IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. 31 (2014) 075020 (24pp) doi:10.1088/0264-9381/31/7/075020

On the space of light rays of a spacetime and a reconstruction theorem by Low

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Received 9 October 2013, revised 19 February 2014
Accepted for publication 19 February 2014
Published 20 March 2014

Abstract

A reconstruction theorem in terms of the topology and geometrical structures on the spaces of light rays and skies of a given spacetime is discussed. This result can be seen as a part of Penrose and Low's programme intending to describe the causal structure of a spacetime M in terms of the topological and geometrical properties of the space of light rays, i.e., unparametrized time-oriented null geodesics, $\mathcal{N}$. In the analysis of the reconstruction problem, the structure of the space of skies, i.e., of congruences of light rays, becomes instrumental. It will be shown that the space of skies Σ of a strongly causal non-refocusing spacetime M carries a canonical differentiable structure diffeomorphic to the original manifold M. Celestial curves, this is, curves in $\mathcal{N}$ which are everywhere tangent to skies, play a fundamental role in the analysis of the geometry of the space of light rays. It will be shown that a celestial curve is induced by a past causal curve of events iff the Legendrian isotopy defined by it is non-negative. This result extends in a nontrivial way some recent results by Chernov *et al* on Low's Legendrian conjecture. Finally, it will be shown that a celestial causal map between the space of light rays of two strongly causal spaces (provided that the target space is null non-conjugate) is necessarily induced from a conformal immersion and conversely. These results make explicit the fundamental role played by the collection of skies, a collection of Legendrian spheres with respect to the canonical contact structure on $\mathcal{N}$, in characterizing the causal structure of spacetime.

Keywords: causal structure, strongly causal spacetime, null geodesic, light rays, contact structures, Legendrian knots, space of skies
PACS numbers: 02.30.Yy, 03.65.−w, 03.67.−a

1. Introduction

In this paper the problem of reconstructing a spacetime M from the topology and geometry of its space of future oriented, unparametrized null geodesics $\mathcal{N}$ or, for brevity, light rays, will be addressed. This problem can be seen as part of a programme proposed by R Penrose and developed partially by R Low in which a systematic discussion of causality properties of Lorentzian spacetime in terms of the topology of the corresponding spaces of null geodesics [7, 9, 10, 13] is intended. Low's conjecture which states that two events in a time-oriented Lorentzian manifold are causally related iff their corresponding skies, which are Legendrian knots with respect to the canonical contact structure in the space of null geodesics, are linked, constitutes one of its most salient outcomes. Recently, it was shown by Chernov and Rudyak [2] and Chernov and Nemirovski [3] that Low's conjecture is actually true in a globally hyperbolic space with a Cauchy surface whose universal covering is diffeomorphic to an open domain in $\mathbb{R}^n$. Thus the exploration of the relation between the causal properties of a conformal class of Lorentzian metrics and the topological properties of skies in the manifold of light rays opens a new and exciting relation between the topology and causality relations of Lorentzian spacetime and the topology of contact manifolds.

In this paper, we will analyze a theorem sketched in Low's papers on the possibility of recovering the conformal structure of the original spacetime from the space of skies which constitutes a family of Legendrian (possibly linked) spheres in the contact manifold of light rays of the original manifold. Such theorem provides a way to 'come back' from the space of light rays to the conformal structure that could contribute to clarify the relation between causality and topological linking.

In the analysis presented here, a paramount role is played by the space of skies Σ of the spacetime M where the sky $S(x)$ of a given point $x \in M$ is the congruence of light rays passing through it. It is well-known that if the spacetime M, i.e., a time-oriented Lorentzian manifold, is strongly causal then the space of light rays has a smooth structure [8].

Moreover, if we assume that the space M is non-refocusing (see for instance [6] for the relevant definitions), in particular it distinguishes skies, then it will be shown (section 3, theorem 2 and corollary 2) that the space of skies Σ carries a canonical topology as well as a canonical differentiable structure defined using exclusively the structure of the manifold $\mathcal{N}$, such that it is diffeomorphic to the smooth structure of the original spacetime (corollary 3.9).

The proof of these results are based on the construction of a basis for the topology of the space of skies by using a family of open subsets of Σ called regular possessing the property that the corresponding tangent spaces to the skies elements of the open set 'pile up' nicely defining a regular submanifold on the tangent space to $\mathcal{N}$. The proof of this statement constitutes the main part of section 3, theorem 3.6, where a new technique of convergence of families of Jacobi fields is used.

Now, we will turn our strategy to study under what circumstances a smooth map between the spaces of light rays corresponding to two spacetimes induces a conformal transformation among them or, in other words, we would like to explore in what sense the space of light rays of a given spacetime characterizes it. It is clear that such a map should satisfy strong conditions. We will show that such analysis relies heavily on the study of celestial curves. A celestial curve is a regular curve in $\mathcal{N}$ whose velocity vector is always tangent to some sky. These curves induce Legendrian isotopies between skies. It will be shown in sections 5 and 6, theorems 3 and 4, that a curve Γ in $\mathcal{N}$ is a causal celestial curve iff it defines a non-negative Legendrian isotopy of skies. This result extends in a non-trivial way results obtained by Chernov *et al* in their analysis of Low's Legendrian conjecture [3].

Finally, the uniqueness of the reconstruction will be discussed in section 6. It is clear that diffeomorphisms on $\mathcal{N}$ preserving skies, i.e., inducing a diffeomorphism in the original spacetime, obviously preserve celestial curves. Then it will be shown that, if we have two strongly causal spacetime M_1 and M_2 such that their spaces of light rays are diffeomorphic by a diffeomorphism that transforms causal celestial curves into causal celestial curves, then it induces a conformal immersion $M_1 \subset M_2$ provided that the space M_2 is null non-conjugate, this is there are no conjugate points along null geodesic segments. This theorem provides the uniqueness result we were looking for and it is the best that can be obtained as the discussion of the example at the end of this section shows.

2. The space of light rays of a spacetime: its differentiable and contact structure

Throughout this section, in the wake of [8, 12] and [13], we will describe the space of light rays of a spacetime, its contact structure and a convenient atlas for its tangent bundle that will be useful in what follows.

2.1. The smooth structure of the space of light rays

Let us consider a time-oriented m-dimensional Lorentz manifold M with metric $\mathbf{g}$ and conformal metric class $\mathcal{C}$ (we will call $(M, \mathcal{C})$ a spacetime in what follows). Given the metric $\mathbf{g}$ we will denote, as indicated in the introduction, by $\mathcal{N}$ the space of its future oriented unparametrized null geodesics, or simply light rays. We are interested in the causal structure $\mathcal{C}$ and the selected metric $\mathbf{g} \in \mathcal{C}$ should be considered as an auxiliary tool to study $\mathcal{C}$.

Let us denote by TM the tangent bundle of M and by $\pi_M : TM \to M$ the corresponding canonical projection. The set $\mathbb{N}^+ = \{\xi \in TM : \mathbf{g}(\xi, \xi) = 0, \xi \neq 0, \xi \text{ future}\} \subset TM$ defines the subbundle of future null vectors on M. Any element $\xi \in \mathbb{N}^+$ defines a unique future oriented null geodesic γ in M such that $\gamma(0) = \pi_M(\xi)$ and $\gamma'(0) = \xi$. Consider the quotient space of $\mathbb{N}^+$ with respect to positive scale transformations, i.e., the quotient space with respect to the dilation, or Euler vector field Δ on $\mathbb{N}^+$, that is the space of leaves of the vector field whose flow is given by $\mathrm{e}^t\xi$, $t \in \mathbb{R}$. In this way, we obtain the bundle $\mathbb{PN}^+$ of future null directions

$$\mathbb{PN}^+ = \{[\xi] : u \in [\xi] \Leftrightarrow u = \lambda\xi \quad \text{where} \quad 0 \neq \lambda \in \mathbb{R}^+, \xi \in \mathbb{N}\}. \tag{2.1}$$

Now, any $[\xi] \in \mathbb{PN}^+$ defines an unparametrized future oriented null geodesic γ, i.e., a light ray, in M which is the oriented graph in M of the null geodesic defined by $\xi \in \mathbb{N}^+$. We denote by $\pi : \mathbb{PN}^+ \to M$ the canonical projection of the bundle $\mathbb{PN}^+$ over M. The fiber $\pi^{-1}(p)$ is diffeomorphic to the standard sphere $\mathbb{S}^{m-2}$. We observe that the bundle $\mathbb{PN}^+$ is foliated by the lifts of these light rays to $\mathbb{PN}^+$, which are projections to $\mathbb{PN}^+$ of integral curves of the geodesic spray $X_\mathbf{g}$ restricted to $\mathbb{N}^+$. We will denote by $\mathcal{F}$ this foliation. Then, the space of light rays $\mathcal{N}$ can be identified with the quotient space $\mathbb{PN}^+/\mathcal{F}$ or, equivalently, as the quotient space of $\mathbb{N}^+$ by the foliation $\mathcal{K}$ whose leaves are the maximal integral submanifolds in $\mathbb{N}^+$ of the integrable distribution defined by Δ and $X_\mathbf{g}$, this is $\mathcal{N} \cong \mathbb{PN}^+/\mathcal{F} = \mathbb{N}^+/\mathcal{K}$. We will denote by σ the canonical projection $\sigma : \mathbb{PN}^+ \to \mathcal{N}$.

The quotient space $\mathbb{PN}^+/\mathcal{F}$ is not a differentiable manifold in general. It is not hard to construct examples (see for instance examples 2.1 and 2.2 in [8]) of spaces of light rays whose topology cannot be induced by any differentiable structure or, that are non-Hausdorff. Sufficient conditions guaranteeing that $\mathcal{N}$ inherits a differentiable structure are given in [8, proposition 2.1 and 2.2]. We will summarize them as follows.

Proposition 1. *Let M be a strongly causal spacetime of dimension m. Then $\mathbb{PN}^+/\mathcal{F}$ inherits a canonical differentiable structure from $\mathbb{PN}^+$ of dimension $2m-3$ such that σ is a smooth submersion. Moreover, if M is not nakedly singular, then $\mathbb{PN}^+/\mathcal{F}$ is Hausdorff.*

Hence, for any strongly causal spacetime M without naked singularities, the space of light rays $\mathcal{N}$ inherits the structure of a Hausdorff smooth $(2m-3)$-dimensional differentiable manifold via the natural identification of $\mathcal{N}$ with $\mathbb{PN}^+/\mathcal{F}$ and $\sigma : \mathbb{PN}^+ \to \mathcal{N}$ is a submersion. In what follows, we will assume that M is a strongly causal not nakedly singular spacetime and we call the space of light rays $\mathcal{N}$ equipped with the smooth structure above, the space of light rays of M (see also for instance [17] for a recent discussion on the topology of the space of all causal curves and its separation axioms properties).

Given a point (or event) $x \in M$, the set of light rays passing through x will be called *the sky of x* and will be denoted by $S(x)$ or X, i.e.

$$S(x) = \{\gamma \in \mathcal{N} : x \in \gamma \subset M\}. \tag{2.2}$$

Note that the light rays $\gamma \in S(x)$ are in one-to-one correspondence with the elements in the fiber $\pi^{-1}(x) \subset \mathbb{PN}^+$, hence the sky $S(x)$ of any point $x \in M$ is diffeomorphic to the standard sphere $\mathbb{S}^{m-2}$. Now, it is possible to define the *space of skies* as

$$\Sigma = \{X \subset \mathcal{N} : X = S(x) \text{ for some } x \in M\} \tag{2.3}$$

and the *sky map* as the application $S : M \to \Sigma$ that maps every x to $S(x) \in \Sigma$. This sky map S is, by definition of Σ, surjective. If the sky map S is a bijection, its inverse map denoted by $P = S^{-1} : \Sigma \to M$ will be called the *parachute map*. An important part of this paper will be devoted to the study of the natural topological and differentiable structures induced in the sky space Σ considered as a collection of subsets of $\mathcal{N}$. In order to understand better the structures inherited by Σ we need to analyze the structure of $T\mathcal{N}$ and in particular the canonical contact distribution carried by it.

2.2. The tangent bundle and the contact structure on the space of light rays

Let us consider $\gamma \in \mathcal{N}$, a tangent vector to $\mathcal{N}$ at γ is defined by an equivalence class $\Gamma'(0)$ of smooth curves $\Gamma(s) = \gamma_s \in \mathcal{N}$, $s \in (-\epsilon, \epsilon)$ such that $\Gamma(0) = \gamma$. The auxiliary metric $\mathbf{g}$ in $\mathcal{C}$, will allow us to consider the space $\mathcal{J}(\gamma)$ of Jacobi fields $J(t)$ along the parametrized geodesics $\gamma(t)$, i.e., vector fields along the curve $\gamma(t)$ which are tangent to geodesic variations $\Gamma(s,t) = \gamma_s(t)$ of $\gamma(t)$, $J(t) = \partial\gamma_s(t)/\partial s \mid_{s=0}$. Then there is a canonical projection $\pi_\gamma : \mathcal{J}(\gamma) \to T_\gamma\mathcal{N}$ given by $\pi_\gamma(J) = \Gamma'(0)$, however such map has a two-dimensional kernel defined by the Jacobi fields of the form $(at+b)\gamma'(t)$. If we denote such Jacobi fields by $\mathcal{J}_{\tan}(\gamma)$, then a tangent vector to $\mathcal{N}$ at γ can be identified with an equivalence class $[J] = J + \mathcal{J}_{\tan}(\gamma)$, $J \in \mathcal{J}(\gamma)$. Note that a vector field J along the curve $\gamma(t)$ is a Jacobi field if and only if it satisfies the Jacobi equation:

$$J'' + R(J, \gamma')\gamma' = 0 \tag{2.4}$$

where 'prime' in J means the covariant derivative with respect the Levi-Civita connection defined by $\mathbf{g}$ along the curve $\gamma(t)$. Then it follows immediately that any Jacobi vector field $J(t)$ defined by a geodesic variation $\gamma_s(t)$ in $\mathcal{N}$ satisfies

$$\mathbf{g}(J(t), \gamma'(t)) = \text{constant}. \tag{2.5}$$

In what follows we will identify a Jacobi field $J(t)$ along $\gamma(t)$ with a tangent vector at γ understanding by it the equivalence class $[J]$, i.e, $J(\mathrm{mod}\,\gamma')$.

There exists a contact structure in $\mathcal{N}$ which arises from the canonical 1-form θ on T^*M but that can be described explicitly in terms of Jacobi fields [11, 13]. Define for each $\gamma \in \mathcal{N}$ the hyperplane $\mathcal{H}_\gamma \subset T_\gamma\mathcal{N}$ given by:

$$\mathcal{H}_\gamma = \{J \in T_\gamma\mathcal{N} : \mathbf{g}(J, \gamma') = 0\}. \tag{2.6}$$

Proposition 2. *The distribution $\mathcal{H} = \bigcup_{\gamma\in\mathcal{N}} \mathcal{H}_\gamma$ defines a contact structure on $\mathcal{N}$.*

The proof of the previous proposition takes advantage of the fact that $\mathcal{N}$ has been constructed from TM, but it is more convenient to start from T^*M via the diffeomorphism defined by the metric $\mathbf{g}$. Hence, if $\hat{\mathbf{g}} : TM \to T^*M$ denotes the canonical diffeomorphism defined by the metric $\mathbf{g}$, then $\hat{\mathbf{g}}(X_\mathbf{g}) = X_H$ is just the Hamiltonian vector field corresponding to the kinetic energy Hamiltonian $H(x, p)$ on T^*M and $\hat{\mathbf{g}}(\Delta)$ is just the Euler field onT^*M. But T^*M carries a canonical 1-form θ, its Liouville 1-form. Then we may restrict θ to $\mathbb{N}^{+*} := \hat{\mathbf{g}}(\mathbb{N}^+)$, whose kernel defines a field $\ker\theta$ of hyperplanes on $\mathbb{N}^{+*}$. The distribution $\ker\theta$ is invariant with respect to the flow of the Euler vector field Δ on T^*M because $L_\Delta\theta = \theta$ and it is also invariant under the flow of X_H because $L_{X_H}\theta = 0$, so $\ker\theta$ descends to $\mathbb{PN}^{+*}$ and then to $\mathcal{N}$. This defines the contact structure (2.6) on $\mathcal{N}$.

Actually, if we denote by $\tilde{\sigma}$ the canonical projection $\tilde{\sigma} : \mathbb{N}^{+*} \to \mathcal{N}$, $\tilde{\sigma}(x, p) = \gamma$ where γ is the projection on M of the integral curve of X_H passing at time 0 through (x, p), i.e., γ is the geodesic such that $\gamma(0) = x$ and $\gamma'(0) = v$ with $\hat{\mathbf{g}}(v) = p$, then a tangent vector $(\dot{x}, \dot{p}) \in T_{(x,p)}\mathbb{N}^{+*}$ will be in the $\ker\theta$ iff $\langle p, \dot{x}\rangle = 0$. The tangent vector $(\dot{x}, \dot{p})$ is mapped by $\tilde{\sigma}$ into a tangent vector J to $\mathcal{N}$, hence we obtain equation (2.6).

Moreover, if $\gamma \in X = S(x)$ where X is the sky of $x \in M$ with $\gamma(s_0) = p$, then

$$T_\gamma X = \{J \in T_\gamma\mathcal{N} : J(s_0) = 0(\operatorname{mod}\gamma')\}. \tag{2.7}$$

For any $J \in T_\gamma X$, since $\mathbf{g}(J, \gamma')$ must be constant and $J(s_0) = 0(\operatorname{mod}\gamma')$, then $\mathbf{g}(J, \gamma') = 0$ and therefore $T_\gamma X \subset \mathcal{H}_\gamma$. This implies that any $T_\gamma X$ is a subspace of $\mathcal{H}_\gamma$ and moreover because $\dim X = m - 2$, X is a Legendrian manifold of the contact structure on $\mathcal{N}$.

2.3. A smooth atlas for the tangent bundle of the space of light rays

We will construct now an atlas for the tangent bundle $T\mathcal{N}$ that is well adapted to the causal structure of M in the sense that in its definition we will take advantage that given an event p in a strongly causal spacetime M we can always choose a globally hyperbolic causally convex normal neighborhood V of p (see for instance [14, theorem 2.1 and definition 3.22]). Note that being V causally convex then for any null geodesic γ we have that $\gamma \cap V$ is connected.

First we will consider an atlas for M whose local charts are $(V, \varphi = (x^1, \ldots, x^m))$ with V a globally hyperbolic causally convex normal neighborhood such that, without lack of generality, the local hypersurface $C \subset V$ defined by $x^1 = 0$ is a smooth spacelike (local) Cauchy surface, hence each null geodesic cutting V intersects C at exactly one point. Let $\{E_1, \ldots, E_m\}$ be an orthonormal frame in V such that E_1 is a future oriented timelike vector field in V. If $\xi \in T_pV$ is written as $\xi = \sum_{j=1}^m u^j E_j(p)$ then (TV, Φ) with:

$$\Phi : TV \to \mathbb{R}^m; \quad \xi \mapsto (x^1, \ldots, x^m, u^1, \ldots, u^m) \tag{2.8}$$

is a local coordinate chart in TM. Let us denote by $\mathbb{N}^+(V)$ the restriction of the bundle $\mathbb{N}^+$ to V and by $\mathbb{PN}^+(V) = \{[\xi] \in \mathbb{PN}^+ : \pi_M([\xi]) \in V\}$ the same for $\mathbb{PN}^+$. For $\xi \in \mathbb{N}^+(V)$ we have $(u^1)^2 = \sum_{j=2}^m (u^j)^2$ so, a coordinate chart in $\mathbb{N}^+(V)$ is given by the map

$$\xi \mapsto (x^1, \ldots, x^m, u^2, \ldots, u^m) \in \mathbb{R}^{2m-1}. \tag{2.9}$$

Taking now homogeneous coordinates $[u^1, \ldots, u^m]$ for $[\xi] \in \mathbb{PN}^+(V)$ in (2.9), or equivalently, fixing $u^1 = 1$ then $(u^2, \ldots, u^m)$ lies in $\mathbb{S}^{m-2}$ and describes a null direction. So, in this way, taking for example $u^2 = \sqrt{1-(u^3)^2 - \cdots (u^m)^2}$ we obtain the coordinate chart $[\Phi] : \mathbb{PN}^+(V) \to \mathbb{R}^{2m-2}$ defined as:

$$[\xi] \mapsto (x^1, \ldots, x^m, u^3, \ldots, u^m) \in \mathbb{R}^{2m-2} \tag{2.10}$$

for $\mathbb{PN}^+(V)$. Let $\mathcal{U}$ be the image of the projection $\sigma : \mathbb{PN}^+(V) \mapsto \mathcal{N}$. Clearly $\mathcal{U} \subset \mathcal{N}$ is open. By global hyperbolicity of V, every null geodesic passing through V intersects C at a unique point and this ensures that $\sigma(\mathbb{PN}^+(V)) = \sigma(\mathbb{PN}^+(C)) = \mathcal{U}$. We have assumed that the Cauchy surface C is a smooth regular submanifold of V, this implies that the bundle $\mathbb{PN}^+(C)$ is a smooth regular submanifold of $\mathbb{PN}^+(V)$, moreover the map $\sigma|_{\mathbb{PN}^+(C)} : \mathbb{PN}^+(C) \mapsto \mathcal{U}$ is a differentiable bijection. The map σ is a submersion such that, for any $[\xi] \in \mathbb{PN}^+(V)$, the kernel of $d\sigma_{[\xi]}$, is the one-dimensional subspace generated by tangent vectors to curves defining light rays, i.e. curves $\lambda(s) = [\gamma'(s)] \in \mathbb{PN}^+_{\gamma(s)}$ where γ is a null geodesic and $[\gamma'(s)] = \{\lambda\gamma'(s) : \lambda \in \mathbb{R}\}$. Because C is a spacelike surface, the kernel of $d\sigma_{[\xi]}|_{\mathbb{PN}^+(C)}$ is trivial, hence $d\sigma_{[\xi]}|_{\mathbb{PN}^+(C)}$ is a surjection between vector spaces of the same dimension, therefore $\sigma|_{\mathbb{PN}^+(C)}$ is a diffeomorphism. We have the following diagram:

$$\begin{array}{ccc} \mathbb{PN}^+(V) & \xrightarrow{\sigma} & \mathcal{U} \\ i\uparrow & \nearrow & \\ \mathbb{PN}^+(C). & & \end{array}$$

So, we can use the restriction of the chart $[\Phi]$, (2.10), to $\mathbb{PN}^+(C)$ as a coordinate chart in $\mathcal{U} \subset \mathcal{N}$. This coordinate chart in $\mathcal{U}$ is given by the map $\psi : \mathcal{U} \to \mathbb{R}^{2m-3}$:

$$\psi(\gamma) = (x^2, \ldots, x^m, u^3, \ldots, u^m) = (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{m-1} \times \mathbb{R}^{m-2} = \mathbb{R}^{2m-3} \tag{2.11}$$

with $\mathbf{x} = (x^2, \ldots, x^m)$ and $\mathbf{u} = (u^3, \ldots, u^m)$, where $\gamma(0) = p \in C \subset V$ have coordinates $\mathbf{x}$ and $\gamma'(0) = \xi = \sum_{i=1}^m u^i E_i \in \mathbb{N}^+(C)$.

We will define an atlas on $T\mathcal{N}$ by using the open sets $T\mathcal{U}$ over the open sets $\mathcal{U}$ defined above. Thus, in order to complete a chart in $T\mathcal{U}$, we will add the coordinates for the tangent vectors at every null geodesic $\gamma \in \mathcal{N}$ with coordinates $\mathbf{x}, \mathbf{u}$. This can be done by using the initial values at $t = t_0 = 0$ for Jacobi's equation (2.4) whose solutions are the Jacobi fields along γ. Thus if $J \in T_\gamma \mathcal{N}$ then $J(t_0) = \sum_{j=1}^m w^j E_j(p)$ and $J'(t_0) = \sum_{j=1}^m v^j E_j(p)$ define J, so a chart in $T\mathcal{U}$ is given by the map $\overline{\psi} : T\mathcal{U} \to \mathbb{R}^{4m-6}$:

$$\overline{\psi}(J) = (\mathbf{x}, \mathbf{u}; \langle v^1, \ldots, v^m \rangle, \langle w^1, \ldots, w^m \rangle) = (\mathbf{x}, \mathbf{u}; \mathbf{v}, \mathbf{w}) \in \mathbb{R}^{4m-6}, \tag{2.12}$$

with $\mathbf{v} = \langle v^1, \ldots, v^m \rangle$ and $\mathbf{w} = \langle w^1, \ldots, w^m \rangle$ denoting respectively,

$$\mathbf{v} = (v^1, \ldots, v^m)(\operatorname{mod} \gamma'), \quad \mathbf{w} = (w^1, \ldots, w^m)(\operatorname{mod} \gamma'),$$

where $(a^1, \ldots, a^m)(\operatorname{mod} \gamma') = \sum_{j=1}^m a^j E_j(p)(\operatorname{mod} \gamma'(t_0))$. We may define $m-2$ independent coordinates from $(v^1, \ldots, v^m)$ and $m-1$ from $(w^1, \ldots, w^m)$. Note that because of (2.5), $J'(t_0)$ is orthogonal to $\gamma'(t_0)$, so $v^1 = v^2 u^2 + \cdots + v^m u^m$. Then, we may consider the representatives $\overline{J}, \overline{J}' \in T\mathcal{N}$ of $J(t_0)$ and $J'(t_0)$ respectively as

$$\overline{J} = J(t_0) - w^1 \gamma'(t_0) = (w^2 - w^1 u^2) E_2 + \cdots + (w^m - w^1 u^m) E_m \tag{2.13}$$

$$\overline{J}' = J'(t_0) - v^1 \gamma'(t_0) = (v^2 - v^1 u^2) E_2 + \cdots + (v^m - v^1 u^m) E_m \tag{2.14}$$

therefore the coordinates $\mathbf{v}$ and $\mathbf{w}$ can be written as

$$\begin{cases} \mathbf{v} = (\overline{v}^3, \ldots, \overline{v}^m) \\ \mathbf{w} = (\overline{w}^2, \ldots, \overline{w}^m) \end{cases} \tag{2.15}$$

where $\overline{v}^k = v^k - v^1 u^k$ and $\overline{w}^k = w^k - w^1 u^k$ for $k = 1, \ldots, m$. Finally note that if, for instance, $u^2 \neq 0$ then $\overline{v}^2 = -\frac{1}{u^2} \sum_{j=3}^m \overline{v}^j u^j$ since $v^1 = v^2 u^2 + \cdots + v^m u^m$. So, we will denote, with an slight abuse of notation, by $(\mathbf{x}, \mathbf{u}; \mathbf{v}, \mathbf{w})$ the $4m - 6$ independent coordinates thus constructed on $T\mathcal{U}$.

It is possible to show the compatibility between the canonical atlas defined on the tangent bundle $T\mathcal{N}$ over the open sets $T\mathcal{U}$ with canonical coordinates $(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}})$ and the atlas previously defined by the local charts $(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$. Actually in doing so we will show that the local charts $(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ define an atlas. We prove first the following lemma.

Lemma 1. *Let M be a Lorentz manifold, $\gamma : [0, 1] \to M$ a null geodesic, $\lambda : (-\epsilon, \epsilon) \to M$ a curve verifying that $\lambda(0) = \gamma(0)$, and $W(s)$ a null vector field along λ such that $W(0) = \gamma'(0)$. Then the family of curves:*

$$\mathbf{f}(s, t) = \exp_{\lambda(s)}(tW(s)) \tag{2.16}$$

is a geodesic variation of $\gamma(t)$ through light rays with $\mathbf{f}(0, t) = \gamma(t)$ and if $J(t) = \frac{\partial \mathbf{f}}{\partial s}(0, t)$, then

$$\frac{DW}{\mathrm{d}s}(0) = \frac{DJ}{\mathrm{d}t}(0). \tag{2.17}$$

Proof. On one hand, $\frac{\partial \mathbf{f}}{\partial s}(0, 0)$ is the tangent vector to the curve $\mathbf{f}(s, 0)$ at $s = 0$, and since $\mathbf{f}(s, 0) = \exp_{\lambda(s)}(0 \cdot W(s)) = \exp_{\lambda(s)}(0) = \lambda(s)$, then we have

$$J(0) = \frac{\partial \mathbf{f}}{\partial s}(0, 0) = \frac{\mathrm{d}\lambda}{\mathrm{d}s}(0) = \lambda'(0).$$

On the other hand, $\frac{D}{\mathrm{d}s}\frac{\partial \mathbf{f}}{\partial t}(0, 0)$ is the covariant derivative of the vector field $\frac{\partial \mathbf{f}}{\partial t}(s, 0) = W(s)$ for $s = 0$ along the curve $\mathbf{f}(s, 0) = \lambda(s)$. Then we can write

$$\frac{DJ}{\mathrm{d}t}(0) = \frac{D}{\mathrm{d}t}\frac{\partial \mathbf{f}}{\partial s}(0, 0) = \frac{D}{\mathrm{d}s}\frac{\partial \mathbf{f}}{\partial t}(0, 0) = \frac{DW}{\mathrm{d}s}(0)$$

therefore J is the Jacobi field of the geodesic variation $\mathbf{f}$. □

Let us consider the coordinate chart $(\psi, \mathcal{U})$ in $\mathcal{N}$ given by (2.11) where $\gamma(0) \in C$ for each $\gamma \in \mathcal{U}$. Now, let $\Gamma_1(s) \in \mathcal{U} \subset \mathcal{N}$, $s \in (-\epsilon, \epsilon)$, be a curve such that its coordinates are

$$\psi(\Gamma_1(s)) = \left(x_0^2, \ldots, x_0^m, \alpha^3(s), \ldots, \alpha^m(s)\right).$$

This curve corresponds to a geodesic variation $\mathbf{f}(s, t)$ such that

$$\lambda(s) = \mathbf{f}(s, 0) = p \in M$$

for every s because the coordinates $\mathbf{x}^k = x_0^k$ remain constant. Moreover $\beta(s) = \partial \mathbf{f}(s, t)/\partial t \in T_pM$ is the curve given by

$$\beta(s) = E_1(p) + \alpha^2(s)E_2(p) + \alpha^3(s)E_3(p) + \cdots + \alpha^m(s)E_m(p).$$

Hence $\mathbf{f}$ can be written by the expression similar to the one in lemma 1

$$\mathbf{f}(s, t) = \exp_p(t\beta(s)).$$

Calling J the Jacobi field of $\mathbf{f}$, then by lemma 1 we have that

$$\begin{cases} J(0) = 0 \\ J'(0) = \beta'(0). \end{cases} \tag{2.18}$$

Now, if we consider a curve $\Gamma_2 \subset \mathcal{N}$ such that its coordinates are

$$\psi(\Gamma_2(s)) = \left(x^2(s), \ldots, x^m(s), u_0^3, \ldots, u_0^m\right).$$

This curve corresponds to a geodesic variation $\mathbf{f}(s,t)$ verifying

$$\lambda(s) = \mathbf{f}(s,0) \in C \subset M.$$

The fact of the coordinates $\mathbf{u}^k = u_0^k$ remain constant implies that

$$W(s) = \frac{\partial \mathbf{f}}{\partial t}(s,0) = E_1(\lambda(s)) + u_0^2 E_2(\lambda(s)) + \cdots + u_0^m E_m(\lambda(s)) \tag{2.19}$$

and $W(s)$ belongs to $T_{\lambda(s)}M$. So the geodesic variation $\mathbf{f}$ corresponding to Γ_2 can be written by

$$\mathbf{f}(s,t) = \exp_{\lambda(s)}(tW(s)).$$

Again, if J is the Jacobi field of $\mathbf{f}$, then by lemma 1

$$\begin{cases} J(0) = \lambda'(0) \\ J'(0) = \frac{DW}{\mathrm{d}s}(0)\,. \end{cases} \tag{2.20}$$

If we choose the curves Γ_1 and Γ_2 such that $\Gamma_1'(0) = (\frac{\partial}{\partial u^i})_{\Gamma_1(0)}$ y $\Gamma_2'(0) = (\frac{\partial}{\partial x^j})_{\Gamma_2(0)}$ respectively with $i = 3, \ldots, m$ and $j = 2, \ldots, m$, then we have that the change from canonical coordinates $(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}})$ to the coordinates $(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ verifies:

$$\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \overline{v}^i \\ \overline{w}^j \end{pmatrix} = \begin{pmatrix} B & I_{m-2} \\ A & 0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \end{pmatrix} \tag{2.21}$$

with $i = 3, \ldots, m$ and $j = 2, \ldots, m$. The matrix $I_{m-2} \in \mathbb{R}^{(m-2)\times(m-2)}$ is the identity matrix and $B \in \mathbb{R}^{(m-2)\times(m-1)}$ is the matrix whose $(k-1)$th column is the vector containing the $\mathbf{v}$ -coordinates of $\frac{DW_k}{\mathrm{d}s}(0)$ with $k = 2, \ldots, m$ with

$$W_k(s) = E_1(\lambda_k(s)) + u_0^2 E_2(\lambda_k(s)) + \cdots + u_0^m E_m(\lambda_k(s)) \tag{2.22}$$

and $\lambda_k(s)$ a curve such that $x^j(\lambda_k(s)) = x_0^j$ are constant for $j \neq k$ and $x^k(\lambda_k(s)) = x_0^k + s$.

Since $J(0) = \lambda_k'(0) = (\partial/\partial x_k)_{\Gamma_2(0)} = \sum_{j=1}^m w_k^j E_j$ then we have that $\overline{w}^j = w_k^j - w_k^1 u^j$ for $j = 2, \ldots, m$. This implies that the matrix A is given by

$$A = \left(w_k^j - w_k^1 u^j\right); \quad j,k = 2, \ldots, m. \tag{2.23}$$

Calling $\mathbb{V} = \mathrm{span}\{E_j(\lambda_k(0))\}_{j=2,\ldots,m}$, the projection $\pi_{\mathbf{u}} : T_{\lambda_k(0)}M \to \mathbb{V}$ is given by

$$\pi_{\mathbf{u}}(\eta) = \eta - \mathbf{g}(\eta, E_1)\gamma'(0),$$

where we have taken $\gamma'(0) = E_1 + u^2 E_2 + \cdots + u^m E_m$. The matrix $\widetilde{A}$ of $\pi_{\mathbf{u}}$ relative to the basis $\{(\partial/\partial x_k)_{\Gamma_2(0)}\}_{k=1,\ldots,m}$ in $T_{\lambda_k(0)}M$ and $\{E_j(\lambda_k(0))\}_{j=2,\ldots,m}$ in $\mathbb{V}$ is

$$\widetilde{A} = \left(w_k^j - w_k^1 u^j\right); \quad j = 2, \ldots, m, \quad k = 1, \ldots, m.$$

We have that $\mathbb{V}$ and $\mathbb{V}_2 = \mathrm{span}\{(\frac{\partial}{\partial x_k})_{\Gamma_2(0)}\}_{k=2,\ldots,m}$ are spacelike by construction, $\ker \pi_{\mathbf{u}} = \mathrm{span}\{\gamma'(0)\}$ and the matrix of the restriction $\pi_{\mathbf{u}}|_{\mathbb{V}_2}$ is A, then $\pi_{\mathbf{u}}|_{\mathbb{V}_2}$ is an isomorphism and therefore A is regular. Hence, the matrix in (2.21) describing the change of coordinates along the fibers of the tangent bundle $T\mathcal{N}$ is regular and differentiable, then the change of coordinates

$$(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}}) \longleftrightarrow (\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$$

is also differentiable. This also shows that $(\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is a coordinate chart of the canonical differentiable structure of $T\mathcal{N}$.

3. The space of skies: its topology and differentiable structure

As it was explained in the introduction, henceforth all the strongly causal manifolds $(M, \mathcal{C})$ that we will consider verify, in addition, that they are non-refocusing, that is M for any $x \in M$ and for any open neighborhood V of x there exists an open set $x \in U \subset V$ such that for all $y \notin V$, there is at least one null geodesic passing through y that does not cut U. Note that if M is non-refocusing it must satisfy the property that *skies distinguish points*, i.e., if $x \neq y$ are two different events, then $S(x) \neq S(y)$ or, in other words, the sky map $S : M \to \Sigma$ is injective, hence a bijection (see [6] for equivalent definitions of the refocusing property).

We will start by defining a natural topology on the space of skies Σ induced by the topology of $\mathcal{N}$. Let $\mathcal{U} \subset \mathcal{N}$ be an open set, then we denote by U or $\Sigma(\mathcal{U}) \subset \Sigma$, the set of all skies $X \in \Sigma$ such that $X \subset \mathcal{U}$. It is obvious that $\Sigma(\mathcal{U} \cap \mathcal{V}) = \Sigma(\mathcal{U}) \cap \Sigma(\mathcal{V})$ for any two open sets $\mathcal{U}, \mathcal{V} \in \mathcal{N}$.

Definition 1. *The topology $\mathfrak{T}$ in Σ generated by the basis $\{\Sigma(\mathcal{U}) \mid \mathcal{U} \subset \mathcal{N} \text{open}\}$ will be called the reconstructive or Low's topology of Σ.*

Note that a set in Σ is open iff it is a union of sets of the form $\Sigma(\mathcal{U})$ with $\mathcal{U}$ open in $\mathcal{N}$ and, given any point $X \in \Sigma$, the family of sets $\Sigma(\mathcal{U})$ with $X \subset \mathcal{U} \subset \mathcal{N}$ define a basis for the neighborhood system of X.

We will prove next that a strongly causal non-refocusing spacetime is homeomorphic to its sky space. The proof we offer here simplifies previous ones.

Proposition 3. *Given the reconstructive topology in Σ, then the sky map $S : M \to \Sigma$ is a homeomorphism.*

Proof. First, we will show that S is continuous. If suffices to show that if $\mathcal{U} \subset \mathcal{N}$ is an open subset, then $V = S^{-1}(U) \subset M$ is open. Thus if $S(x) \subset \mathcal{U}$ we must show that there exists an open subset $V^x \subset M$ such that $S(y) \subset \mathcal{U}$ for all $y \in V^x$. If this were not the case, we can choose of family of compact globally hyperbolic convex normal neighborhoods[3] $\{V_n^x\}$ such that $V_{n+1} \subset V_n$ with local Cauchy surfaces C_n, $C_{n+1} \subset C_n$, such that $\{x\} = \bigcap_n C_n$, and points $y_n \in V_n^x$ with $S(y_n) \nsubseteq \mathcal{U}$. Hence, there will exist $\gamma_n \in \mathcal{N}$ with $y_n \in \gamma_n$, but $\gamma_n \notin \mathcal{U}$. If $\gamma_n \cap C_n = \{x_n\}$ and $x_n = \gamma_n(0)$, then $\lim x_n = x$ and $\lim [\gamma_n'(0)] = [u]$ (note that the space of directions $\mathbb{PN}^+$ over a compact set is compact). Denoting by γ the light ray defined by x and $[u]$, we have shown that $\lim \gamma_n = \gamma = \sigma[u]$, but then $\gamma \in S(x) \subset \mathcal{U}$, and because $\mathcal{U}$ is open, there exists n such that $\gamma_n \in \mathcal{U}$ and we get a contradiction.

Next, we will show that S is an open map. If $V \subset M$ is open, then for all $x \in V$, there exists an open set $\mathcal{U} \subset \mathcal{N}$ such that $S(x) \in \mathcal{U}$, and $U = \Sigma(\mathcal{U}) \subset S(V)$. Suppose this is not true. Taking a family of open sets $\{V_n^x\}$ as before with $V_n^x \subset V$ for all n, the sets

$$\mathcal{U}_n = \{\gamma \in \mathcal{N} \mid \gamma \cap V_n^x \neq \emptyset\}$$

are such that $U_n \nsubseteq S(V)$, hence there exists $x_n \in M$ with $S(x_n) \subset \mathcal{U}_n$ and $x_n \notin V$. Then for all V_n^x there exists $x_n \notin V_n^x$ such that for all $\gamma \in \mathcal{N}$ with $x_n \in \gamma$, then $\gamma \cap V_n^x \neq \emptyset$ and M is refocusing at x. This concludes the proof. □

If $V \subset M$ is an open convex normal neighborhood (see footnote) and $x, y \in V$, then there exists a unique geodesic segment joining x and y. Let us consider the open set $U = S(V) = \{S(x) \mid x \in V\}$, then for every $S(x) = X \neq Y = S(y) \in U$ and $\gamma \in X \cap Y$ verifying $T_\gamma X \cap T_\gamma Y \neq \{0\}$ there exist a Jacobi field J such that $J(s_0) = J(s_1) = 0$ where

[3] A classical theorem due to Whitehead guarantees the existence of convex normal neighborhoods V at any point $x \in M$, (see [16, chapter 5] and [14, theorem 2.1 and definition 3.22] for a treatment of this result in Lorentz manifolds).

$x = \gamma(s_0)$ and $y = \gamma(s_1)$, but that is not possible in a convex normal neighborhood V (see [16, proposition 10.10]). So, in this case we have that $X = Y$ and the next definition is justified.

Definition 2. *An open set $U \subset \Sigma$ in the reconstructive topology is called normal if for every $X, Y \in U$ and every $\gamma \in X \cap Y$ such that $T_\gamma X \cap T_\gamma Y \neq \{0\}$ implies that $X = Y$.*

All the convex normal neighborhoods at $x \in M$ set up a basis for the topology of M at x, then by proposition 3, all the normal neighborhoods also constitute a basis for the topology of Σ.

Normal neighborhoods are not good enough to construct a differentiable structure on Σ. The following definition states the condition that will be required on open sets of Σ to define a smooth atlas. If N is manifold, we denote by $\widehat{T}N$ its reduced tangent bundle, this is, $\widehat{T}N = \cup_{x\in N}\widehat{T}_x N$ where $\widehat{T}_x N = T_x N - \{0\}$.

Definition 3. *A normal open set $U \subset \Sigma$ is said to be a regular open set if U verifies that $\widehat{U} = \bigcup_{X\in U} \widehat{T}X \subset T\mathcal{N}$ is a regular submanifold of $\widehat{T}\mathcal{U}$, where $\mathcal{U} = \bigcup_{X\in U} X$.*

We will prove that regular open sets constitute a basis for the reconstructive topology of Σ.

Theorem 1. *For every $X \in \Sigma$ there exists a regular open neighborhood $U \subset \Sigma$ of X.*

Proof. Let $V \subset M$ be a relatively compact, globally hyperbolic, causally convex normal neighborhood of $q \in M$ and $U = S(V) \subset \Sigma$ be the normal neighborhood of $Q = S(q)$, in the sense of definition 2, image of V under the sky map S. We will use the local coordinate chart $\psi : \mathcal{U} \to \mathbb{R}^{2m-3}$ described by equation (2.11) on $\mathcal{U}$, with $\mathcal{U} = \bigcup_{X\in U} X = \bigcup_{x\in V} S(x)$.

Without any lack of generality, because of the properties of V, we can assume the existence of a coordinate chart $\varphi = (x^1, \ldots, x^m)$ and a orthonormal frame $\{E_1, \ldots, E_m\}$ in V such that the map $\overline{\varphi} : \widehat{U} \to \mathbb{R}^{3m-4}$ (actually we may use the same orthonormal frame $\{E_1, \ldots, E_m\}$ and coordinate chart φ used to construct the coordinates $\overline{\psi} = (\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$ of $T\mathcal{N}$) given by:

$$\overline{\varphi}(J) = (\mathbf{x}, \mathbf{u}; \mathbf{v}) = (x^1, x^2, \ldots, x^m, u^3, \ldots, u^m, \langle v^1 \ldots, v^m\rangle) \in \mathbb{R}^{3m-4} \tag{3.1}$$

defines a coordinate chart for $\widehat{U} = \bigcup_{X\in U} \widehat{T}X$ in an analogous way to the chart $\overline{\psi}$ in (2.12), where $J_0' = \sum_{j=1}^m v^j E_j(x)$ and again $v = \langle v^1 \ldots, v^m\rangle = (v^1, \ldots, v^m)(\mathrm{mod}\, \gamma')$. Note that because of equation (2.7) if J is tangent to a sky $S(q)$, $\gamma(0) = q$, then $J(0) = 0$, hence the local chart $\overline{\varphi}$ is analogous to the chart $\overline{\psi}$ setting $\mathbf{w} = 0$ but the coordinate $\mathbf{x}$ describes the point $q \in V$ where J vanishes.

We will show now that the map $\overline{\varphi}$ gives a differentiable structure to $\widehat{U}$ which does not depend on the chart φ nor the orthonormal frame chosen in V.

(i) First, we will prove that the inclusion $i : \widehat{U} \hookrightarrow T\mathcal{U} \subset T\mathcal{N}$ is differentiable. By construction of the coordinates (x, u) of $\widehat{U}$ and $(\mathbf{x}, \mathbf{u})$ of $T\mathcal{N}$ from the coordinates of $\mathbb{PN}^+(V)$ and $\mathbb{PN}^+(C)$ in equations (2.10) and (2.11) respectively, we have shown that $\sigma|_{\mathbb{PN}^+(C)} : \mathbb{PN}^+(C) \to \mathcal{U}$ is a diffeomorphism and therefore $\mathbf{x}(x, u)$ and $\mathbf{u}(x, u)$ are differentiable functions since they are the equations in coordinates of the submersion

$$\sigma_{VC} = \sigma|^{-1}_{\mathbb{PN}^+(C)} \circ \sigma|_{\mathbb{PN}^+(V)} : \mathbb{PN}^+(V) \mapsto \mathbb{PN}^+(C).$$

If $\mathbf{x} = (x^2, \ldots, x^m)$, we will denote $(0, \mathbf{x}) = (0, x^2, \ldots, x^m)$. Consider then

$$p(x, u) = \varphi^{-1}(0, \mathbf{x}(x, u)) \in C \subset V$$

and

$$W(x, u) = E_1(p(x, u)) + u^2(x, u)E_2(p(x, u)) + \cdots + u^m(x, u)E_m(p(x, u))$$

where $\mathbf{u}(x,u) = (u^3(x,u),\ldots,u^m(x,u))$ and $u^2 = \sqrt{1-(u^3)^2-\cdots-(u^m)^2}$. For any (x,u) we define the following map

$$h(t,x,u) = \exp_{p(x,u)}(tW(x,u)).$$

It is clear that h is differentiable by composition of differentiable maps, and for fixed (x_0,u_0) the curve $\gamma_{(x_0,u_0)}(t) = h(t,x_0,u_0)$ is a null geodesic such that $\gamma_{(x_0,u_0)}(0) \in C$. For any of these geodesics, we have the initial value problem of Jacobi fields given by equation (2.4) with initial data

$$J(\tau) = 0, \quad J'(\tau) = \xi, \tag{3.2}$$

with τ in the domain of $\gamma_{(x,u)}$ and $\xi \in T_{\gamma_{(x,u)}(\tau)}M$.

If we express the Jacobi field J as $J = \alpha^k \partial/\partial x^k$, then equation (2.4) can be written as a system of differential equations

$$\frac{\mathrm{d}^2\alpha^k}{\mathrm{d}t^2} + \frac{\mathrm{d}\alpha^i}{\mathrm{d}t}\left(\Gamma^k_{ij}\frac{\partial h^j}{\partial t}\right) + \alpha^i\frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma^k_{ij}\frac{\partial h^j}{\partial t}\right) \\ +\Gamma^k_{ln}\left(\frac{\mathrm{d}\alpha^l}{\mathrm{d}t} + \Gamma^l_{ij}\alpha^i\frac{\partial h^j}{\partial t}\right)\frac{\partial h^n}{\partial t} - \alpha^n\frac{\partial h^i}{\partial t}\frac{\partial h^j}{\partial t}R^k_{jni} = 0$$

for $k = 1,\ldots,m$ where, for brevity, we write $\Gamma^k_{ij} = \Gamma^k_{ij}(h(t,x,u))$, $R^k_{jni} = R^k_{jni}(h(t,x,u))$ and $h^j = x^j \circ h$.

If we transform this second order system into a first order one by using the standard transformation $y^k = \alpha^k$ and $y^{m+k} = \mathrm{d}\alpha^k/\mathrm{d}t$ for $k = 1,\ldots,m$ then, the initial value problem (2.4)–(3.2) has the form:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y,x,u), \quad y(\tau) = \overline{\xi}. \tag{3.3}$$

Let us denote as $y(t,x,u,\tau,\overline{\xi})$ the solution of (3.3), corresponding to a Jacobi field $J_{\tau,\overline{\xi}} \in \widehat{U}$ along the null geodesic $\gamma_{(x,u)}$ with $J_{\tau,\overline{\xi}}(\tau) = 0$ and $J'_{\tau,\overline{\xi}}(\tau) = \xi$. By construction, for each (x,u) there exists a unique τ such that $\varphi(h(\tau,x,u)) = x$. We will write this function as $\tau(x,u)$ and it is possible to show easily that this τ is differentiable.[4] The solution $y(0,x,u,\tau(x,u),\overline{\xi})$ gives us the values of $J_{\tau,\overline{\xi}}(0)$ and $J'_{\tau,\overline{\xi}}(0)$, and therefore it provides the coordinates $\mathbf{v}(x,u,v)$ and $\mathbf{w}(x,u,v)$. Because of the theorem on the regular dependence of solutions of initial value problems with parameter (see for instance [4, chapter 5]), $y(0,x,u,\tau(x,u),\overline{\xi})$ is a differentiable function depending smoothly on the data$(x,u,\overline{\xi})$ and hence $\mathbf{v}(x,u,v)$ and $\mathbf{w}(x,u,v)$ are differentiable functions of (x,u,v). This proves that $i : \widehat{U} \hookrightarrow T\mathcal{U}$ is differentiable.

(ii) The second step in this proof is to show that $i : \widehat{U} \hookrightarrow T\mathcal{U}$ is an immersion. For this purpose we will show that any regular curve in $\widehat{U}$ is transformed by i into a regular curve in $T\mathcal{U}$. Let us consider a regular curve $c(s) \in \widehat{U}$ with $s \in (-\varepsilon,\varepsilon)$. This means that $c(s) = J_s$ is a Jacobi field along a null (parametrized) geodesic γ_s verifying $J_s(t_s) = 0$, and $J'_s(t_s) = \xi(s)$ is not proportional to $\gamma'_s(t_s)$. We will prove that $i_*(c'(0)) \neq 0$ if $c'(0) \neq 0$, that is

$$c'(0) \neq 0 \Rightarrow (i\circ c)'(0) \neq 0.$$

This curve c can be written in coordinates as $\overline{\varphi}(c(s)) = (x(s),u(s),v(s))$ with $\overline{\varphi}(c(0)) = (x_0,u_0,v_0)$ and it has a differentiable image in $T\mathcal{U}$. The inclusion i transforms the coordinates of c as

$$\overline{\psi}\circ i\circ(\overline{\varphi})^{-1}(x(s),u(s),v(s)) = (\mathbf{x}(x(s),u(s)),\mathbf{u}(x(s),u(s)), \\ \mathbf{v}(x(s),u(s),v(s)),\mathbf{w}(x(s),u(s),v(s))).$$

[4] It can be done applying the implicit function theorem to the map $F(t,x,u) = \varphi(h(t,x,u)) - x$.

The map $(\mathbf{x}(x,u), \mathbf{u}(x,u))$ coincides with the map $\sigma_{VC} = \sigma|^{-1}_{\mathbb{PN}^+(C)} \circ \sigma|_{\mathbb{PN}^+(V)} : \mathbb{PN}^+(V) \mapsto \mathbb{PN}^+(C)$ in coordinates, which is a submersion, then its differential has maximal rank $2m-3$ and codimension 1. If the curve with coordinates $(x(s), u(s))$ is transversal to the fiber of σ_{VC} at $s=0$, then obviously $(i \circ c)'(0) \neq 0$. In other case, we can take c (defining $c'(0)$) as a regular curve verifying that $c(s) = J_s$ lies on a fixed null geodesic γ, then

$$\overline{\psi} \circ i \circ (\overline{\varphi})^{-1}(x(s), u(s), v(s)) = (\mathbf{x}(x_0,u_0), \mathbf{u}(x_0,u_0), \mathbf{v}(x_0,u_0,v(s))\mathbf{w}(x_0,u_0,v(s)))$$

where $(\mathbf{x}, \mathbf{u})$ remains constant for every s. Then the differential

$$(\mathrm{d}\mathbf{x}_{c(0)}(c'(0)), \mathrm{d}\mathbf{u}_{c(0)}(c'(0))) = (0,0).$$

This regular curve c is a curve of Jacobi fields $J_s \in \widehat{U}$ along the null geodesic γ such that $J_s(t_0+s) = 0$ and $J_s'(t_0+s) = \xi(s)$ for $s \in (-\epsilon, \epsilon)$ and hence $\xi(s)$ is a vector field along γ non-proportional to γ' at $s=0$. We can assume, without any lack of generality that $t_0 = 0$ and the local Cauchy surface C associated with the chart $\overline{\psi}$ contains $\gamma(0)$. We have that $J_0(0) = 0$. So,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} J_s(0) = \lim_{s\mapsto 0} \frac{J_s(0) - J_0(0)}{s} = \lim_{s\mapsto 0} \frac{J_s(0)}{s}.$$

By [1, proposition 10.16], we have that $J_s(t) = (\exp_{\gamma(s)})_*((t-s)\tau_{(t-s)\gamma'(s)}J_s'(s))$ where for $v \in T_{\gamma(s)}M$, the map $\tau_v : T_{\gamma(s)}M \to T_vT_{\gamma(s)}M$ is the canonical isomorphism. Then

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} J_s(0) &= \lim_{s\mapsto 0} \frac{1}{s}(\exp_{\gamma(s)})_*((-s)\tau_{(-s)\gamma'(s)}\xi(s)) \\
&= \lim_{s\mapsto 0}(\exp_{\gamma(s)})_*\left(\left(\frac{-s}{s}\right)\tau_{(-s)\gamma'(s)}\xi(s)\right) = \lim_{s\mapsto 0}(\exp_{\gamma(s)})_*(-\tau_{(-s)\gamma'(s)}\xi(s)) \\
&= (\exp_{\gamma(0)})_*(-\tau_0\xi(0)) = -\xi(0).
\end{aligned}$$

Hence, we state that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} J_s(0) = -\xi(0).$$

Since $\xi(0)$ is not proportional to $\gamma'(0)$, then $\mathrm{d}\mathbf{w}_{c(0)}(c'(0)) \neq 0$, and this implies that $i \circ c$ is a regular curve for $s=0$. Therefore i is an immersion.

(iii) In the last step of this proof, we will show that $\widehat{U} \subset T\mathcal{U}$ is a regular submanifold. Let us consider the system of ordinary differential equations 3.3 for Jacobi fields in $\widehat{U}$. We will denote its solution as $y(t, x, u, \tau, \overline{\xi})$. If the origin of the parameter t of 3.3 is lying in the local Cauchy surface C, we can write de Jacobi field J such that $J(\tau) = 0$ and $J'(\tau) = \xi$ as the solution $y(t, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi})$, where $\mathbf{x} = (0, x^2, \ldots, x^m)$ which can be identified with the adapted coordinates $\mathbf{x}$ to C in 2.11. Then, the pair $(\mathbf{x}, \mathbf{u})$ are the coordinates of a point in $\mathbb{PN}^+(C)$ and therefore, they determine the null geodesic $\gamma_{(\mathbf{x},\mathbf{u})}$. In fact, $y(\tau, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi})$ corresponds to the values $J(\tau) = 0$ and $J'(\tau) = \xi$. Moreover, $y(0, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi})$ represents the values $J(0)$ and $J'(0)$ which are lying in C, therefore $y(0, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi})$ is equivalent to give the coordinates $\overline{\psi}(J) = (\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w})$ of J in $T\mathcal{N}$. Since V is relatively compact and due to the existence of flow boxes of non-vanishing differentiable vector fields, we can assume, without any lack of generality, that there exist a compact interval I neighborhood of 0 such that the parameter of any null geodesic defined by $\eta = E_1(p) + u^2E_2(p) + \cdots + u^mE_m(p) \in \mathbb{N}_p^+(V)$ with $p \in V$ through V is defined for $t \in I$. Now, let us consider an arbitrary sequence $\{J_n\} \subset \widehat{U} \subset T\mathcal{N}$ converging to $J_\infty \in \widehat{U} \subset T\mathcal{N}$ in $T\mathcal{N}$. Proving that $\{J_n\}$ converges to J_∞ in $\widehat{U}$ is sufficient to show that $\widehat{U} \subset T\mathcal{U}$ is a regular submanifold.

The Jacobi fields J_n and J_∞ are fields along the null geodesics $\gamma_{(\mathbf{x}_n,\mathbf{u}_n)}$ and $\gamma_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}$ respectively and moreover there exist $t_n, t_\infty \in I$ such that $J_n(t_n)$ and $J_\infty(t_\infty)$ are proportional to $\gamma'_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)$ and $\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)$ respectively for every $n \in \mathbb{N}^+$. If their coordinates in $T\mathcal{N}$ are $\overline{\psi}(J_n) = (\mathbf{x}_n, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n)$ and $\overline{\psi}(J_\infty) = (\mathbf{x}_\infty, \mathbf{u}_\infty, \mathbf{v}_\infty, \mathbf{w}_\infty)$ respectively, then we have that

$$\lim_{n\mapsto\infty} \overline{\psi}(J_n) = \overline{\psi}(J_\infty)$$

or equivalently

$$\lim_{n\mapsto\infty} y(0, \mathbf{x}_n, \mathbf{u}_n, t_n, \overline{\xi}_n) = y(0, \mathbf{x}_\infty, \mathbf{u}_\infty, t_\infty, \overline{\xi}_\infty).$$

Again because of the theorem on the regular dependence of solutions of initial value problems with parameters, the solution $y(t, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi})$ differentiably depends on the variables $(t, x, u, \tau, \overline{\xi})$, therefore

$$\lim_{n\mapsto\infty} y(t, \mathbf{x}_n, \mathbf{u}_n, t_n, \overline{\xi}_n) = y(t, \mathbf{x}_\infty, \mathbf{u}_\infty, t_\infty, \overline{\xi}_\infty).$$

This implies that

$$\lim_{n\mapsto\infty} J_n(t) = J_\infty(t).$$

Since I is compact, the sequence $\{t_n\} \subset I$ has a convergent subsequence, so we can assume that $\{t_n\}$ itself verifies that $\lim_{n\mapsto\infty} t_n = \overline{t} \in I$. Then we have that

$$\lim_{n\mapsto\infty} y(t_n, \mathbf{x}_n, \mathbf{u}_n, t_n, \overline{\xi}_n) = y(\overline{t}, \mathbf{x}_\infty, \mathbf{u}_\infty, t_\infty, \overline{\xi}_\infty)$$

hence

$$\begin{aligned} &\lim_{n\mapsto\infty} J_n(t_n) = J_\infty(\overline{t}) \\ &\lim_{n\mapsto\infty} J'_n(t_n) = J'_\infty(\overline{t}). \end{aligned}$$

Since $J_n(t_n)$ is proportional to $\gamma'_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)$ for every $n \in \mathbb{N}^+$, then $J_\infty(\overline{t})$ is also proportional to $\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)$, but $\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}$ is a null geodesic without conjugate points, therefore $\overline{t} = t_\infty$. This gives us

$$\lim_{n\mapsto\infty} J'_n(t_n) = J'_\infty(t_\infty).$$

Recall that the coordinates of $\widehat{U}$ are given by $\overline{\varphi} = (x, u, v)$ where $\varphi = (x^1, \ldots, x^m)$ is the chart in V. Then

$$\begin{aligned} \lim_{n\mapsto\infty} \overline{\varphi}(J_n) &= \lim_{n\mapsto\infty} (\varphi(\gamma_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)), [\gamma'_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)], \langle J'_n(t_n)\rangle) \\ &= (\varphi(\gamma_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)), [\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)], \langle J'_\infty(t_\infty)\rangle) = \overline{\varphi}(J_\infty). \end{aligned}$$

So, the sequence $\{J_n\}$ converges to J_∞ in $\widehat{U}$.

This completes the proof. □

Corollary 1. *The family of regular open sets constitutes a basis for the topology of* Σ.

Proof. From theorem 1 it follows that $S(V)$ is a regular open set for any local coordinate chart V which is normal, relatively compact, globally hyperbolic and causally convex. But such sets form a basis for the topology of M. Since S is a homeomorphism, this concludes the proof. □

Theorem 2. *Let $V \subset M$ be a globally hyperbolic convex normal open set such that $U = S(V) \subset \Sigma$ is a regular open set. Then U has a canonical differentiable structure depending only on $\mathcal{N}$. Moreover, the restricted sky map $S : V \to U$ is a diffeomorphism.*

Proof. Any $X \in U$ is a regular submanifold of $\mathcal{N}$, therefore $\widehat{T}X$ is a regular submanifold of $\widehat{T}\mathcal{N}$. Denote $\widetilde{U} = \{\widetilde{X} = \widehat{T}X : X \in U\}$ and define the map $\widetilde{S} : V \to \widetilde{U}$ given by $\widetilde{S}(x) = \widetilde{S(x)}$. By definition 3, $\widehat{U}$ is a regular submanifold of $\widehat{T}\mathcal{U}$ which is an open set of $\widehat{T}\mathcal{N}$ and since $\widehat{U} = \bigcup_{X \in U} \widehat{T}X$ then $\widehat{U}$ is foliated by $\{\widehat{T}X : X \in U\}$, i.e. by $\widetilde{U}$.

Denoting the distribution induced by that foliation as $\mathcal{D}$, we have that $\widetilde{U} = \widehat{U}/\mathcal{D}$ inherits a smooth structure because the chart $\overline{\varphi}$ defined by equation (3.1) along the proof of theorem 1 is adapted to $\mathcal{D}$. Hence $\widetilde{S} : V \to \widetilde{U}$ is a diffeomorphism. Moreover, by normality of U then the map $U \to \widetilde{U}$ defined by $X \mapsto \widetilde{X}$ is a bijection, and it allows to identify U with $\widetilde{U}$. Therefore U inherits from $\widetilde{U}$ its structure of differentiable manifold and this implies that $S : V \to U$ is a diffeomorphism. □

An important consequence of corollary 1 and theorem 2 is that, since $\widehat{U}$ is a regular submanifold of $T\mathcal{N}$, then the differentiable structure given in $\widehat{U}$ coincides with the inherited from $T\mathcal{N}$ on $\widehat{U}$. This allows us to disregard the differentiable structure built in $\widehat{U}$ from the one involving M, but considering it inherited from $T\mathcal{N}$. In this way, the differentiable structure of U is inherited from $\widetilde{U} = \widehat{U}/\mathcal{D}$, and then the spacetime M is not necessary to obtain a differentiable structure for Σ, because it is canonically obtained from $\mathcal{N}$. So, in order to recover the strongly causal manifold M from $\mathcal{N}$ and Σ we will not need M itself but only $\mathcal{N}$ and Σ and their corresponding structures.

Corollary 2. *There exists a unique differentiable structure in Σ compatible with the differentiable structure of any regular open set $U \subset \Sigma$ given in theorem 2. Moreover both, the sky map $S : M \to \Sigma$ and the parachute map $P : \Sigma \to M$ are diffeomorphisms.*

Proof. For every $X \in \Sigma$ there exists a regular open set $W \subset \Sigma$. If $x \in M$ verifies that $S(x) = X$, we can consider a globally hyperbolic convex normal neighborhood $V \subset M$ of x such that $U = S(V) \subset W$. By corollary 1, the set U is also a regular open set containing X, and therefore, by theorem 2, $S : V \to U$ is a local diffeomorphism in X. The bijectivity of S provides us the global diffeomorphism $S : M \to \Sigma$. □

4. The reconstruction theorem

We will start discussing in this section under what conditions a spacetime can be reconstructed from its spaces of light rays and skies. A space that could be reconstructed from these data should have the property that 'isomorphic' data must provide the same reconstruction. This observation leads to the following definition.

Definition 4. *Let $(M, \mathcal{C})$, $(\overline{M}, \overline{\mathcal{C}})$ be two strongly causal manifolds and $(\mathcal{N}, \Sigma)$, $(\overline{\mathcal{N}}, \overline{\Sigma})$ the corresponding pairs of spaces of light rays and skies. We say that a map $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ preserves skies if $\phi(X) \in \overline{\Sigma}$ for any $X \in \Sigma$. Moreover, $(M, \mathcal{C})$ will be said to be recoverable if for any $(\overline{\mathcal{N}}, \overline{\Sigma})$, the spaces of light rays and skies corresponding to another strongly causal manifold $(\overline{M}, \overline{\mathcal{C}})$, and $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ a diffeomorphism preserving skies, then the map*

$$\varphi = \overline{P} \circ \phi \circ S : M \to \overline{M}$$

is a conformal diffeomorphism on its image, where $\overline{P} : \overline{\Sigma} \to \overline{M}$ is the parachute map to $\overline{M}$.

Lemma 2. *Let $(M, \mathcal{C})$ and $(\overline{M}, \overline{\mathcal{C}})$ be two strongly causal manifolds and let $(\mathcal{N}, \Sigma)$ and $(\overline{\mathcal{N}}, \overline{\Sigma})$ be the corresponding pairs of spaces of light rays and skies. If $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ is a diffeomorphism preserving skies then the induced map $\Phi : \Sigma \to \overline{\Sigma}$ defined by $\Phi(X) = \phi(X)$ is injective, open, continuous and a diffeomorphism onto its range.*

Proof. Obviously, Φ is well defined and injective. To show that Φ is continuous, consider and open set $\overline{U} \subset \overline{\Sigma}$ and let U be $\Phi^{-1}(\overline{U})$. Since $\overline{U}$ is open, there exists an open set $\overline{\mathcal{W}} \subset \overline{\mathcal{N}}$ such that any sky $\overline{X} \subset \overline{\mathcal{W}}$ is in $\overline{U}$. Since ϕ is a diffeomorphism, then $\mathcal{W} = \phi^{-1}(\overline{\mathcal{W}})$ is an open set in $\mathcal{N}$ and every sky $X \subset \mathcal{W}$ verifies that $\phi(X) \subset \overline{\mathcal{W}}$ and, therefore $\Phi(X) \in \overline{U}$. This implies that $U = \Sigma(\mathcal{W})$ and U is open in Σ.

Now, we show Φ is an open map. Consider $X \in \Sigma$ and $\overline{X} = \phi(X) \in \overline{\Sigma}$. Because of corollary 1 and the continuity of Φ there exist regular neighborhoods $U \subset \Sigma$ of X and $\overline{U} \subset \overline{\Sigma}$ of $\overline{X}$ such that $\Phi(U) \subset \overline{U}$. Then $\phi(\mathcal{U}) \subset \overline{\mathcal{U}}$ with $U = \Sigma(\mathcal{U})$ and $\overline{U} = \Sigma(\overline{\mathcal{U}})$. Hence, because $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ is a diffeomorphism, then $\phi_* : T\mathcal{N} \to T\overline{\mathcal{N}}$ is a diffeomorphism too and the restriction $\phi_* : \widehat{T\mathcal{U}} \to \widehat{T\overline{\mathcal{U}}}$ is a diffeomorphism onto its image. It can be restricted again to $\phi_* : \widehat{U} \to \widehat{\overline{U}}$ since

$$\phi_*(\widehat{U}) = \phi_*\left(\bigcup_{\overline{X}\in U} \widehat{T}X\right) = \bigcup_{X\in U} \phi_*(\widehat{T}X) = \bigcup_{X\in U} \widehat{T}\phi(X) \subset \widehat{\overline{U}},$$

and the regularity of U and $\overline{U}$, i.e., the fact that $\widehat{U}$ and $\widehat{\overline{U}}$ are regular submanifolds of $\widehat{T\mathcal{U}}$ and $\widehat{T\overline{\mathcal{U}}}$, respectively.

Denoting by $\mathcal{D} = \{\widehat{T}X : X \in U\}$, and $\overline{\mathcal{D}} = \{\widehat{T}\overline{X} : \overline{X} \in \overline{U}\}$ the distributions in $\widehat{U}$ and $\widehat{\overline{U}}$, we see that $\phi_*\mathcal{D} = \overline{\mathcal{D}}$. Therefore $\phi_* : \widehat{U} \to \widehat{\overline{U}}$ induces a smooth map

$$\overline{\phi_*} : \widehat{U}/\mathcal{D} \to \widehat{\overline{U}}/\overline{\mathcal{D}},$$

and we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\phi_*} & \widehat{\overline{U}} \\ \downarrow & & \downarrow \\ \widehat{U}/\mathcal{D} & \xrightarrow{\overline{\phi_*}} & \widehat{\overline{U}}/\overline{\mathcal{D}} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\Phi} & \overline{U} \end{array}$$

(note that the lower vertical arrows are diffeomorphisms because of theorem 2). Therefore we conclude that $\Phi : U \to \overline{U}$, is injective, smooth with nonsingular differential, hence it is open and a diffeomorphism onto its image. □

Restricting the map Φ of lemma 2 to its image, $\Phi : \Sigma \to \Phi(\Sigma)$ then it is clear that Φ is bijective, open and continuous, hence is a homeomorphism. This homeomorphism induces, in virtue of lemma 3 or corollary 2, the homeomorphism $\varphi = \overline{P} \circ \Phi \circ S$ onto an open set of $\overline{M}$. So, we can assume, with no lack of generality that $\overline{\Sigma} = \Phi(\Sigma)$ and $\overline{M} = \overline{P} \circ \Phi(\Sigma)$.

Theorem 3. *Let $(M, \mathcal{C})$ be a strongly causal manifold, then M is recoverable.*

Proof. Let $(\overline{M}, \overline{\mathcal{C}})$ be another strongly causal manifold with $(\overline{\mathcal{N}}, \overline{\Sigma})$ its corresponding spaces of light rays and skies, and $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ a diffeomorphism such that $\phi(\Sigma) = \overline{\Sigma}$. Then because of lemma 2 we conclude that $\Phi : \Sigma \to \overline{\Sigma}$ is a diffeomorphism. So, in virtue of corollary 2, the map $\varphi = \overline{P} \circ \Phi \circ S : M \to \overline{M}$ is a diffeomorphism too.

Now, we need to show that φ maps light rays of M into light rays of $\overline{M}$. We can consider all the light rays in the skies of a given light ray γ, denoted as

$$S(\gamma) = \{\beta \in \mathcal{N} : \exists X \in \Sigma \text{ such that } \gamma, \beta \in X\}.$$

Then $\Phi(S(\gamma)) = \phi(S(\gamma)) = \{\phi(\beta) \in \overline{\mathcal{N}} : \exists X \in \Sigma \text{ such that } \gamma, \beta \in X\}$, and since ϕ is a diffeomorphism preserving skies:

$$\Phi(S(\gamma)) = \{\phi(\beta) \in \overline{\mathcal{N}} : \exists \Phi(X) \in \overline{\Sigma} \text{ such that } \phi(\gamma), \phi(\beta) \in \Phi(X)\}.$$

Therefore $\Phi(S(\gamma)) = \overline{S}(\phi(\gamma))$. So, it implies $\varphi(\gamma) = \overline{P} \circ \Phi \circ S(\gamma) = \overline{P} \circ \overline{S} \circ \phi(\gamma) = \phi(\gamma) \in \overline{\mathcal{N}}$ is a null geodesic. By [5, section 3.2], φ is a conformal diffeomorphism. □

5. Causality and Legendrian isotopies

Let us recall first some basic concepts from contact geometry that are going to be related to causality properties of spacetime.

Let $(Y, \mathcal{H})$ be a co-oriented $(2n-1)$-dimensional contact manifold with contact distribution $\mathcal{H} = \ker \alpha$ where $\alpha \in T^*Y$ is a contact 1-form which defines the co-orientation. A differentiable family $\{\Lambda_s\}_{s\in[0,1]}$ of Legendrian submanifolds is called a *Legendrian isotopy*. It is possible to describe a Legendrian isotopy by a parametrization $F : \Lambda_0 \times [0, 1] \to Y$ verifying $F(\Lambda_0 \times \{s\}) = \Lambda_s \subset Y$ where $s \in [0, 1]$. Note that we are assuming that the map $F_s : \Lambda_0 \to \Lambda_s$, given by $F_s(\lambda) = F(s, \lambda)$ is a diffeomorphism for all $s \in [0, 1]$.

Definition 5. *A parametrization F of a Legendrian isotopy is said to be non-negative if $(F^*\alpha)(\frac{\partial}{\partial s}) \geqslant 0$ and non-positive if $(F^*\alpha)(\frac{\partial}{\partial s}) \leqslant 0$.*

Definition 6. *We will say that two Legendrian isotopies are equivalent if their corresponding parametrizations $F, \widetilde{F} : \Lambda_0 \times [0, 1] \to Y$ verify $F(\Lambda_0 \times \{s\}) = \widetilde{F}(\Lambda_0 \times \{s\})$ for every $s \in [0, 1]$.*

Lemma 3. *Let $F, \widetilde{F} : \Lambda_0 \times [0, 1] \to Y$ be two parametrizations of a Legendrian isotopy $\{\Lambda_s\}_{s\in[0,1]}$. If F is non-negative (respectively non-positive) then so is $\widetilde{F}$.*

Proof. Let us consider a Legendrian isotopy $\{\Lambda_s\}_{s\in[0,1]}$ given by two parametrizations $F, \widetilde{F} : \Lambda_0 \times [0, 1] \to Y$. Let us define the maps $F_s, \widetilde{F}_s : \Lambda_0 \to \Lambda_s \subset Y$ for $s \in [0, 1]$ by $F_s(\lambda) = F(\lambda, s)$ as before. Then we have that

$$F(\lambda, s) = \widetilde{F}(\varphi(\lambda, s), s)$$

where $\varphi(\lambda, s) = \widetilde{F}_s^{-1} \circ F(\lambda, s)$. To check that φ is differentiable, consider the differentiable map $\Upsilon : \Lambda_0 \times [0, 1] \to \mathcal{N} \times [0, 1]$ defined by $\Upsilon(z, s) = (\widetilde{F}(z, s), s)$ whose differential at any (z, s) is given by:

$$\mathrm{d}\Upsilon_{(z,s)} = \begin{pmatrix} \mathrm{d}\widetilde{\mathrm{F}}_{(\mathrm{z,s})} \\ \mathrm{Id}_s \end{pmatrix} = \begin{pmatrix} (\mathrm{d}\widetilde{F}_s)_z & * \\ 0 & \mathrm{Id}_s \end{pmatrix}$$

and since $\widetilde{F}_s$ is a diffeomorphism, then $(\mathrm{d}\Upsilon)_{(z,s)}$ is a isomorphism, therefore because of the inverse function theorem, Υ is a local diffeomorphism onto its image in (z, s) and φ can be written locally as:

$$\varphi(z, s) = \pi \circ \Upsilon^{-1}(F(z, s), s)$$

where $\pi : \Lambda_0 \times [0, 1] \to \Lambda_0$ is the canonical projection.

Defining $\phi : \Lambda_0 \times [0, 1] \to \Lambda_0 \times [0, 1]$ as $\phi(\lambda, s) = (\varphi(\lambda, s), s)$, we have

$$\begin{aligned} \mathrm{d}F_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)} &= d(\widetilde{F} \circ \phi)_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)} \\ &= \mathrm{d}\widetilde{F}_{(\varphi(\lambda,s),s)}\left(\mathrm{d}\phi_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right) \\ &= \mathrm{d}\widetilde{F}_{(\varphi(\lambda,s),s)}\left(\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)} + \mathrm{d}\varphi_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right). \end{aligned} \tag{5.1}$$

Note that $\alpha(\mathrm{d}\widetilde{F}_{(\varphi(\lambda,s),s)}\,\mathrm{d}\varphi_{(\lambda,s)}(\partial/\partial s)) = 0$, since $\mathrm{d}\widetilde{F}_{(\varphi(\lambda,s),s)}\mathrm{d}\varphi_{(\lambda,s)}(\partial/\partial s) \in T_{(\varphi(\lambda,s),s)}\Lambda_s$ because $\mathrm{d}\varphi_{(\lambda,s)}(\partial/\partial s) \in T_{\varphi(\lambda,s)}\Lambda_0$. Now, applying α to both sides of equation (5.1) we obtain:

$$\alpha\left(\mathrm{d}F_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right) = \alpha\left(\mathrm{d}\widetilde{F}_{(\varphi(\lambda,s),s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right)$$

hence

$$(F^{*}\alpha)\left(\frac{\partial}{\partial s}\right) = \alpha\left(F_{*}\left(\frac{\partial}{\partial s}\right)\right) = \alpha\left(\widetilde{F}_{*}\left(\frac{\partial}{\partial s}\right)\right) = (\widetilde{F}^{*}\alpha)\left(\frac{\partial}{\partial s}\right)$$

therefore the sign of the parametrizations F and $\widetilde{F}$ coincides. □

As it was discussed in the introduction we are interested in the study of Legendrian isotopies in the space of null geodesics $\mathcal{N}$ of a Lorentz manifold M. Recall that, in this case, the co-orientation is defined by using the criterion that the sign of $J(\mathrm{mod}\,\gamma') \in T_{\gamma}\mathcal{N}$ is the sign of $\mathbf{g}(J,\gamma')$, which is unambiguously determined for vectors J in the class $[J] = J + \mathcal{J}_{\tan}(\gamma)$, where $\gamma \in \mathcal{N}$ and $\mathbf{g} \in \mathcal{C}$.

Again, because of the remark after equation (2.7) the sky $X_0 = S(x_0) \in \Sigma$ for any $x_0 \in M$ is a Legendrian submanifold of $\mathcal{N}$ diffeomorphic to $S_0 = \{[u] : u \in \mathbb{N}^{+}_{x_0}\} = \mathbb{PN}^{+}_{x_0} \cong S^{m-2}$, then given a Legendrian isotopy $\{X_s\}_{s\in[0,1]}$ where X_s is the sky of $x_s \in M$ for $s \in [0,1]$, a parametrization F for it can be found of the form:

$$F : S_0 \times [0,1] \to \mathcal{N}.$$

Lemma 4. *Any differentiable curve $\mu : [0,1] \to M$ defines a Legendrian isotopy parametrized by the function $F^{\mu} : S_0 \times [0,1] \to \mathcal{N}$ given by:*

$$F^{\mu}([u],t) = \gamma_{[u_s]}$$

with $S_0 = \{[u] : u \in \mathbb{N}^{+}_{\mu(0)}\}$ and $u_s \in \mathbb{N}^{+}_{\mu(s)}$ the parallel transport of $u \in \mathbb{N}^{+}_{\mu(0)}$ along γ. Moreover F^{μ} is a Legendrian isotopy of skies and $F^{\mu}_s(S_0) = S(\mu(s))$.

Proof. Let $\mathbf{g} \in \mathcal{C}$ be a metric in the spacetime M and let $\mathcal{P} : T_{\mu(0)}M \times [0,1] \to TM$ be the parallel transport with respect to the Levi-Civita connection defined by $\mathbf{g}$ along μ given by $\mathcal{P}(u,s) = u_s \in T_{\mu(s)}M$. It is widely known that $\mathcal{P}$ is differentiable and the map $\mathcal{P}_s : T_{\mu(0)}M \to T_{\mu(s)}M$ defined by $\mathcal{P}_s(u) = \mathcal{P}(u,s)$ is a linear isometry. Let us also consider the submersion $p_{\mathbb{N}^+} : \mathbb{N}^{+} \to \mathcal{N}$ given by $p_{\mathbb{N}^+}(u) = \gamma_{[u]}$. By composition of differentiable maps, $p_{\mathbb{N}^+} \circ \mathcal{P}$ is differentiable and because of the linearity of $\mathcal{P}$ it induces a map F^{μ} on the quotient space $\mathbb{PN}^{+}$.

Moreover, since $\mathcal{P}_s$ is a linear isometry, then

$$\mathbf{g}(u_s,u_s) = \mathbf{g}(u,u) = 0, \quad u \in \mathbb{N}^{+}$$

for any metric $\mathbf{g} \in \mathcal{C}$, therefore $u_s \in \mathbb{N}^{+}_{\mu(s)}$ and $\mathcal{P}_s(\mathbb{N}^{+}_{\mu(0)}) = \mathbb{N}^{+}_{\mu(s)}$. For $s \in [0,1]$ we have

$$\begin{aligned} F^{\mu}(S_0 \times \{s\}) &= \{F^{\mu}([u],s) \in \mathcal{N} : u \in \mathbb{N}^{+}_{\mu(0)}\} = \{\gamma_{[u_s]} \in \mathcal{N} : u \in \mathbb{N}^{+}_{\mu(0)}\} \\ &= \{\gamma_{[v]} \in \mathcal{N} : v \in \mathbb{N}^{+}_{\mu(s)}\} = S(\mu(s)). \end{aligned}$$

Hence, F^{μ} is a Legendrian isotopy. □

Lemma 5. *Let $F : S_0 \times [0,1] \to \mathcal{N}$ be a Legendrian isotopy such that $F(S_0 \times \{s\}) = S(\mu(s)) \in \Sigma$. Then the curve $\mu : [0,1] \to M$ is differentiable and F is equivalent to F^{μ}.*

Proof. Let us define the map $F_s : S_0 \to S(\mu(s)) \subset \mathcal{N}$ given by $F_s(z) = F(z, s)$ for $s \in [0, 1]$. It is clear that F_s is differentiable for any $s \in [0, 1]$. Now, take any $z_0 \in S_0$ and $\xi \in T_{z_0}S_0$. Since F and F_s are differentiable maps, then the curve

$$j(s) = (\mathrm{d}F_s)_{z_0}(\xi) \in T_{F(z_0,s)}S(\mu(s))$$

is also differentiable in $\widehat{T\mathcal{N}}$ and $j(s)$ is a Jacobi field along the null geodesic $F(z_0, s) \in \mathcal{N}$ for each $s \in [0, 1]$. Let $s_0 \in [0, 1]$ and $U = S(V)$ be a regular open neighborhood of $\mu(s_0)$. Let $(\widehat{U}, \overline{\varphi} = (x, u, v))$ and $(V, \varphi = x)$ be coordinate charts as in theorem 1. Then, since j is differentiable, and $\widehat{U}$ is a neighborhood of $j(s_0)$ in $\widehat{T\mathcal{N}}$ we conclude that $j(s) \in \widehat{U}$ for s close to s_0, is differentiable and $\mu(s) = \varphi^{-1} \circ x(j(s)) \in V$. Therefore μ is differentiable. □

Now, we need a simple result on the geometry of causal vectors on Lorentz manifolds that we state as the following technical lemma.

Lemma 6. *Let M be a Lorentz manifold and $p \in M$. If $v \neq 0$ is a vector in T_pM verifying $\mathbf{g}(u, v) \geqslant 0$ for any $u \in \mathbb{N}_p^+$ future, then v is causal past.*

Proof. First, we will see that if $v \in T_pM$ is spacelike, then there exists $u \in T_pM$ null future verifying $\mathbf{g}(u, v) < 0$. So, let $v \in T_pM$ be spacelike and take some $z \in T_pM$ timelike future, then since $\mathbf{g}(z, z) < 0$ and $\mathbf{g}(v, v) > 0$, the equation

$$\mathbf{g}(z + \lambda v, z + \lambda v) = \mathbf{g}(z, z) + 2\lambda\mathbf{g}(z, v) + \lambda^2\mathbf{g}(v, v) = 0$$

has two solutions λ_1, λ_2 due to $(2\mathbf{g}(z, v))^2 - 4\mathbf{g}(z, z)\mathbf{g}(v, v) > 0$. These solutions can be written as

$$\lambda_1 = -\frac{\mathbf{g}(z, v)}{\mathbf{g}(v, v)} + \sqrt{\frac{\mathbf{g}(z, v)^2}{\mathbf{g}(v, v)^2} - \frac{\mathbf{g}(z, z)}{\mathbf{g}(v, v)}}$$

$$\lambda_2 = -\frac{\mathbf{g}(z, v)}{\mathbf{g}(v, v)} - \sqrt{\frac{\mathbf{g}(z, v)^2}{\mathbf{g}(v, v)^2} - \frac{\mathbf{g}(z, z)}{\mathbf{g}(v, v)}}.$$

For $i = 1, 2$, let $u_i = z + \lambda_i v$ be the corresponding null vectors. We have that

$$\mathbf{g}(u_i, v) = \mathbf{g}(z, v) + \lambda_i\mathbf{g}(v, v) = (-1)^{i+1}\mathbf{g}(v, v)\sqrt{\frac{\mathbf{g}(z, v)^2}{\mathbf{g}(v, v)^2} - \frac{\mathbf{g}(z, z)}{\mathbf{g}(v, v)}}$$

hence $\mathbf{g}(u_2, v) < 0$.

Let us see now that u_2 is null future. Since

$$\mathbf{g}(u_1, u_2) = 2\left[\mathbf{g}(z, z) - \frac{\mathbf{g}(v, z)^2}{\mathbf{g}(v, v)}\right] < 0$$

therefore u_1 and u_2 are in the same time-cone. Moreover

$$\mathbf{g}(u_i, z) = \mathbf{g}(v, v)\left[\frac{\mathbf{g}(z, z)}{\mathbf{g}(v, v)} - \frac{\mathbf{g}(z, v)^2}{\mathbf{g}(v, v)^2}\right] \pm \sqrt{\frac{\mathbf{g}(z, v)^2}{\mathbf{g}(v, v)^2} - \frac{\mathbf{g}(z, z)}{\mathbf{g}(v, v)}}\mathbf{g}(z, v)$$

with the positive sign corresponding to $i = 1$ and the negative to $i = 2$. It can be observed that if $\mathbf{g}(z, v) > 0$ then $\mathbf{g}(u_2, z) < 0$ therefore u_2 is in the same time–cone of z, hence u_2 is null future. In case of $\mathbf{g}(z, v) < 0$ we have that $\mathbf{g}(u_1, z) < 0$, then u_1 (and also u_2) is in the same time-cone of z, therefore u_1 and u_2 are null future.

At this point, we have proven the equivalent result: If for any $u \in T_pM$ null future $\mathbf{g}(u, v) \geqslant 0$ is verified, then $v \in T_pM$ is causal. But if v is causal future, then $\mathbf{g}(u, v) \leqslant 0$, hence $v = 0$ contradicting the hypothesis, therefore v must be causal past. □

Let us recall that a curve $\mu : [a, b] \to M$ is a *null curve* if it is differentiable and $\mathbf{g}(\mu', \mu') = 0$. Note that this is a conformal property and μ doesn't have to be a regular curve.

Definition 7. *The set of all null curves $\mu : I \to M$ will be denoted as $\mathfrak{L}(M)$. The subset of $\mathfrak{L}(M)$ consisting of all time-orientable (future or past) null curves μ will be denoted as $\mathfrak{L}_c(M)$, i.e., $\mu \in \mathfrak{L}_c(M)$ if μ is differentiable, $\mathbf{g}(\mu', \mu') = 0$ and either $\mu'(s) \in \mathbb{N}^+$ or $\mu'(s) \in \mathbb{N}^-$ wherever μ is regular.*

Proposition 4. *The curve μ is causal past (respectively causal future) if and only if F^μ is a non-negative (respectively non-positive) Legendrian isotopy.*

Proof. Let us suppose that μ is causal past. Since $F^\mu([u], s) = \gamma_{[u_s]}$ then giving parameters to the geodesics $\gamma_{[u_s]}$ we can write

$$F^\mu([u], s)(t) = \gamma_{[u_s]}(t) = \exp_{\mu(s)}(tu_s)$$

which is a null geodesic variation of the null geodesic $\gamma_{[u_{s_0}]}$ for every $s_0 \in [0, 1]$. By lemma 1, we have that the Jacobi field $J_{s_0}(t)$ defined by this geodesic variation verifies that $J_{s_0}(0) = \mu'(s_0)$ and $J'_{s_0}(0) = \frac{D}{\mathrm{d}s}|_{s=s_0} u_s$, and since u_s is the parallel transport of u along μ, then $J'_{s_0}(0) = 0$. Hence, since

$$F^\mu_*\left(\frac{\partial}{\partial s}\right)_{([u],s_0)} = \left.\frac{\partial}{\partial s}\right|_{([u],s_0)} F^\mu([u], s) = \left.\frac{\partial}{\partial s}\right|_{(s_0,t)} (\exp_{\mu(s)}(tu_s)) = J_{s_0}(t)$$

we have that

$$\begin{aligned}\alpha\left(F^\mu_*\left(\frac{\partial}{\partial s}\right)\right)_{([u],s_0)} &= \alpha(J_{s_0}(t)) = \mathbf{g}(J_{s_0}(t), \gamma'_{[u_{s_0}]}(t)) \\ &= \mathbf{g}(J_{s_0}(0), \gamma'_{[u_{s_0}]}(0)) = \mathbf{g}(\mu'(s_0), u_{s_0}) \geqslant 0\end{aligned}$$

since $\mu'(s_0)$ is causal past where it does not vanish and u_{s_0} null future. This shows that F^μ is a non-negative Legendrian isotopy.

Now, let us suppose that F^μ is non-negative. So, we have as before

$$F^\mu([u], s)(t) = \gamma_{[u_s]}(t) = \exp_{\mu(s)}(tu_s)$$

then if $\alpha(F^\mu_*(\frac{\partial}{\partial s}))_{([u],s_0)} \geqslant 0$ for any $([u], s_0)$, we have that

$$0 \leqslant \alpha\left(F^\mu_*\left(\frac{\partial}{\partial s}\right)\right)_{([u],s_0)} = \mathbf{g}(\mu'(s_0), u_{s_0}).$$

Then because of lemma 6 we obtain that $\mu'(s_0)$ is causal past provided that $\mu'(s_0) \neq 0$ for every $s_0 \in [0, 1]$. □

Corollary 3. *A Legendrian isotopy of skies $\{S(\mu(s))\}_{s\in[0,1]}$ is non-negative if and only if the curve $\mu : [0, 1] \to M$ is causal past.*

Proof. By lemma 5, a Legendrian isotopy of skies $F : S_0 \times [0, 1] \to \mathcal{N}$ defines a differentiable curve $\mu : [0, 1] \to M$ such that F is equivalent to F^μ. By lemma 3, F^μ is non-negative, then proposition 4 shows that every regular segment of μ is causal past, therefore μ is causal past because is the union of causal past segments. □

6. Celestial curves and reconstruction theorem

We will start this section by introducing a class of curves that are going to play a fundamental role in characterizing when the spaces of light rays and skies of a given strongly causal spacetime are 'isomorphic' regarding the reconstruction problem.

Definition 8. *A tangent vector $J \neq 0$ at $T_\gamma \mathcal{N}$ will be called a celestial vector if there exists a sky $S \in \Sigma$ such that $J \in T_\gamma S \subset T\mathcal{N}$. We will denote the set of all celestial vectors by $\widehat{\Sigma} \subset T\mathcal{N}$. With the notation introduced in section 3, $\widehat{\Sigma} = \bigcup_{X \in \Sigma} \widehat{T}X \subset \widehat{T}\mathcal{N}$.*

A differentiable curve $\Gamma : I \to \mathcal{N}$ is called a celestial curve if $\Gamma'(s) \in \widehat{\Sigma}$ for every $s \in I$. We denote the set of celestial curves as $\mathfrak{C}(\mathcal{N})$.

We would like now to understand if celestial curves can be described as a particular instance of geodesic variations. In order to achieve that we will note first that any differentiable curve in $\mathcal{N}$ can be obtained from a geodesic variation, then we can characterize celestial curves in terms of a specific class of geodesic variations.

Proposition 5. *If the curve $\Gamma : [0, 1] \to \mathcal{N}$ with $\Gamma(s) = \gamma_s \in \mathcal{N}$ is celestial then there exists a null curve $\mu : [0, 1] \to M$ such that $\gamma_s(\tau) = \exp_{\mu(s)}(\tau\sigma(s))$ where $\sigma(s) \in \mathbb{N}^+_{\mu(s)}$ is a differentiable curve proportional to $\mu'(s)$ wherever μ is regular.*

Proof. Let $\Gamma : [0, 1] \to \mathcal{N}$ be a celestial curve with $\Gamma(s) = \gamma_s$. Let $s_0 \in [0, 1]$ and $t_0 \in \mathbb{R}$ such that $\Gamma'(s_0) \in T_{\gamma_{s_0}} S(\gamma_{s_0}(t_0))$ and a local chart $(\widehat{U}, \overline{\varphi})$, with $\overline{\varphi} = (\mathbf{x}, \mathbf{u}, \mathbf{v})$ as in (3.1) with $\Gamma'(s_0) \in \widehat{U}$ such that (V, φ) is the local chart containing $\gamma_{s_0}(t_0) \in M$ used to define $\overline{\varphi}$. We will denote again by $\{E_1, \ldots, E_m\}$ the orthonormal frame in V used to define the components $\mathbf{u}$ and $\mathbf{v}$ in $\overline{\varphi}$.

Consider the neighborhood $I \subset \mathbb{R}$ of s_0 such that $\Gamma'(s) \in \widehat{U}$ for all $s \in I$, thus we have that

$$\overline{\varphi}(\Gamma'(s)) = (\mathbf{x}(\Gamma'(s)), \mathbf{u}(\Gamma'(s)), \mathbf{v}(\Gamma'(s))) \in \mathbb{R}^m \times \mathbb{R}^{m-2} \times \mathbb{R}^{m-2}$$

is a smooth curve. The components $\mathbf{x}$ and $\mathbf{u}$ describe the light rays supporting the Jacobi fields, thus we can reconstruct from them, the curve Γ. Note that the curve $\mu(s) = \varphi^{-1} \circ \mathbf{x}(\Gamma'(s)) \in M$ is smooth. Then consider the curve in $\mathbb{N}^+$ given by:

$$\sigma(s) = E_1(\mu(s)) + u^2(\Gamma'(s))E_2(\mu(s)) + \cdots + u^m(\Gamma'(s))E_m(\mu(s)) \in \mathbb{N}^+_{\mu(s)}.$$

Clearly, $\sigma(s)$ is smooth, then the geodesic variation:

$$\mathbf{f}(s, \tau) = \exp_{\mu(s)}(\tau\sigma(s)) = \overline{\gamma}_s(\tau),$$

reconstructs the curve $\Gamma(s)$.

Because the Jacobi field $\overline{J}_s$ along $\overline{\gamma}_s$ defined by $\mathbf{f}(s, \tau)$ satisfies that $\overline{J}_s(0) = \mu'(s)$ (we choose now $t_0 = 0$) and Γ is a celestial curve, hence tangent to $S(\mu(s))$ at $\Gamma(s)$, then $\overline{J}_s(0) = \lambda_s \overline{\gamma}'_s(0)$ for some $\lambda_s \in \mathbb{R}$. Then we conclude that μ' is proportional to $\overline{\gamma}'_s(0)$, this is to $\sigma(s)$.

Finally, because of the compactness of Γ, the curves μ and σ can be extended to the full interval $[0, 1]$. □

The previous proposition describes a celestial curve Γ as a pair $(\mu, \sigma) \subset M \times \mathbb{N}^+$ where μ is a null curve that cannot be geodesic because in this case Γ would not be regular. Moreover the regularity of μ is not guaranteed at all, in fact, it is possible to exhibit examples of celestial curves such that μ stops for $s \in [a, b] \subset \mathbb{R}$ where $a = b$ is not excluded. While μ remains at $\mu(s) = p \in M$, the curve $\sigma(s)$ moves smoothly in $\mathbb{N}^+_p$. The time-orientation of μ is not guaranteed neither, as the next example shows.

Example 1. Let $\mathbb{M}^3$ be the 3-dimensional Minkowski spacetime with coordinates given by $(t, x, y) \in \mathbb{R}^3$ and metric $\mathbf{g} = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$. Let us denote its space of light rays as $\mathcal{N}$. We consider the curve $\Gamma : [-\varepsilon, \varepsilon] \to \mathcal{N}$ defined by the geodesic variation

$$\mathbf{f}(s,\tau) = \gamma_s(\tau) = \left(\tau + \tfrac{1}{2}s^2, s\sin s + (1+\tau)\cos s, -s\cos s + (1+\tau)\sin s\right)$$

as $\Gamma(s) = \gamma_s$. An easy calculation shows that Γ is a celestial curve. For this curve, μ is defined as

$$\mu(s) = \mathbf{f}(s, \tau(s)) = \mathbf{f}(s, 0) = \left(\tfrac{1}{2}s^2, s\sin s + \cos s, -s\cos s + \sin s\right)$$

hence,

$$\mu'(s) = (s, s\cos s, s\sin s) = s(1, \cos s, \sin s)$$

and μ is a null curve since

$$\mathbf{g}(\mu'(s), \mu'(s)) = 0$$

but the s factor in μ' changes the time-orientation of μ : if $s < 0$ then μ is past-oriented and if $s > 0$ then μ is future-oriented. It is trivial to observe that μ is not a regular curve when $s = 0$.

The previous example motivates the following definitions.

Definition 9. *With the same notations used in proposition 5, a celestial curve $\Gamma \subset \mathcal{N}$ is called a sky curve if $\Gamma \subset X$ for some sky $X \in \Sigma$. We denote the set of all sky curves as $\mathfrak{C}_s(\mathcal{N})$.*

Definition 10. *We say that $(M, \mathcal{C})$ is null non-conjugate if there are no conjugate points in any null geodesic segment or, equivalently, if $\widehat{T}X \cap \widehat{T}Y \neq \emptyset$ for two skies X, Y lying on a null geodesic segment, then $X = Y$.*

Note that the previous definition is equivalent to say that Σ is normal in the sense of definition 2. A convex normal neighborhood V at any point $x \in M$ is null non-conjugate because it is normal (recall definition 2) and similarly, a neighborhood 'small' enough of any closed spacial surface has this property too.

By convention, we can consider $M \subset \mathfrak{L}(M)$ since any point $p \in M$ can be identified with a constant curve. Moreover, if M is null non–conjugate, then the map $\pi_{CL} : \mathfrak{C}(\mathcal{N}) \to \mathfrak{L}(M)$ given by $\pi_{CL}(\Gamma) = \mu$ is well defined and μ is characterized by $\Gamma'(s) \in \widehat{T}_{\Gamma(s)}S(\mu(s))$ for every s.[5] We call $\{S(\mu(s))\}$ the Legendrian isotopy of Γ.

Definition 11. *Let $(\mathcal{N}, \Sigma)$ the space of rays and skies of a null non-conjugate strongly causal spacetime M. We define the set of causal celestial curves as*

$$\mathfrak{C}_c(\mathcal{N}) = \{\Gamma \in \mathfrak{C}(\mathcal{N}) : \mu = \pi_{CL}(\Gamma) \in \mathfrak{L}_c(M)\}.$$

The previous definition of the class of causal celestial curves in $\mathcal{N}$ uses explicitly the space M, however because of the results of section 5 we can provide a characterization of $\mathfrak{C}_c(\mathcal{N})$ without making any reference to M. In fact, using corollary 3 and propositions 4 and 5, we see that $\mu \in \mathfrak{L}_c(M)$ if and only if μ is a null curve defining a non-positive (or non-negative) Legendrian isotopy and we obtain the following corollary that could be used as an alternative definition of $\mathfrak{C}_c(\mathcal{N})$.

Corollary 4. *A celestial curve $\Gamma \in \mathfrak{C}(\mathcal{N})$ is a past (future) causal celestial curve if and only if Γ defines a non-negative (non-positive) Legendrian isotopy of skies.*

[5] In general $\Gamma \in \mathfrak{C}(\mathcal{N})$ can be defined by several curves μ_i with $i = 1, 2, \ldots$, and so $\pi_{CL}(\Gamma)$ should be interpreted as the family $\{\mu_i\}$.

Definition 12. *Let M_1 and M_2 be two strongly causal spaces and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be their corresponding spaces of light rays. A diffeomorphism $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ will be called a celestial map if it preserves celestial vectors, (i.e. $\phi_*(\widehat{\Sigma}_1) \subset \widehat{\Sigma}_2$).*

The following lemma is a direct consequence of the definitions.

Lemma 7. *Any celestial map $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ preserves celestial curves.*

Proof. If $\Gamma : I \to \mathcal{N}_1$ is a celestial curve, then $\Gamma'(s) \in \widehat{\Sigma}_1$ for every $s \in I$. Since ϕ is celestial then $(\phi \circ \Gamma)'(s) = \phi_*(\Gamma'(s)) \in \widehat{\Sigma}_2$ and hence, $\phi \circ \Gamma : I \to \mathcal{N}_2$ is a celestial curve. Moreover ϕ induces a map $\phi : \mathfrak{C}(\mathcal{N}_1) \to \mathfrak{C}(\mathcal{N}_2)$. □

Finally we have the following definition:

Definition 13. *Let M_1 and M_2 be two strongly causal spaces and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be their corresponding spaces of light rays. A celestial map $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ will be called a causal celestial map if ϕ preserves causal celestial curves, that is*

$$\phi : \mathfrak{C}_c(\mathcal{N}_1) \to \mathfrak{C}_c(\mathcal{N}_2).$$

Theorem 4. *Let M_1 and M_2 be two strongly causal spaces, suppose that M_2 is null non-conjugate, and let $(\mathcal{N}_1, \Sigma_1)$ and $(\mathcal{N}_2, \Sigma_2)$ be their corresponding pairs of spaces of light rays and skies. Let $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ be a celestial map. Then the following conditions are equivalent:*

(i) ϕ is a causal celestial map, that is $\phi \circ \Gamma_1 \in \mathfrak{C}_c(\mathcal{N}_2)$, for all $\Gamma_1 \in \mathfrak{C}_c(\mathcal{N}_1)$,
(ii) ϕ is a celestial sky map, that is $\phi \circ \Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_2)$, for all $\Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_1)$.
(iii) There exists a conformal immersion $\Phi : M_1 \to M_2$ such that $\phi(\gamma) = \Phi \circ \gamma$ for every $\gamma \in \mathcal{N}_1$.

Proof. (i) ⇒ (ii) and (iii) ⇒ (i) are trivial.

(ii) ⇒ (iii) consider $X_1 \in \Sigma_1$ and a closed sky curve $\Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_1)$ such that $\Gamma_1 : [0, 1] \to X_1 \subset \mathcal{N}_1$. Since ϕ is a diffeomorphism and by lemma 7, then $\Gamma_2 = \phi \circ \Gamma_1$ is a closed celestial curve. Let μ_2 and σ_2 be the curves defining Γ_2, according to proposition 5. Then, the endpoints verify

$$\mu_2(0), \mu_2(1) \in \Gamma_2(0) = \Gamma_2(1) = \gamma_2 \in \mathcal{N}_2.$$

By the hypothesis we have that $\Gamma_2 \in \mathfrak{C}_c(\mathcal{N}_2)$ and therefore $\mu_2 \in \mathfrak{L}_c(M)$. We will show that μ_2 is a constant, and therefore that Γ_2 is a sky curve. Suppose that μ_2 is future non-constant, then we can construct a future causal curve $\overline{\mu}_2$ such that $\mathrm{Im}(\overline{\mu}_2) = \mathrm{Im}(\mu_2)$ and $\mu_2(0), \mu_2(1) \in \gamma_2 \cap \overline{\mu}_2$. Since M_2 is strongly causal, then $\mu_2(0) \neq \mu_2(1)$ and by [16, proposition 10.51], $\mu_2(0)$ and $\mu_2(1)$ are timelikely related and there exists a conjugate point of $\mu_2(0)$ in γ_2 before $\mu_2(1)$ contradicting that M_2 is conformal non-conjugate. Therefore μ_2 must be constant. This shows that ϕ preserves sky curves and hence also skies. Then theorem 3 gives us the desired result. □

The following example illustrates that the existence of a contactomorphism preserving celestial vectors between the spaces of light rays of two spacetime is not sufficient to induce a conformal diffeomorphism (on its image) between them, showing that condition (1) in theorem 4 cannot be weakened.

Example 2. Let $M = \mathbb{M}^3$ be the 3-dimensional Minkowski spacetime with coordinates given by $(t, x, y) \in \mathbb{R}^3$ and let $\mathcal{N}$ be its space of light rays. The hypersurface $C \equiv \{t = 0\}$ is a Cauchy surface, then $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ are coordinates in $\mathcal{N}$ for any null geodesic $\gamma(s) = (s, x + s\cos\theta, y + s\sin\theta)$. Then $\{(\frac{\partial}{\partial x})_\gamma, (\frac{\partial}{\partial y})_\gamma, (\frac{\partial}{\partial \theta})_\gamma\}$ is a basis of $T_\gamma\mathcal{N}$. The contact hyperplane $\mathcal{H}_\gamma$ is generated by the tangent spaces of two different skies containing γ, therefore

$$\mathcal{H}_\gamma = \operatorname{span}\left\{\left(\frac{\partial}{\partial\theta}\right)_\gamma, \sin\theta\left(\frac{\partial}{\partial x}\right)_\gamma - \cos\theta\left(\frac{\partial}{\partial y}\right)_\gamma\right\}$$

and a contact form α can be written as

$$\alpha = \cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y.$$

For this γ, we have that $T_\gamma S(\gamma(s)) = \operatorname{span}\{s(\sin\theta(\frac{\partial}{\partial x})_\gamma - \cos\theta(\frac{\partial}{\partial y})_\gamma) + (\frac{\partial}{\partial\theta})_\gamma\}$ with $s \in \mathbb{R}$ and hence the celestial vectors at γ are given by $\widetilde{\gamma} = \bigcup_{s\in\mathbb{R}} T_\gamma S(\gamma(s))$. It can be easily observed that the whole $\mathcal{H}_\gamma$ is covered by $\widetilde{\gamma}$ except the subspace $\operatorname{span}\{\sin\theta(\frac{\partial}{\partial x})_\gamma - \cos\theta(\frac{\partial}{\partial y})_\gamma\}$.

We can restrict this space to $M_0 = \{(t, x, y) \in \mathbb{M}^3 : t < 0\}$ denoting $\mathcal{N}_0$ its corresponding space of light rays. By global hyperbolicity of M and M_0, every null geodesic $\gamma_0 \in \mathcal{N}_0$ can be written as $\gamma_0 = \gamma \cap M_0$ for a unique null geodesic $\gamma \in \mathcal{N}$, then we can define the restriction map

$$\begin{array}{rccl} \rho: & \mathcal{N} & \longrightarrow & \mathcal{N}_0 \\ & \gamma & \longmapsto & \gamma_0 = \gamma \cap M_0 \end{array}$$

and the extension map

$$\begin{array}{rccl} \varepsilon: & \mathcal{N}_0 & \longrightarrow & \mathcal{N} \\ & \gamma_0 & \longmapsto & \gamma. \end{array}$$

Both ρ and ε are contactomorphisms and they verify $\varepsilon = \rho^{-1}$ and hence we have that $\mathcal{N} \simeq \mathcal{N}_0$.

Now, let us consider $M_\epsilon = \{(t, x, y) \in \mathbb{R}^3 : t < \epsilon\}$ for $\epsilon > 0$, equipped with the metric

$$\mathbf{g}_\epsilon = -(1 + f(t))\,\mathrm{d}t \otimes \mathrm{d}t + 2f(t)\,\mathrm{d}t \otimes \mathrm{d}x + (1 - f(t))\,\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y$$

where f is a smooth function verifying $f(t) = 0$ for every $t \leqslant 0$. We can see $\mathbf{g}_\epsilon$ as a small perturbation of the metric $\mathbf{g}$ of M for $0 < t < \epsilon$. Trivially, we observe that M and M_ϵ are two spacetime extending M_0. By [15], the value of ϵ can be chosen small enough such that M_ϵ remains globally hyperbolic, then we can consider $\mathcal{N}_\epsilon \simeq \mathcal{N}$ and therefore $\mathcal{H}_\gamma \simeq \mathcal{H}_{\gamma_0} \simeq \mathcal{H}_{\gamma_\epsilon}$ for $\gamma_0 = \gamma \cap M_0$ and $\gamma_\epsilon = \gamma \cap M_\epsilon$. This extension is independent from the coordinates x and y. Denoting by $\widetilde{\gamma_\epsilon}$, $\widetilde{\gamma_0}$ the celestial vectors at the corresponding curve, and working at $\mathcal{N}$ with certain abuse of notation we have that $\widetilde{\gamma_0} = \bigcup_{s\in(-\infty,0)} T_\gamma S(\gamma(s)) \subset \widetilde{\gamma} \cap \widetilde{\gamma_\epsilon}$ then the value ϵ also can be selected small enough such that $\widetilde{\gamma_\epsilon} \subset \widetilde{\gamma}$ and therefore the contactomorphism $\Phi : \mathcal{N}_\epsilon \to \mathcal{N}$ preserves celestial vectors. In spite of the existence of Φ preserving celestial vectors, the spacetime M and M_ϵ cannot be conformally equivalent. Observe that 3–dimensional Minkowski spacetime M is flat. Denoting as R_{ij}, R and g^ϵ_{ij} the Ricci curvature, the scalar curvature and the metric in M_ϵ respectively, then the components of the Cotton tensor $\mathbf{C}_\epsilon$ in M_ϵ are given by $C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{4}(\nabla_j R g^\epsilon_{ik} - \nabla_k R g^\epsilon_{ij})$. It is widely known that one 3-dimensional manifold is locally conformally flat if its Cotton tensor vanishes. A straightforward calculation shows that $\mathbf{C}_\epsilon \neq 0$, then M_ϵ is not conformally flat and therefore it cannot be conformal to M.

Acknowledgments

The authors would like to thank the referees for their valuable suggestions that have really helped to improve the paper and by detecting an error in a previous proof of lemma 2. This work has been partially supported by the Spanish MICIN grant MTM 2010-21186-C02-02 and QUITEMAD P2009 ESP-1594. AI wants to thank the program 'Salvador de Madariaga' for partial support during the stay at the Deptartment of Mathematics, University of California at Berkeley where part of this work was done.

References

[1] Beem J K, Ehrlich P E and Easley K L 1996 *Global Lorentzian Geometry* (New York: Marcel Dekker)
[2] Chernov V and Rudyak Yu 2008 Linking and causality in globally hyperbolic space-times *Commun. Math. Phys.* **279** 309–54
[3] Chernov V and Nemirovski S 2010 Legendrian links, causality, and the Low conjecture *Geom. Funct. Anal.* **19** 1320–33
[4] Hartman P 1964 *Ordinary Differentiable Equations* (New York: Wiley)
[5] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press)
[6] Kinlaw P A 2011 Refocusing of light rays in space-time *J. Math. Phys.* **52** 052505
[7] Low R J 1988 Causal relations and spaces of null geodesics *PhD Thesis* Oxford University
[8] Low R J 1989 The geometry of the space of null geodesics *J. Math. Phys.* **30** 809–11
[9] Low R J 1990 Twistor linking and causal relations *Class. Quantum Grav.* **7** 177–87
[10] Low R J 1994 Twistor linking and causal relations in exterior Schwarzschild space *Class. Quantum Grav.* **11** 453–6
[11] Low R J 1998 Stable singularities of wave-fronts in general relativity *J. Math. Phys.* **39** 3332–5
[12] Low R J 2001 The space of null geodesics *Proceedings of the Third World Congress of Nonlinear Analysts, Part 5 (Catania, 2000) Nonlinear Anal.* **47** 3005–17
[13] Low R J 2006 *The Space of Null Geodesics (And A New Causal Boundary) (Lecture Notes in Physics* vol 692*)* (Berlin: Springer) pp 35–50
[14] Minguzzi E and Sánchez M 2008 The causal hierarchy of spacetimes *Recent developments in pseudo-Riemannian geometry (ESI Lectures in Mathematics and Physics)* ed D Alekseevsky and H Baum (Zurich: European Mathematical Society Publishing House) pp 299–358 arXiv:gr-qc/0609119
[15] Navarro J J Benavides and Minguzzi E 2011 Global hyperbolicity is stable in the interval topology *J. Math. Phys.* **52** 112504
[16] O'Neill B 1983 *Semi-Riemannian Geometry with Applications to Relativity* (New York: Academic)
[17] Pourkhandani R and Bahrampour Y 2012 The space of causal curves and separation axioms *Class. Quantum Grav.* **29** 015014