1, 1)$
0	>2	2	$D(k-2)$	0		0	$(k+2N-1)/N$	2	$((k-2)/N, N)(2, 1)$
1	>0	2	$D(k)$	0		1	k/N	2	$(k/N, N)$
1	>0	2	$3 \in D(k)$	0		0	$(k+6)/3$	2	$((k-3)/3, 3)(3, 1)$
1	>0	2	$3 \in D(k)$	3	3	0	$k/3$	2	$(k/3, 3)$
1	>0	2	$3 \in D(k-1)$	2	3	0	$(k+2)/3$	2	$((k-1)/3, 3)(1, 1)$
1	>0	2	$3 \in D(k-2)$	1	3	0	$(k+4)/3$	2	$((k-2)/3, 3)(2, 1)$
1	>1	1	$D(k)$	1	N	1	k/N	1	$(k/N, N)$
1	>1	1	$D(k-1)$	0		1	$(k-1)/N$	1	$((k-1)/N, N)(1, 1)$
2	>0	1	$D(k)$	0		2	k/N	1	$(k/N, N)$

Proof. As an example we explain the case $g = 0$, $k > 2$, $\alpha = 2$ in Table 2. The other ones are left to the reader. Let f be a homeomorphism of odd order N on an orientable surface S of genus 0 whose boundary has k connected components. Then a homeomorphism f^* of order N exists on the sphere S^* obtained from S by filling with disks the k holes of S. By part (b) in Theorem 3.1, $\mathrm{Fix}(f^*) = \{p, q\}$ where $p, q \in S^*$ and $p \neq q$. Let $D_1, \ldots, D_k$ be the disks added to S to construct S^*. Then if neither p nor q occurs in $D = \bigcup_{i=1}^{k} D_i$, clearly N divides k, while if $p \in D$ but $q \notin D$, then N divides $k - 1$. Finally, if $p \in D_1$ and $q \in D_2$, then N must divide $k - 2$. This explains the tree possibilities occurring in Table 2 in this case, and it is now straightforward computation to check the other data. □

4. Final Remarks

1. Note that Tables 1 and 2, together with Theorems 3.1 and 3.2, describe explictly the sets $O(g, k, \alpha)$, the topological types of all quotients $S/\langle f \rangle$ where $S \in K(g, k, \alpha)$ and f is a homeomorphism on S or order $N \in O(g, k, \alpha)$, the ramification indices on the inner points of $S/\langle f \rangle$ of the projection $\pi: S \to S/\langle f \rangle$, and the action of f on the boundary components of S. This together with the algorithm in Section 2 solves completely the problem stated in the Introduction.

2. The argument used in Section 2 to produce an algorithm allows us to decide if a given odd integer $N \in O(g, k, \alpha)$, $g + \alpha > 3$, consists essentially of two parts. First we look for a set (as small as possible) of candidates and secondly we check if those candidates actually occur in $O(g, k, \alpha)$. Although this second step is unavoidable, there are some procedures to select sets of candidates which, combined with Steps 1, 2, and 4, could shorten the number of computations. For example, the argument used to produce Tables 1 and 2 is also valid for bigger genera. In other words, for all $k > 0$, $O(g, k, \alpha) \subset O(g, 0, \alpha)$ and so this second set is a set of candidates to check if some N occurs in $O(g, k, \alpha)$. Analogously, from Lemma 3.2, $O(g, 0, 1) \subset O(g - 1, 0, 2)$ for all $g \geq 1$ and more generally, using the orientable compact unbordered Riemann surface associated to a Klein surface $S \in K(g, k, \alpha)$, see [1], we know that

$$O(g, k, \alpha) \subset O(\alpha g + k - 1, 0, 2).$$

3. In Step 4 of our algorithm in Section 2 we have lexicographically ordered the set T of candidates actually to occur in $O(g, k, \alpha)$. We think it is interesting to decide if a different ordering in T produces a better algorithm.

4. As explained in Section 2 to compute $O(g, k, \alpha)$, $g + \alpha > 3$, we first look for all primary solutions, i.e., all elements in $O(g, k, \alpha)$ of the form p^d where p is prime and $p^{d+1} \notin O(g, k, \alpha)$. Afterwards, we calculate $O(g, k, \alpha)$ looking among all numbers of the form $p_1^{e_1} \cdots p_s^{e_s}$ where $p_1^{d_1}, \ldots, p_s^{d_s}$ are the primary elements in $O(g, k, \alpha)$ and $0 \leq e_i \leq d_i$. We could attack the problem directly: first look for the set

$$T' = \{N \text{ odd}: N \leq 2(\alpha g + k - 1)\}.$$

Then, for every element $N \in T'$, we check if conditions (a″)–(d″) in Step 4 are fulfilled. We believe that our procedure is more effective. First, because this bound $N \leq 2(\alpha g + k - 1)$ is not sharp in general while the condition "$\mu(p, e, k, \alpha) \leq \alpha g + k - 1$ if $(\alpha, k) \neq (2, 0)$ or $\mu \leq g$ if $(\alpha, k = (2, 0)$" is the best one, from [5]. Secondly, because the previous calculation of primary solutions allows us to employ suitable notation that simplies the analysis of conditions (a″)–(d″) in Step 4.

5. If we fix integers $g \geq 0$, $k \geq 0$, $\alpha = 1$ or 2, $g + \alpha > 3$, to decide if a prime power p^e occurs in $O(g, k, \alpha)$ involves solving some diophantine equations whose solutions should be nonnegative integers (Step 3 of our algorithm).

This is handled in the following way:

Case 1. Given positive integers a, b and a nonnegative integer c look for all pairs of nonnegative integers (x, y) such that

$$ax + by = c. \tag{1}$$

Using Euclid's algorithm we compute the greatest common divisor d of a and b, and integers α, β such that $a\alpha + b\beta = d$. Of course, if d does not divide c, the given equation has no solutions. Assume $c = Q \cdot d$ for some integer Q. It is obvious that the integer solutions of (1) are $x = Q\alpha + bt$, $y = Q\beta - at$, $t \in \mathbf{Z}$. We look for nonnegative solutions x and y, i.e., $-Qa\alpha \leq abt \leq Qb\beta$

Case 2. Let $r \geq 2$, let $a_0, \ldots, a_r$ be positive integers, and let c be a nonnegative integer. To look for all r-tuples of nonnegative integers $(x_0, \ldots, x_r)$ such that

$$a_0 x_0 + \cdots + a_r x_r = c \tag{2}$$

we use Euclid's algorithm to produce $d = \text{g.c.d.}(a_1, \ldots, a_r)$. If the equation

$$a_0 x_0 + dx = c \tag{3}$$

has no nonnegative integer solutions, the same holds true for (2). On the other hand, for every solution (l_0, l) of (3) we consider the equation

$$(a_1/d)x_1 + \cdots + (a_r/d)x_r = l \tag{4}$$

and $(l_1, \ldots, l_r)$ solves (4) if and only if $(l_0, l_1, \ldots, l_r)$ solves (2). Since in Step 3 of Section 2 the number of variables is bounded above in terms of g, k, α and the number of unknowns in (4) is smaller than in (2), the process finishes.

6. This algorithm has been implemented by Lafuente in the language C and using it, for example, in the case $g = 20$, $k = 90$, $\alpha = 2$ the time needed to compute $O(20, 90, 2)$ directly, i.e., using the algorithm described in Section 2, is considerably longer than the time needed if we assume $O(20, 0, 2)$ to be known, as we commented in Remark 2.

As an example consider the surface of Fig. 3.

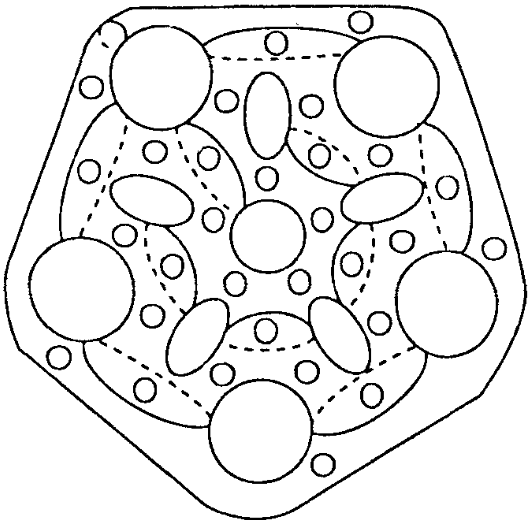

Fig. 3. An example.

Consider the set P of vertices and edges of a regular dodecahedron. Let U be a regular neighborhood of P and let T be the boundary of U. We produce a hole in T for each edge in P and we get an orientable surface S of genus 11 with 30 boundary components. From the construction we remark athat S admits a free action of order 5 (given by the rotation of order 5 in P) and an action of order 3 with four branching points with ramification index equal to 3 (given by the rotation

Table 3

$O(11, 30, 2)$	B.p.	R.i.	g'	k'	α'	Orb. dis.
3	1	3	0		2	(6, 3)(12, 1)
3	4	3	0		2	(7, 3)(9, 1)
3	1	3	1		2	(7, 3)(9, 1)
3	7	3	0		2	(8, 3)(6, 1)
3	4	3	1		2	(8, 3)(6, 1)
3	1	3	2		2	(8, 3)(6, 1)
3	10	3	0		2	(9, 3)(3, 1)
3	7	3	1		2	(9, 3)(3, 1)
3	4	3	2		2	(9, 3)(3, 1)
3	1	3	3		2	(9, 3)(3, 1)
3	13	3	0		2	(10, 3)
3	10	3	1		2	(10, 3)
3	7	3	2		2	(10, 3)
3	4	3	3		2	(10, 3)
9	2	3, 9	0		2	(3, 9)(3, 1)
9	4	9	0		2	(3, 9)(1, 3)
5	0		1		2	(5, 5)(5, 1)
5	5	5	1		2	(6, 5)
5	0		3		2	(6, 5)
15	4	15, 15, 3, 5	0		2	(2, 15)

of order 3 in P). Table 3 gives us not only this information but describes explicitly the set $O(11, 30, 2)$ and the topological data of the corresponding coverings for the automorphisms.

References

1. Alling, N. L., Greenleaf, N. *Foundations of the Theory of Klein Surfaces.* Lecture Notes in Mathematics, Vol. 219. Springer-Verlag, Berlin, 1971.
2. Broughton, S. A. Classifying finite group actions on surfaces of low genus. *J. Pure Appl. Algebra* **69** (1990), 233–270.
3. Bujalance, E. Cyclic groups of automorphisms of compact non-orientable surfaces without boundary. *Pacific J. Math.* **109** (1983), 279–289.
4. Bujalance, E., Etayo, J. J., Gamboa, J. M., Gromadzki, J. M. *Automorphism Groups of Compact Bordered Klein Surfaces.* Lecture Notes in Mathematics, Vol. 1439. Springer-Verlag, Berlin, 1990.
5. Bujalance, E., Gamboa, J. M., Maclachlan, C. Minimum topological genus of compact bordered Klein surfaces admitting a prime-power automorphism. Preprint.
6. Greenleaf, N., May, C. L. Bordered Klein surfaces with maximal symmetry. *Trans. Amer. Math. Soc.* **274** (1982), 265–283.
7. Harvey, W. J. Cyclic groups of automorphishs of a compact Riemann surface. *Quart. J. Math. Oxford Ser.* (2) **17** (1966), 86–97.
8. Hoare, A. H. M., Singerman, D. *Subgroups of Plane Groups.* London Mathematical Society Lecture Note Series, Vol. 71. Cambridge University Press, Cambridge, 1982, pp. 221–227.
9. Kerckhoff, S. P. The Nielsen realization problem. *Ann. of Math.* **117** (1983), 235–265.
10. Maclachlan, C. Ph.D. Thesis, University of Birmingham, 1966.
11. May, C. L. Automorphisms of compact Klein surfaces with boundary. *Pacific J. Math.* **59** (1975), 199–210.
12. Scherk, F. A. The regular maps on a surface of genus three. *Canad. J. Math.* **11** (1959), 452–480.
13. Wiman, A. Über die hyperelliptischen Kurven und diejenigen vom Geschlecht $p = 3$, welche eindeutige Transformationen in sich besitzen. *Bihang Till Kolgl. Svenka Vetenskaps-Akademiens Handlingar* **21**, 1, no. 3 (1895), 41 pp.

Received May 29, 1993, and in revised form December 1, 1993.