SMOOTH APPROXIMATIONS IN PL GEOMETRY

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Abstract. Let $Y \subset \mathbb{R}^n$ be a triangulable set and let r be either a positive integer or $r = \infty$. We say that Y is a $\mathscr{C}^r$-approximation target space, or a $\mathscr{C}^r$-`ats` for short, if it has the following universal approximation property: *For each $m \in \mathbb{N}$ and each locally compact subset X of $\mathbb{R}^m$, each continuous map $f : X \to Y$ can be approximated by $\mathscr{C}^r$ maps $g : X \to Y$ with respect to the strong Whitney $\mathscr{C}^0$ topology*. Taking advantage of new approximation techniques we prove: *if Y is weakly $\mathscr{C}^r$ triangulable, then Y is a $\mathscr{C}^r$-`ats`*. This result applies to relevant classes of triangulable sets, namely: (1) every locally compact polyhedron is a $\mathscr{C}^\infty$-`ats`, (2) every set that is locally $\mathscr{C}^r$ equivalent to a polyhedron is a $\mathscr{C}^r$-`ats` (this includes $\mathscr{C}^r$ submanifolds with corners of $\mathbb{R}^n$) and (3) every locally compact locally definable set of an arbitrary o-minimal structure is a $\mathscr{C}^1$-`ats` (this includes locally compact locally semialgebraic sets and locally compact subanalytic sets). In addition, we prove: *if Y is a global analytic set, then each proper continuous map $f : X \to Y$ can be approximated by proper $\mathscr{C}^\infty$ maps $g : X \to Y$*. Explicit examples show the sharpness of our results.

1. Introduction, main theorems and corollaries. Approximation is a tool of great importance in many areas of mathematics. It allows to understand objects and morphisms of a certain category taking advantage of the corresponding properties of objects and morphisms in other categories that enjoy a better behavior and are dense inside the one we want to study.

In the geometrical context a remarkable example of an approximation result with thousand of applications concerns Whitney's approximation theorem [W] for continuous maps whose target space is a $\mathscr{C}^r$ submanifolds Y of $\mathbb{R}^n$ for either a positive integer r or $r = \infty$. An important fact is the existence of a system of $\mathscr{C}^r$ tubular neighborhoods of Y in $\mathbb{R}^n$ (together with the corresponding $\mathscr{C}^r$ retractions onto Y). Whitney's approximation theorem can be used for instance to prove the existence of a unique $\mathscr{C}^\infty$ manifold structure on each differentiable manifold of class $\mathscr{C}^r$ for each positive integer r (see [H3]).

This paper deals with the problem of approximating continuous maps by differentiable maps when the target space $Y \subset \mathbb{R}^n$ may have "singularities". Actually, we require that Y is at least triangulable.

The special case when the target space $Y \subset \mathbb{R}^n$ is a Nash set was already treated by Coste, Ruiz and Shiota in [CRS1]. In fact, they approximate real analytic maps on a compact Nash manifolds by a very restrictive class of approximating maps, the so-called Nash maps, see [BCR, Ch. 8]. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is

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(real) Nash if it is of class $\mathscr{C}^\omega$ (that is, real analytic) and algebraic over the real polynomials, that is, there exists a non-zero polynomial $P \in \mathbb{R}[x_1, \ldots, x_n, y]$ such that $P(x, f(x)) = 0$ for each $x \in \mathbb{R}^n$. In addition, $Y \subset \mathbb{R}^n$ is a *Nash set* if there exist a Nash function f on $\mathbb{R}^n$ such that $Y = \{f = 0\}$. A Nash set $X \subset \mathbb{R}^m$ that is in addition a smooth manifold is called a *Nash manifold*. A map $F : \mathbb{R}^m \to \mathbb{R}^n$ is *Nash* if its components are Nash functions. The *Nash maps* $f : X \to Y$ are the restrictions from X to Y of Nash maps $F : \mathbb{R}^m \to \mathbb{R}^n$ such that $F(X) \subset Y$. The authors proved in [CRS1, Thm 0.0] a global version of Artin's approximation theorem [Ar], which implies the following:

THEOREM 1.1. [CRS1] *Let $Y \subset \mathbb{R}^n$ be a Nash set and let $X \subset \mathbb{R}^m$ be any compact Nash manifold. Then every real analytic map $f : X \to Y$ can be uniformly approximated by (real) Nash maps $g : X \to Y$.*

The proof of the previous theorem is based on a deep result on commutative algebra: the so-called general Néron desingularization, see the survey [CRS2] for further references. Lempert proved in [Le] the counterpart of Theorem 1.1 for the complex setting taking advantage again of the general Néron desingularization:

THEOREM 1.2. [Le] *Let $Y \subset \mathbb{C}^n$ be a complex algebraic set and let X be any holomorphically convex compact subset X of a complex algebraic subset of $\mathbb{C}^m$. Then every holomorphic map $f : X \to Y$ can be uniformly approximated by complex Nash maps $g : X \to Y$.*

In [BP] the authors provide a simpler proof of the previous statement based on strategies with a more geometric flavor.

Our main results in this work (Theorems 1.6 and 1.15) are of a different nature. We show: if $Y \subset \mathbb{R}^n$ belongs to a wide class of triangulable sets including differentiable manifolds, polyhedra, semialgebraic sets, subanalytic sets and definable sets of an o-minimal structure, then Y enjoys the following approximation property as target space: *Let $X \subset \mathbb{R}^m$ be any locally compact set. Then each continuous map $f : X \to Y$ can be approximated by arbitrarily close $\mathscr{C}^r$ maps $g : X \to Y$ for either suitable positive integers r or $r = \infty$, with respect to the strong $\mathscr{C}^0$ topology of* $\mathscr{C}^0(X,Y)$. As the set $\mathscr{C}^0_*(X,Y)$ of proper maps between X and Y is an open subset of $\mathscr{C}^0(X,Y)$ (see [H3, Ch. 2., Thm. 1.5]), the above $\mathscr{C}^r$ approximation property implies the "proper $\mathscr{C}^r$ approximation target space property". We will revisit the latter property when dealing with C-analytic sets in Section 1.B below.

The preceding approximation by $\mathscr{C}^r$ maps is always possible if $Y \subset \mathbb{R}^n$ is a $\mathscr{C}^r$ submanifold with boundary (even if it has corners) or if Y is a locally compact polyhedron. Our proofs introduce new approximation strategies that make use of a variant of the general simplicial approximation theorem, a "shrink-widen" covering and approximation technique and $\mathscr{C}^r$ weak retractions. Our constructions provide also certain relative versions of the preceding approximation results, that is, results of the following form: *if in addition $X' \subset X$ is non-empty and $f|_{X'}$ is a $\mathscr{C}^r$ map,* *there exist arbitrarily close* $\mathscr{C}^r$ *maps* $g : X \to Y$ *to* f *with respect to the strong* $\mathscr{C}^0$ *topology of* $\mathscr{C}^0(X,Y)$ *such that* $g|_{X'} = f|_{X'}$. Examples 1.20 show that we have to be quite restrictive with the hypotheses about X' (even if Y is as simple as a $\mathscr{C}^r$ manifold with non-empty boundary).

In the literature there are many celebrated results concerning the existence of obstructions to approximate homeomorphisms between differentiable manifolds by diffeomorphisms. This obstruction theory is a central topic in differential topology, which was mainly developed by names like Milnor, Thom, Munkres and Hirsch in the fifties and sixties [H1, H2, Mi1, M1, M2, M3, Th]. We refer the reader also to [DP, HM, IKO, MP, Mu] for some recent developments. Additional obstructions where found by Milnor in [Mi2] when he constructed two homeomorphic compact polyhedra which are not PL homeomorphic. Our results state that there are no obstructions to approximate continuous maps $f : X \to Y$ by differentiable maps $g : X \to Y$ when Y admits a "good" triangulation. Of course, one cannot expect that the approximating map g is a diffeomorphism or a PL homeomorphism if the map f we want to approximate is a homeomorphism. In fact, the approximating maps g we construct in this paper are far from being injective (see Remark 3.12).

1.A. Weakly $\mathscr{C}^r$ triangulable sets. We assume in the whole article that every subset of $\mathbb{R}^n$ is endowed with the relative Euclidean topology (where $n \in \mathbb{N} := \{0,1,2,\dots\}$).

Let $Y \subset \mathbb{R}^n$ be a (non-empty) set. We say that Y is *triangulable* if it is homeomorphic to a locally compact polyhedron of some $\mathbb{R}^q$. A *locally compact polyhedron* of $\mathbb{R}^q$ is defined as the realization $|L|$ of a locally finite simplicial complex L of $\mathbb{R}^q$. For related notions concerning simplicial complexes we refer the reader to [Hu, M5].

Let $X \subset \mathbb{R}^m$ be a (non-empty) locally compact set and let $\mathscr{C}^0(X,Y)$ be the set of all continuous maps from X to Y. We endow $\mathscr{C}^0(X,Y)$ with the *strong (Whitney)* $\mathscr{C}^0$ *topology*. A fundamental system of neighborhoods of $f \in \mathscr{C}^0(X,Y)$ in such a topology is given by the sets

$$\mathcal{N}(f,\varepsilon) = \big\{g \in \mathscr{C}^0(X,Y) : \|g(x) - f(x)\|_n < \varepsilon(x)\ \forall x \in X\big\},$$

where $\|\cdot\|_n$ denotes the Euclidean norm of $\mathbb{R}^n$ and $\varepsilon : X \to \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\}$ is a strictly positive continuous function on X.

Denote $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the set of all positive integers and fix $r \in \mathbb{N}^* \cup \{\infty\}$. A map $g : X \to Y$ is a $\mathscr{C}^r$ *map* if there exist an open neighborhood $U \subset \mathbb{R}^m$ of X (in which X is closed) and a differentiable map $G : U \to \mathbb{R}^n$ of class $\mathscr{C}^r$ (in the standard sense) such that $g(x) = G(x)$ for each $x \in X$. Denote $\mathscr{C}^r(X,Y)$ the subset of $\mathscr{C}^0(X,Y)$ of all $\mathscr{C}^r$ maps from X to Y.

Definition 1.3. Let $r \in \mathbb{N}^* \cup \{\infty\}$. A triangulable set $Y \subset \mathbb{R}^n$ is a $\mathscr{C}^r$-*approximation target space* or a $\mathscr{C}^r$-`ats` for short if $\mathscr{C}^r(X,Y)$ is dense in

$\mathscr{C}^0(X,Y)$ for each locally compact subset X of each Euclidean space $\mathbb{R}^m$, where $m \in \mathbb{N}$ is any natural number.

If $Y \subset \mathbb{R}^n$ is a $\mathscr{C}^r$-`ats`, it is triangulable, so by [Ha, Cor.3.5] it is an absolute neighborhood retract. This implies that if $X \subset \mathbb{R}^m$ is an arbitrary locally compact set, $f \in \mathscr{C}^0(X,Y)$ and $g \in \mathscr{C}^r(X,Y)$ is any close enough approximation of f, then g is homotopic to f (see [Ha, Thm.4.1]). Thus, close enough approximations of f are also homotopic between them. In fact, $\mathscr{C}^r(X,Y)$ is not only dense in $\mathscr{C}^0(X,Y)$ but it is also "homotopically dense" in $\mathscr{C}^0(X,Y)$ in the following sense: *for each $f \in \mathscr{C}^0(X,Y)$ and each strictly positive continuous function $\varepsilon : X \to \mathbb{R}^+$ there exists $g \in \mathcal{N}(f,\varepsilon) \cap \mathscr{C}^r(X,Y)$ that is homotopic to f.*

A natural question consists of determining if *homotopic maps of $\mathscr{C}^r(X,Y)$* and *$\mathscr{C}^r$ homotopic maps of $\mathscr{C}^r(X,Y)$* coincide. *In case X is a locally compact set and Y is a $\mathscr{C}^r$ submanifold with boundary, then homotopic maps of $\mathscr{C}^r(X,Y)$ are also $\mathscr{C}^r$ homotopic maps of $\mathscr{C}^r(X,Y)$* (see [ORR, Ch. III, Thm. 8.3]). The proof of this result uses the following $\mathscr{C}^r$ approximation result relative to a closed subset X' of X (when $Y \subset \mathbb{R}^n$ is a $\mathscr{C}^r$ submanifold with boundary).

THEOREM 1.4. [ORR, Ch. III, Thm. 6.1] *Let $X \subset \mathbb{R}^m$ be a locally compact set, let $r \in \mathbb{N}^* \cup \{\infty\}$ and let $Y \subset \mathbb{R}^n$ be a $\mathscr{C}^r$ submanifold with boundary. Let $X' \subset X$ be a closed set and let $f : X \to Y$ be a continuous map such that $f|_W$ is a $\mathscr{C}^r$ map for some neighborhood $W \subset X$ of X'. Then there exists $g \in \mathscr{C}^r(X,Y)$ arbitrarily close to f in the strong $\mathscr{C}^0$ topology such that g coincides with f in a neighborhood of X' in X.*

In this work we analyze when a triangulable set $Y \subset \mathbb{R}^n$ is a $\mathscr{C}^r$-`ats`. Classical examples of $\mathscr{C}^r$-`ats` are the $\mathscr{C}^r$ submanifolds of Euclidean spaces. Indeed, if Y is any $\mathscr{C}^r$ submanifold of $\mathbb{R}^n$, then it is triangulable ($\mathscr{C}^r$ triangulable indeed) by Cairns-Whitehead's triangulation theorem [Ca]. In addition, by Whitney's approximation theorem one can approximate each continuous map $f : X \to Y$ by an arbitrarily close $\mathscr{C}^r$ map $g^* : X \to \mathbb{R}^n$ and then one can use a $\mathscr{C}^r$ tubular neighborhood $\rho : U \to Y$ of Y in $\mathbb{R}^n$ to define the $\mathscr{C}^r$ map $g : X \to Y, x \mapsto g(x) = \rho(g^*(x))$ arbitrarily close to f in the strong $\mathscr{C}^0$ topology of $\mathscr{C}^0(X,Y)$.

If $Y \subset \mathbb{R}^n$ is an arbitrary triangulable set, a serious difficulty arises: Y does not have $\mathscr{C}^r$ tubular neighborhoods in $\mathbb{R}^n$ (recall that $r \geq 1$)! An easy counterexample is the $\mathscr{C}^r$ triangulable set $Y := \{xy = 0\} \subset \mathbb{R}^2$. If there were a $\mathscr{C}^1$ retraction $\rho : U \to Y$, then $d_0\rho$ would be the identity on $\mathbb{R}^2$ and the origin should be an interior point of Y by the inverse function theorem, which is a contradiction. In fact, (boundaryless) $\mathscr{C}^r$ submanifolds of Euclidean spaces can be characterized by the existence of $\mathscr{C}^r$ retractions, namely: *if a subset of an Euclidean space can be covered by local $\mathscr{C}^r$ retractions, it is a (boundaryless) $\mathscr{C}^r$ submanifold* (see [Mic, Thm. 1.15]).

In order to overcome this problem concerning the lack of $\mathscr{C}^r$ tubular neighborhoods (when Y is not a boundaryless $\mathscr{C}^r$ submanifold of $\mathbb{R}^n$), we introduce the key concept of weakly $\mathscr{C}^r$ triangulable set.

Fix $r \in \mathbb{N}^* \cup \{\infty\}$ for a while.

Definition 1.5. We say that $Y \subset \mathbb{R}^n$ is *weakly $\mathscr{C}^r$ triangulable* if there exist a locally finite simplicial complex L of some $\mathbb{R}^q$ and a homeomorphism $\Psi : |L| \to Y$ such that the restriction $\Psi|_\xi : \xi \to Y$ is a $\mathscr{C}^r$ map for each simplex $\xi \in L$.

Our first main result reads as follows.

THEOREM 1.6. *Every weakly $\mathscr{C}^r$ triangulable set is a $\mathscr{C}^r$-*`ats`.

A natural matter that arises from the preceding statement is to reveal large and relevant families of weakly $\mathscr{C}^r$ triangulable sets. A first relevant example is the collection of $\mathscr{C}^r$ submanifolds with boundary of $\mathbb{R}^n$ (also treated in [ORR, Ch. III, Thm. 6.1]):

COROLLARY 1.7. *Every $\mathscr{C}^r$ submanifold with boundary of $\mathbb{R}^n$ is a $\mathscr{C}^r$-*`ats`.

Another important example is given by the family of locally compact polyhedra itself. Indeed, each locally compact polyhedron is weakly $\mathscr{C}^\infty$ triangulable by definition. Consequently:

COROLLARY 1.8. *Every locally compact polyhedron is a $\mathscr{C}^\infty$-*`ats`.

Let $Y \subset \mathbb{R}^n$ be a triangulable set, let L be a locally finite simplicial complex of some $\mathbb{R}^q$ and let $\Psi : |L| \to Y$ be a homeomorphism between the realization $|L|$ of L and Y. Given $w \in L$ the *star* $\mathrm{St}(w, L)$ *of w in L* is the union of the interiors of those simplices of L that have w as a vertex [M5, §2, p. 11]. Recall that the interior of a simplex σ is defined as σ with its proper faces removed [M5, §1, p. 5].

Recall that $Y \subset \mathbb{R}^n$ is said to be $\mathscr{C}^r$ *triangulable* if there exists a homeomorphism $\Psi : |L| \to Y$ between the realization $|L|$ of a locally finite simplicial complex of some $\mathbb{R}^q$ and Y such that:

- the restriction $\Psi|_\xi : \xi \to Y$ is a $\mathscr{C}^r$ map for each simplex ξ of L and
- the map $d_w\Psi : \mathrm{St}(w, L) \to \mathbb{R}^n$, $y \mapsto d_w(\Psi|_\xi)(y - w)$ (where ξ is a simplex of L such that $y \in \xi$) is a homeomorphism onto its image for each $w \in L$.

Every $\mathscr{C}^r$ triangulable set is weakly $\mathscr{C}^r$ triangulable by definition. For further reference concerning $\mathscr{C}^r$ *triangulations* see for instance [Ca], [M4, §II.8] and [Sh, §I.3, pp. 72–94].

A set $Y \subset \mathbb{R}^n$ is called *locally $\mathscr{C}^r$ equivalent to a polyhedron*, or *locally $\mathscr{C}^r$ polyhedral* for short, if for each point $x \in Y$ there exist two open neighborhoods U and V of x in $\mathbb{R}^n$, a $\mathscr{C}^r$ diffeomorphism $\phi : U \to V$ and a locally compact polyhedron P of $\mathbb{R}^n$ such that $x \in P$, $\phi(x) = x$ and $\phi(U \cap Y) = V \cap P$. In [Sh, Prop. I.3.13] Shiota proved that every locally $\mathscr{C}^r$ polyhedral set is $\mathscr{C}^r$ triangulable. We deduce:

COROLLARY 1.9. *Every $\mathscr{C}^r$ triangulable set is a $\mathscr{C}^r$-*`ats`. *In particular, every locally $\mathscr{C}^r$ polyhedral set is a $\mathscr{C}^r$-*`ats`.

A celebrated family of locally $\mathscr{C}^r$ polyhedral sets is the collection of $\mathscr{C}^r$ submanifolds with corners of Euclidean spaces, which includes the above mentioned collection of $\mathscr{C}^r$ submanifolds with boundary of Euclidean spaces. A subset $Y \subset \mathbb{R}^n$ is a $\mathscr{C}^r$ *submanifold with corners* of dimension d if for each point $x \in Y$ there exist an integer $k \in \{0,\dots,d\}$ and open neighborhoods $U \subset \mathbb{R}^n$ of x and $V \subset \mathbb{R}^n$ of the origin together with a $\mathscr{C}^r$ diffeomorphism $\varphi : U \to V$ such that $\varphi(U \cap Y) = V \cap \{x_1 \geq 0,\dots,x_k \geq 0, x_{d+1} = 0,\dots,x_n = 0\} \subset \mathbb{R}^n$, see [Jo, Me1, Me2]:

COROLLARY 1.10. *Every $\mathscr{C}^r$ submanifold with corners of $\mathbb{R}^n$ is a $\mathscr{C}^r$*-`ats`.

Another well-known family of locally $\mathscr{C}^r$ polyhedral sets arises when considering subsets Y of a $\mathscr{C}^r$ submanifold M of dimension d of some $\mathbb{R}^n$ with the following property: *for each point $x \in Y$ there exists an open neighborhood $W \subset M$ of x endowed with a $\mathscr{C}^r$ diffeomorphism $\psi : W \to \mathbb{R}^p$ that maps x to the origin and satisfies that $\psi(Y \cap W)$ is a union of coordinate vector subspaces of $\mathbb{R}^p$*. Inside the preceding family appears unions of locally finite families of $\mathscr{C}^r$ submanifolds of M that meet transversally (in the preceding sense). Sets obtained in this way are called *sets with (only) $\mathscr{C}^r$ monomial singularities* [BFR, FGR]. A particular case concerns $\mathscr{C}^r$ *normal-crossing divisors*, that is, unions of locally finite families of $\mathscr{C}^r$ hypersurfaces of M that meet transversally.

A very relevant class of triangulable sets is certainly the one of subanalytic sets, which includes semialgebraic sets. See [BCR, BM1, Sh] for basic facts concerning the geometry of these sets. Let us recall the main definitions.

A set $Y \subset \mathbb{R}^n$ is *semialgebraic* if it admits a description as a finite Boolean combination of polynomial equalities and inequalities. The set Y is called *locally semialgebraic* if the intersection $Y \cap B$ is semialgebraic for each compact ball B of $\mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be a (non-empty) open set. A set $Y \subset U$ is *analytic* if for each point $x \in U$ there exists an open neighborhood $V \subset U$ of x such that $Y \cap V = \{f_1 = 0,\dots,f_r = 0\} = \{f_1^2 + \cdots + f_r^2 = 0\}$ for some real analytic functions $f_i \in \mathscr{C}^\omega(V,\mathbb{R})$. More generally, a set $Y \subset U$ is *semianalytic* if for each point $x \in U$ there exists an open neighborhood $V \subset U$ of x such that $Y \cap V$ is a finite Boolean combination of real analytic equalities and inequalities on V. The subanalytic sets are roughly speaking the images of semianalytic sets under proper real analytic maps. More precisely, $Y \subset U$ is a *subanalytic set* if there exist an open subset W of some $\mathbb{R}^p$, a real analytic map $f : W \to U$ and a semianalytic set $T \subset W$ such that the restriction $f|_{\mathrm{Cl}_W(T)} : \mathrm{Cl}_W(T) \to U$ is proper and $f(T) = Y$. Here $\mathrm{Cl}_W(T)$ is the closure of T in W. Locally semialgebraic sets are semianalytic and hence subanalytic.

The Hironaka-Hardt triangulation theorem [Hi, Hr] asserts that each locally finite family $\mathfrak{Y} := \{Y_i\}_{i \in I}$ of subanalytic subsets of $\mathbb{R}^n$ is "$\mathscr{C}^\omega$ triangulable on open simplices" in the following sense [Hr, Thms. 1 & 2]: *there exist a locally finite simplicial complex L of some $\mathbb{R}^q$ and a homeomorphism $\Psi : |L| \to \mathbb{R}^n$ such that:*

- *the image* $\Psi(\xi^0)$ *of each open simplex* ξ^0 *of* L *is a* $\mathscr{C}^\omega$ *submanifold of* $\mathbb{R}^n$,
- *each restriction* $\Psi|_{\xi^0} : \xi^0 \to \Psi(\xi^0)$ *is a* $\mathscr{C}^\omega$ *diffeomorphism.*
- *each subanalytic set* Y_i *is the (disjoint) union of some of the images* $\Psi(\xi^0)$.

Unfortunately, this result does not ensure that a subanalytic subset Y of $\mathbb{R}^n$ is weakly $\mathscr{C}^r$ triangulable for some $r \in \mathbb{N}^* \cup \{\infty\}$.

The weakly $\mathscr{C}^r$ triangulability of semialgebraic and subanalytic sets is not yet known for $r \geq 2$. However, the situation is completely different for $r = 1$. Indeed, in [OS] the authors have proved recently that every locally compact locally semialgebraic set Y has a triangulating homeomorphism $\Psi : |L| \to Y$ such that $\Psi \in \mathscr{C}^1(|L|, Y)$. In particular, Y is weakly $\mathscr{C}^1$ triangulable. See [CP] for further information concerning the regularity of Ψ.

As it is commented by the authors of [OS] in the first paragraph of the introduction, it is straightforward to check that the techniques developed in [OS] extend to the subanalytic case. It turns out that locally compact subanalytic sets are weakly $\mathscr{C}^1$ triangulable as well. We deduce:

COROLLARY 1.11. *Every locally compact subanalytic set is a* $\mathscr{C}^1$-`ats`. *In particular, each locally compact locally semialgebraic set is a* $\mathscr{C}^1$-`ats`.

Let us recall next the definition of o-minimal structure.

Definition 1.12. An *o-minimal structure* (on the field $\mathbb{R}$) is a collection $\mathfrak{S} := \{\mathfrak{S}_n\}_{n\in\mathbb{N}^*}$ of families of subsets of $\mathbb{R}^n$ satisfying:

- $\mathfrak{S}_n$ contains all the algebraic subsets of $\mathbb{R}^n$.
- $\mathfrak{S}_n$ is a Boolean algebra.
- If $A \in \mathfrak{S}_m$ and $B \in \mathfrak{S}_n$, then $A \times B \in \mathfrak{S}_{m+n}$.
- If $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the natural projection and $A \in \mathfrak{S}_{n+1}$, then $\pi(A) \in \mathfrak{S}_n$.
- $\mathfrak{S}_1$ consists precisely of all the finite unions of points and intervals of any type.

The elements of $\bigcup_{n\in\mathbb{N}^*} \mathfrak{S}_n$ are called *definable sets of* $\mathfrak{S}$. As a consequence of Tarski-Seidenberg theorem, semialgebraic sets constitute an o-minimal structure, which is in fact contained in each o-minimal structure. The collection of "global" subanalytic sets is precisely the collection of definable sets in the remarkable o-minimal structure $\mathbb{R}_{\text{an}}$, see [Wi]. We refer the reader to [vD, vdDM] for further information on the celebrated theory of o-minimal structures. As in the semialgebraic case, we say that a set $Y \subset \mathbb{R}^n$ is a *locally definable set of* $\mathfrak{S}$ if the intersection $Y \cap B$ is a definable set of $\mathfrak{S}$ for each compact ball B of $\mathbb{R}^n$.

Also in the o-minimal setting it is straightforward to adapt the constructions developed in [OS, CP] to show that every locally compact locally definable sets of any o-minimal structure is weakly $\mathscr{C}^1$ triangulable (see the first paragraph of the introduction of [OS]). We deduce the following extension of Corollary 1.11:

COROLLARY 1.13. *Every locally compact locally definable set of an arbitrary o-minimal structure is a* $\mathscr{C}^1$-`ats`.

1.B. ***C*-analytic sets.** Now, we focus on a quite significant subclass of subanalytic sets, the one of C-analytic sets (also known as global analytic sets [C]). We do not know if C-analytic sets are weakly $\mathscr{C}^r$ triangulable for some $r \geq 2$, but we develop an alternative approximation strategy to prove in Theorem 1.15 an analogous result to Theorem 1.6 (with $r = \infty$) under the additional assumption that the involved maps are proper.

Let $U \subset \mathbb{R}^n$ be a (non-empty) open set. A set $Y \subset U$ is said to be a *C-analytic subset of U* if there exist finitely many global real analytic functions $f_1, \dots, f_r \in \mathscr{C}^\omega(U, \mathbb{R})$ such that

$$Y = \{f_1 = 0, \dots, f_r = 0\} = \{f_1^2 + \cdots + f_r^2 = 0\}.$$

By the term *C-analytic set* we mean a C-analytic subset of an open subset of some $\mathbb{R}^n$. Real algebraic sets and Nash sets are particular examples of C-analytic sets.

Let $X \subset \mathbb{R}^m$ be a locally compact set, let $Y \subset \mathbb{R}^n$ be a set and let $\mathscr{C}^0_*(X,Y)$ be the set of all *proper* continuous maps from X to Y endowed with the relative topology inherited from the strong $\mathscr{C}^0$ topology of $\mathscr{C}^0(X,Y)$. Given $r \in \mathbb{N}^* \cup \{\infty\}$, we set $\mathscr{C}^r_*(X,Y) := \mathscr{C}^r(X,Y) \cap \mathscr{C}^0_*(X,Y)$.

Definition 1.14. Let $r \in \mathbb{N}^* \cup \{\infty\}$. A triangulable set $Y \subset \mathbb{R}^n$ is a *$\mathscr{C}^r_*$-approximation target space* or a $\mathscr{C}^r_*$-`ats` for short if $\mathscr{C}^r_*(X,Y)$ is dense in $\mathscr{C}^0_*(X,Y)$ for each locally compact subset X of each Euclidean space $\mathbb{R}^m$, where $m \in \mathbb{N}$ is any natural number.

Our second main result reads as follows.

THEOREM 1.15. *Every C-analytic set is a $\mathscr{C}^\infty_*$-*`ats`.

1.C. Sharpness. The results presented above provide families of triangulable sets $Y \subset \mathbb{R}^n$ for which $\mathscr{C}^r(X,Y)$ is dense in $\mathscr{C}^0(X,Y)$ where $r \in \mathbb{N}^* \cup \{\infty\}$ and $X \subset \mathbb{R}^m$ is an arbitrary locally compact set. If $s > 0$ is any positive integer such that $s < r$, one may ask whether $\mathscr{C}^r(X,Y)$ is also dense in $\mathscr{C}^s(X,Y)$ at least in the case X is a $\mathscr{C}^s$ submanifold of $\mathbb{R}^m$, where $\mathscr{C}^s(X,Y)$ is endowed with the relative topology induced by the strong (Whitney) $\mathscr{C}^s$ topology of $\mathscr{C}^s(X,\mathbb{R}^n)$ via the natural inclusion $\mathscr{C}^s(X,Y) \hookrightarrow \mathscr{C}^s(X,\mathbb{R}^n)$.

The following example points out that there is no hope to obtain general statements if $s > 0$.

Example 1.16. Let $X := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the standard circle, let $s \in \mathbb{N}^*$ and let $Y := \{(x,y,z) \in X \times \mathbb{R} : z^3 - y^{3s+1} = 0\}$. Note that Y is not a $\mathscr{C}^{s+1}$ submanifold of $\mathbb{R}^3$.

Consider the $\mathscr{C}^s$ map $f : X \to Y$, $(x,y) \mapsto (x,y,y^{s+1/3})$. Such a map cannot be $\mathscr{C}^1$ approximated (and hence $\mathscr{C}^s$ approximated) by maps in $\mathscr{C}^{s+1}(X,Y)$. Suppose on the contrary that there exists a $\mathscr{C}^{s+1}$ map $g := (g_1, g_2, g_3) : X \to Y$ arbitrarily close to f in the strong $\mathscr{C}^1$ topology. Thus, $g_* := (g_1, g_2) : X \to X$ is arbitrarily $\mathscr{C}^1$

close to the identity map on X, hence g_* is a $\mathscr{C}^{s+1}$ diffeomorphism by the inverse function theorem. As $g_3 = g_2^{s+1/3}$, it follows that $(g_3 \circ g_*^{-1})(x,y) = y^{s+1/3}$ is a $\mathscr{C}^{s+1}$ function on X, which is a contradiction. Indeed, such a function is not of class $\mathscr{C}^{s+1}$ locally at $(\pm 1, 0)$. This proves that $\mathscr{C}^{s+1}(X,Y)$ is not dense in $\mathscr{C}^s(X,Y)$.

If $s = 0$, it is also sharp our choice $r \in \mathbb{N}^* \cup \{\infty\}$, that is, we cannot choose in general $r = \omega$.

Example 1.17. Let $Y := \{xy = 0\} \subset \mathbb{R}^2$ and let $f : \mathbb{R} \to Y$ be the continuous map defined by $f(t) := (0,t)$ if $t < 0$ and $f(t) := (t,0)$ if $t \geq 0$. Then f cannot be approximated by real analytic maps $g = (g_-, g_+) : \mathbb{R} \to Y$. Otherwise, $g_\pm$ would be a real analytic function on $\mathbb{R}$ vanishing identically locally at $\pm\infty$ and nowhere zero locally at $\mp\infty$, which is impossible by the principle of analytic continuation. The reader may compare this "negative" example with the "positive" approximation theorem [BFR, Thm. 1.7], which is a key result for the proof of the main theorem of [Fe].

Similar examples appeared in our manuscript [FG].

1.D. An unexpected by-product. The techniques involved in the proof of Theorem 1.6 reveal another approximation property of each locally compact polyhedron P that has interest by its own. We fix a convention: *The set $\mathscr{C}^0(P,\mathbb{R}^+)$ of strictly positive continuous functions on P is endowed with the partial ordering $\succcurlyeq$ defined by $\varepsilon \succcurlyeq \delta$ if $\varepsilon(w) \leq \delta(w)$ for each $w \in P$.* Note that $\mathscr{C}^0(P,\mathbb{R}^+)$ is a directed set with such an ordering.

COROLLARY 1.18. *Let K be a locally finite simplicial complex of $\mathbb{R}^p$ and let $P := |K| \subset \mathbb{R}^p$ be its underlying locally compact polyhedron. Then there exists a net $\{\iota_\varepsilon\}_{\varepsilon \in \mathscr{C}^0(P,\mathbb{R}^+)}$ in $\mathscr{C}^\infty(P,P)$ that depends only on K, converges in the Moore-Smith sense to the identity map on P in $\mathscr{C}^0(P,P)$ and satisfies the following universal property:*

$(*)$ Let $r \in \mathbb{N}^* \cup \{\infty\}$, let $Y \subset \mathbb{R}^n$ be any weakly $\mathscr{C}^r$ triangulable set and let $f \in \mathscr{C}^0(P,Y)$ be such that $f|_\sigma \in \mathscr{C}^r(\sigma,Y)$ for each $\sigma \in K$. Then the net $\{f \circ \iota_\varepsilon\}_{\varepsilon \in \mathscr{C}^0(P,\mathbb{R}^+)}$ converges in the Moore-Smith sense to f in $\mathscr{C}^0(P,Y)$ and each composition $f \circ \iota_\varepsilon$ belongs to $\mathscr{C}^r(P,Y)$.

Remark 1.19. If in the preceding statement K is a finite simplicial complex, the net $\{\iota_\varepsilon\}_{\varepsilon \in \mathscr{C}^0(P,\mathbb{R}^+)}$ can be replaced by a sequence $\{\iota_k\}_{k \in \mathbb{N}}$ in $\mathscr{C}^\infty(P,P)$ with the same universal property.

1.E. Smooth relative approximations. A natural question that arises when dealing with approximation problems concerns the existence of relative versions.

Fix $r \in \mathbb{N}^* \cup \{\infty\}$.

Let $X' \subset X \subset \mathbb{R}^m$ be non-empty sets such that X is locally compact and X' is closed in X, let $Y \subset \mathbb{R}^n$ be a set and let $f : X \to Y$ be a continuous map whose restriction $f|_{X'} : X' \to Y$ to X' is a $\mathscr{C}^r$ map. *Are there $\mathscr{C}^r$ maps $g : X \to Y$ that approximate f and satisfy $g|_{X'} = f|_{X'}$?*

As it is well known [M4, p. 42, Ex. (a)]: *If Y admits a system of $\mathscr{C}^r$ tubular neighborhoods in $\mathbb{R}^n$* (that is, Y is a boundaryless $\mathscr{C}^r$ submanifold of $\mathbb{R}^n$), *then relative approximation is always possible*. We sketch here a proof of the previous fact to stress once more the crucial role played by $\mathscr{C}^r$ retractions onto Y.

Sketch of proof. Let $f \in \mathscr{C}^0(X,Y)$ be such that $f|_{X'}$ is a $\mathscr{C}^r$ map. Let (V,ρ) be a $\mathscr{C}^r$ tubular neighborhood of Y in $\mathbb{R}^n$, where $\rho : V \to Y$ is a $\mathscr{C}^r$ retraction. By Whitney's approximation theorem there exists $g_0 \in \mathscr{C}^r(X,Y)$ close to f in the strong $\mathscr{C}^0$ topology. Let $f_1 : X \to \mathbb{R}^n$ be a $\mathscr{C}^r$ extension of $f|_{X'}$ to a small enough open neighborhood $U \subset X$ of X'. Let $\{\theta, 1-\theta\}$ be a $\mathscr{C}^r$ partition of unity associated to $\{U, X \setminus X'\}$. Then $g_1 := \theta f_1 + (1-\theta)g_0 : X \to \mathbb{R}^n$ is close to f in the strong $\mathscr{C}^0$ topology and $g_1|_{X'} = f|_{X'}$. We may assume in addition $g_1(X) \subset V$ and $g := \rho \circ g_1$ is close to $\rho \circ f = f$ in the strong $\mathscr{C}^0$ topology (see Lemma 2.1 below). Observe that $g|_{X'} = g_1|_{X'} = f|_{X'}$, as required. □

For more general spaces Y, which do not have $\mathscr{C}^r$ retractions of open neighborhoods of Y in $\mathbb{R}^n$ onto Y, the situation is more restrictive with both X' and the restriction $f|_{X'}$. Let us see some enlightening counterexamples concerning $\mathscr{C}^r$ manifolds Y with non-empty boundary.

Examples 1.20. Let Y be a $\mathscr{C}^r$ submanifold of $\mathbb{R}^n$ with non-empty boundary. Let $D(Y)$ be the double of Y. Denote T_yY the tangent space of Y at the point $y \in Y$. Let $h : Y \to \mathbb{R}$ be a $\mathscr{C}^r$ equation of the boundary ∂Y of Y such that $\{h > 0\}$ equals the interior $Y \setminus \partial Y$ of Y and $d_yh : T_yY \to \mathbb{R}$ is surjective for each $y \in \partial Y$. A well-known way to endow $D(Y)$ with a $\mathscr{C}^r$ structure is to identify it with the boundaryless $\mathscr{C}^r$ manifold $M := \{(x,t) \in Y \times \mathbb{R} : t^2 = h(x)\}$. We also identify Y with $M \cap \{t \geq 0\}$. Denote $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ the projection $(x,t) \mapsto t$, which satisfies $\pi(Y) \subset [0,+\infty)$ and $\pi(\partial Y) = \{0\}$.

(i) Let $X' \subset X \subset \mathbb{R}^m$ be non-empty sets and let $f : X \to Y$ be a continuous map such that the restriction $f|_{X'} : X' \to Y$ is a $\mathscr{C}^r$ map. Assume that there exists a continuous path $\beta : [-1,1] \to X$ such that $\beta([0,1]) \subset X'$, the restriction $\beta|_{[0,1]} : [0,1] \to X$ is a $\mathscr{C}^r$ map, $f(\beta(0)) \in \partial Y$ and the derivative $(\pi \circ f \circ \beta)'(0)$ of $\pi \circ f \circ \beta$ at 0 is strictly positive. Recall that $\pi(Y) \subset [0,+\infty)$ and $\pi(\partial Y) = \{0\}$. Consider the continuous map

$$f^* : [-1,1] \to [0,+\infty),\ t \mapsto \pi(f(\beta(t))).$$

Note that $f^*|_{[0,1]}$ is a $\mathscr{C}^r$ map, $f^*(0) = 0$ and $(f^*)'(0) > 0$. It follows immediately that there exists no $\mathscr{C}^r$ map $g : X \to Y$ such that $g|_{X'} = f|_{X'}$ (or better $g|_{\beta([0,1])} = f|_{\beta([0,1])}$). Otherwise, the $\mathscr{C}^r$ map $g^* : [-1,1] \to [0,+\infty)$, $t \mapsto \pi(g(\beta(t)))$ would coincide with f^* on $[0,1]$, so $(g^*)'(0)=(f^*)'(0)>0$, so g^* would be strictly increasing locally at $t=0$ in $[-1,1]$. This is impossible because $g^*(0)=f^*(0)=0$ and $g^*\geq 0$ on the whole interval $[-1,1]$. Consequently, there exists no $\mathscr{C}^r$ extension of $f^*|_{[0,1]}$ to $[-1,1]$ whose image is contained in $[0,+\infty)$ and there exists no $\mathscr{C}^r$ extension from X to Y of $f|_{X'}$. In particular, there exists no $\mathscr{C}^r$ approximation g of f such that $g|_{X'}=f|_{X'}$.

(i$'$) An easy example in which the obstruction described in (i) appears is the following: $X:=[-1,1]$, $X':=[0,1]$, $Y:=\{(x,y)\in\mathbb{R}^2 : y\geq 0\}$ and $f:X\to Y$ is given by $f(t):=(t,0)$ if $t\in[-1,0)$, $f(t):=(0,t)$ if $t\in X'=[0,1]$ and $\beta:[-1,1]\to X$ is the identity map.

(ii) We keep the notations fixed above concerning the $\mathscr{C}^r$ submanifold Y of $\mathbb{R}^n$ with non-empty boundary. Suppose that Y has dimension d. Consider the continuous map

$$f_1 : D(Y)\to Y,\quad (x,t)\mapsto (x,|t|).$$

We claim: $f_1|_Y=\mathrm{id}_Y$ *(which is* $\mathscr{C}^r$*), but* $f_1|_Y$ *admits no* $\mathscr{C}^r$ *extension to* $D(Y)$ *whose image is contained in* Y. This implies that *there exists no* $\mathscr{C}^r$ *approximation* g *of* f_1 *such that* $g|_Y=f_1|_Y$.

Pick a point $y_0\in\partial Y$. Let $\epsilon>0$, let $B_n(0,\epsilon^2)$ be the open ball of $\mathbb{R}^n$ centered at 0 with radius ϵ^2 and let $U\subset\mathbb{R}^n$ be an open neighborhood of y_0 endowed with a $\mathscr{C}^r$ diffeomorphism $u:=(u_1,\dots,u_n):U\to B_n(0,\epsilon^2)$ such that $u(y_0)=0$, $u(U\cap Y)=B_n(0,\epsilon^2)\cap\{x_1\geq 0, x_{d+1}=0,\dots,x_n=0\}$ and $u_1(x)=h(x)$ for each $x\in U\cap Y$. Consider the $\mathscr{C}^r$ diffeomorphism

$$u^* : U^* := U\times\mathbb{R}\to B_n(0,\epsilon^2)\times\mathbb{R},\quad (x,t)\mapsto (u(x),t)$$

such that $u^*(y_0,0)=(0,0)$ and

$$u^*(D(Y)\cap U^*)=(B_n(0,\epsilon^2)\times\mathbb{R})\cap\{x_1=t^2, x_{d+1}=0,\dots,x_n=0\}=:M^*.$$

Consider the path $\alpha:(-\epsilon,\epsilon)\to M^*$, $t\mapsto(t^2,0,\dots,0,t)$ and define the function

$$f_1^* : (-\epsilon,\epsilon)\to\mathbb{R},\quad t\mapsto\pi(f_1((u^*)^{-1}(\alpha(t))))=|t|.$$

Observe that $f_1^*|_{[0,\epsilon)}:[0,\epsilon)\to\mathbb{R}$, $t\mapsto t$ is a $\mathscr{C}^r$ function, but there exists no extension of $f_1^*|_{[0,\epsilon)}$ to $(-\epsilon,\epsilon)$ whose image is contained in $\pi(Y)\subset[0,+\infty)$. In particular, there exists no $\mathscr{C}^r$ extension of $f_1|_Y$ to $D(Y)$ whose image is contained in Y, as required.

In order to avoid the obstruction described in the preceding examples, in [ORR, Ch. III, Thm. 6.1] (see Theorem 1.4 above) the authors slightly modify classical relative approximation statement and ask (as a stronger hypothesis) that the restriction of the continuous function f to a small enough open neighborhood $W\subset X$ of X' is a $\mathscr{C}^r$ map. Then they take advantage of the existence of $\mathscr{C}^r$ tubular neighborhoods of $Y\setminus\partial Y$ in $\mathbb{R}^n$ and of $\mathscr{C}^r$ collars of ∂Y in Y. In this way, they obtain a $\mathscr{C}^r$ approximation g of f which coincides with f on a neighborhood $W' \subset W$ of X'. Since $g = f$ on a neighborhood $W' \subset W$ of X', it seems not possible to integrate the constructions in [ORR, Ch. III, Thm. 6.1] with our approximation method to achieve more general situations, unless f is constant on W (see Remark 3.12 below).

Suppose next X' is a discrete and closed subset of X. Our next result states that $\mathscr{C}^r$ approximation of continuous maps $f : X \to Y$ relative to X' is possible.

THEOREM 1.21. *Let $X \subset \mathbb{R}^m$ be a locally compact set, let X' be a discrete and closed subset of X, let $Y \subset \mathbb{R}^n$ be a weakly $\mathscr{C}^r$ triangulable set and let $f : X \to Y$ be a continuous map. If $f(X')$ is discrete and closed in Y, then there exists $g \in \mathscr{C}^r(X,Y)$ arbitrarily close to f in the strong $\mathscr{C}^0$ topology such that $g|_{X'} = f|_{X'}$.*

The latter result implies straightforwardly some properties of weakly $\mathscr{C}^r$ triangulable sets Y concerning connectedness and homotopy. Fix $y_0 \in Y$ and denote $\pi_p(Y,y_0)$ the pth homotopy group of the pointed space (Y,y_0) for each $p \in \mathbb{N}^*$. We understand the elements of $\pi_p(Y,y_0)$ as the homotopy classes of continuous maps $(\mathbb{S}^p,N) \to (Y,y_0)$, where $\mathbb{S}^p$ is the standard p-sphere and N is its north pole. The path-connected components of Y coincide with its $\mathscr{C}^r$ path-connected components and each element of $\pi_p(Y,y_0)$ has a representative of class $\mathscr{C}^r$.

COROLLARY 1.22. *Let $Y \subset \mathbb{R}^n$ be a weakly $\mathscr{C}^r$ triangulable set. We have:*

(i) *Each continuous path $\gamma : [0,1] \to Y$ can be approximated in the strong $\mathscr{C}^0$ topology by $\mathscr{C}^r$ paths $\alpha : [0,1] \to Y$ such that $\alpha(0) = \gamma(0)$ and $\alpha(1) = \gamma(1)$.*

(ii) *Every element of $\pi_p(Y,y_0)$ can be represented by a $\mathscr{C}^r$ map.*

The preceding corollary holds for $r = \infty$ if Y is a locally compact polyhedron and for $r = 1$ if Y is a locally compact locally definable set in an arbitrary o-minimal structure.

Theorem 1.21 can still be extended: the crucial property to have approximations relative to X' is that $f(X')$ has no accumulation points in Y. Before entering into details, we introduce the following definition.

Definition 1.23. Let $X' \subset X \subset \mathbb{R}^m$ be sets such that (X is non-empty and) X' is closed in X. The pair (X,X') is *weakly* $\mathscr{C}^r$ triangulable* if there exist a locally finite simplicial complex K of some $\mathbb{R}^p$, a subcomplex K' of K and a homeomorphism $\Phi : |K| \to X$ such that $\Phi(|K'|) = X'$ and the restriction $\Phi|_{\sigma^0} : \sigma^0 \to X$ is a $\mathscr{C}^r$ map for each open simplex σ^0 of K. We say that X is *weakly* $\mathscr{C}^r$ triangulable* if so is the pair $(X,\emptyset)$.

Let $X' \subset X \subset \mathbb{R}^m$ be such that X is locally compact and X' is closed in X. The pair (X,X') is a *subanalytic pair* if both X and X' are subanalytic subsets of $\mathbb{R}^m$. Analogously, if $\mathfrak{S}$ is an o-minimal structure (on the field $\mathbb{R}$), the pair (X,X') is a *locally definable pair of* $\mathfrak{S}$ if both X and X' are locally definable sets of $\mathfrak{S}$. By Hironaka-Hardt's triangulation theorem a subanalytic pair is a weakly* $\mathscr{C}^\infty$

triangulable pair, whereas by [Sh, Ch. II, Thm. II'] a locally definable pair is a weakly* $\mathscr{C}^\infty$ triangulable pair.

The announced extension of Theorem 1.21 is the following.

THEOREM 1.24. *Let (X, X') be a weakly* $\mathscr{C}^r$ *triangulable pair, let $Y \subset \mathbb{R}^n$ be a weakly $\mathscr{C}^r$ triangulable set and let $f : X \to Y$ be a continuous map such that $f(X')$ is discrete and closed in Y. Then, there exists $g \in \mathscr{C}^r(X, Y)$ arbitrarily close to f in the strong $\mathscr{C}^0$ topology such that $g|_{X'} = f|_{X'}$.*

As an immediate consequence of the preceding result, one deduces the following:

COROLLARY 1.25. *Let (X, X') be either a subanalytic pair or a locally definable pair of an arbitrary o-minimal structure $\mathfrak{S}$, let $Y \subset \mathbb{R}^n$ be a set and let $f : X \to Y$ be a continuous map such that $f(X')$ is discrete and closed in Y. We have:*

(i) *If (X, X') is a subanalytic pair and Y is a locally compact polyhedron, there exists $g \in \mathscr{C}^\infty(X, Y)$ arbitrarily close to f in the strong $\mathscr{C}^0$ topology such that $g|_{X'} = f|_{X'}$.*

(ii) *If (X, X') is a locally definable pair of $\mathfrak{S}$ and Y is a locally compact locally definable set of $\mathfrak{S}$, there exists $g \in \mathscr{C}^1(X, Y)$ arbitrarily close to f in the strong $\mathscr{C}^0$ topology such that $g|_{X'} = f|_{X'}$.*

1.F. Structure of the article. In Section 2 we collect a couple of preliminary results concerning spaces of continuous maps. In the first part of Section 3 we present our variant of the general approximation theorem and our "shrink-widen" covering and approximation technique. In the second part we combine these results with the ones in Section 2 to prove Theorem 1.6. We provide also the proof of Corollary 1.18 and the relative approximation Theorem 1.24 (from which follow readily the other results in Section 1.E). Section 4 is devoted to prove Theorem 1.15, which involves the proof of the existence of $\mathscr{C}^r$ *weak retractions* and the immersion of C-analytic sets as singular sets of coherent C-analytic sets homeomorphic to Euclidean spaces. Weaker and purely semialgebraic versions of some results presented in this article appeared in our preceding manuscript [FG].

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2. Preliminaries on spaces of continuous maps. In this short section we collect a couple of results useful for the sequel. First we fix two notations we will use freely throughout the manuscript. Let $S,T \subset \mathbb{R}^q$ be such that $S \subset T$. Denote respectively $\mathrm{Cl}_T(S)$ and $\mathrm{Int}_T(S)$ the closure of S in T and the interior of S in T. The following result is well known and its proof follows straightforwardly from [H3, §2.5. Ex.10, pp. 64–65] using standard arguments.

LEMMA 2.1. *Let $X \subset \mathbb{R}^m$, $X' \subset \mathbb{R}^{m'}$, $Y \subset \mathbb{R}^n$ and $Y' \subset \mathbb{R}^{n'}$ be locally compact sets, let $f : Y \to Y'$ be an arbitrary continuous map and let $g : X \to X'$ be a proper continuous map. Then the maps*

$$f_* : \mathscr{C}^0(X,Y) \to \mathscr{C}^0(X,Y'), \quad h \mapsto f \circ h$$

and

$$g^* : \mathscr{C}^0(X',Y) \to \mathscr{C}^0(X,Y), \quad h \mapsto h \circ g$$

are continuous.

As a consequence, we deduce:

COROLLARY 2.2. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be locally compact sets. Then there exist closed subsets X' of $\mathbb{R}^{m+1}$ and Y' of $\mathbb{R}^{n+1}$ such that:*

- *X' is homeomorphic to X and Y' is homeomorphic to Y,*

and the following property holds for each $r \in \mathbb{N}^* \cup \{\infty\}$:

- *$\mathscr{C}^r(X,Y)$ is dense in $\mathscr{C}^0(X,Y)$ if and only if $\mathscr{C}^r(X',Y')$ is dense in $\mathscr{C}^0(X',Y')$.*

In addition, if Y is a C-analytic subset of some open subset U of $\mathbb{R}^n$, then there exists also a C-analytic subset Y'' of $\mathbb{R}^{2n+1}$ homeomorphic to Y such that $\mathscr{C}^\infty_*(X,Y)$ is dense in $\mathscr{C}^0_*(X,Y)$ if and only if $\mathscr{C}^\infty_*(X',Y'')$ is dense in $\mathscr{C}^0_*(X',Y'')$.

Proof. As $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are locally compact (or equivalently locally closed) sets, the differences $\mathrm{Cl}_{\mathbb{R}^m}(X) \setminus X$ and $\mathrm{Cl}_{\mathbb{R}^n}(Y) \setminus Y$ are respectively closed in $\mathbb{R}^m$ and in $\mathbb{R}^n$. Let $\theta : \mathbb{R}^m \to \mathbb{R}$ and $\xi : \mathbb{R}^n \to \mathbb{R}$ be $\mathscr{C}^\infty$ functions such that $\theta^{-1}(0) = \mathrm{Cl}_{\mathbb{R}^m}(X) \setminus X$ and $\xi^{-1}(0) = \mathrm{Cl}_{\mathbb{R}^n}(Y) \setminus Y$. Define $X' := \{(x,t) \in X \times \mathbb{R} : t = 1/\theta(x)\}$ and $Y' := \{(y,t) \in Y \times \mathbb{R} : t = 1/\xi(y)\}$ and consider the homeomorphisms $\Theta : X' \to X$ and $\Xi : Y \to Y'$ given by $\Theta(x,t) := x$ and $\Xi(y) := (y, 1/\xi(y))$. The sets X' and Y' are respectively closed in $\mathbb{R}^{m+1}$ and in $\mathbb{R}^{n+1}$. By Lemma 2.1 the map $H : \mathscr{C}^0(X,Y) \to \mathscr{C}^0(X',Y')$ given by $H := \Theta^* \circ \Xi_*$ is a homeomorphism. As Θ, Θ^{-1}, Ξ and Ξ^{-1} are $\mathscr{C}^\infty$ maps, we deduce $H(\mathscr{C}^r(X,Y)) = \mathscr{C}^r(X',Y')$ for each $r \in \mathbb{N}^* \cup \{\infty\}$, so the first part of the statement is proved.

Let us prove the second part. By Whitney's embedding theorem for the real analytic case [Na, 2.15.12], there exists a real analytic embedding $\varphi : U \to \mathbb{R}^{2n+1}$ such that $M := \varphi(U)$ is a closed real analytic submanifold of $\mathbb{R}^{2n+1}$. By Cartan's Theorem B real analytic functions on M are restrictions to M of real analytic functions on $\mathbb{R}^{2n+1}$. Thus, $Y'' := \varphi(Y)$ is a C-analytic subset of $\mathbb{R}^{2n+1}$. Denote

$\Phi : Y \to Y''$ the restriction of φ from Y to Y'' and $H' : \mathscr{C}^0(X,Y) \to \mathscr{C}^0(X',Y'')$ the homeomorphism $H' := \Theta^* \circ \Phi_*$. We conclude $H'(\mathscr{C}^0_*(X,Y)) = \mathscr{C}^0_*(X',Y'')$ and $H'(\mathscr{C}^\infty_*(X,Y)) = \mathscr{C}^\infty_*(X',Y'')$, as required. □

Remark 2.3. Let $X \subset \mathbb{R}^m$ and let $Y \subset \mathbb{R}^n$ be (non-empty) sets such that X is locally compact. Consider a locally finite covering $\{C_\ell\}_{\ell\in L}$ of X by non-empty compact sets and a family $\{\varepsilon_\ell\}_{\ell\in L}$ of positive real numbers. Making use of a suitable $\mathscr{C}^0$ partition of unity on X, one shows the existence of a strictly positive continuous function $\varepsilon : X \to \mathbb{R}^+$ such that $\max_{C_\ell}(\varepsilon) \le \varepsilon_\ell$ for each $\ell \in L$. This implies that a fundamental system of neighborhoods of $f \in \mathscr{C}^0(X,Y)$ for the strong $\mathscr{C}^0$ topology of $\mathscr{C}^0(X,Y)$ is given by the sets

$$\begin{aligned}&\mathcal{N}(f,\{C_\ell\}_{\ell\in L},\{\varepsilon_\ell\}_{\ell\in L})\\&\qquad := \{g \in \mathscr{C}^0(X,Y) : \|g(x)-f(x)\|_n < \varepsilon_\ell \ \forall \ell \in L,\ \forall x \in C_\ell\},\end{aligned}$$

where $\{C_\ell\}_{\ell\in L}$ runs over the locally finite coverings of X by non-empty compact sets and $\{\varepsilon_\ell\}_{\ell\in L}$ runs over the families of positive real numbers with the same set L of indices.

3. Proofs of Theorem 1.6 and Corollary 1.18. In this section we develop first all the machinery we need to prove Theorem 1.6:

- a variant of the general simplicial approximation theorem (that appears in Section 3.A),
- a "shrink-widen" covering and approximation technique (that appears in Section 3.B),

and after we approach its proof (see Section 3.C). Finally, we prove Corollary 1.18 and Theorems 1.21 and 1.24 (see Sections 3.D, 3.E, and 3.F) in the required order. A weaker "finite" version of the "shrink-widen" covering and approximation technique, that we present here in Section 3.B, is contained in our manuscript [FG].

3.A. A variant of the general simplicial approximation theorem. Given a locally finite simplicial complex K of some $\mathbb{R}^p$, a *subdivision K' of K* is a locally finite simplicial complex K' of $\mathbb{R}^p$ such that $|K'| = |K|$ and each simplex of K' is a subset of some simplex of K. A particular case of subdivision of K is the first barycentric subdivision $\mathrm{sd}(K)$ of K. We denote $\mathrm{sd}^k(K) := \mathrm{sd}(\mathrm{sd}^{k-1}(K))$ the *kth barycentric subdivision of K* for $k \ge 1$, where $\mathrm{sd}^0(K) := K$.

Let L be a locally finite simplicial complex of some $\mathbb{R}^q$ and let $F : |K| \to |L|$ be a continuous map. A simplicial map $F^\bullet : |K| \to |L|$ is said to be a *simplicial approximation of F* if $F(\mathrm{St}(v,K)) \subset \mathrm{St}(F^\bullet(v),L)$ for each vertex v of K. If $w \in |L|$, the *carrier of w in L* is the unique simplex $\tau \in L$ such that $w \in \tau^0$. The classical simplicial approximation theorem asserts:

THEOREM 3.1. [Al] *If K and L are finite simplicial complexes, there exists a natural number k such that F has a simplicial approximation $F^\bullet : |\mathrm{sd}^k(K)| \to |L|$. In addition, if $w \in |K|$ and τ is the carrier of $F(w)$ in L, then $F^\bullet(w) \in \tau$.*

As an immediate consequence:

COROLLARY 3.2. *If K and L are finite simplicial complexes and $\varepsilon > 0$ is a positive real number, then there exist two natural numbers κ and ℓ and a simplicial map $F^* : |\mathrm{sd}^\kappa(K)| \to |\mathrm{sd}^\ell(L)|$ such that $\|F^*(w) - F(w)\|_q < \varepsilon$ for each $w \in |\mathrm{sd}^\kappa(K)| = |K|$.*

We need to extend the latter result for locally finite simplicial complexes K and L with respect to the strong $\mathscr{C}^0$ topology of $\mathscr{C}^0(|K|, |L|)$.

Let K and L be arbitrary locally finite simplicial complexes and let $F : |K| \to |L|$ be a continuous map. We say that F satisfies the *star condition (relative to K and L)* if for each vertex v of K there exists a vertex w of L such that $F(\mathrm{St}(v, K)) \subset \mathrm{St}(w, L)$.

LEMMA 3.3. [M5, Lem. 14.1(a)(b)] *If the continuous map $F : |K| \to |L|$ satisfies the star condition, then it has a simplicial approximation $F^\bullet : |K| \to |L|$.*

Given two coverings $\mathcal{A}$ and $\mathcal{B}$ of $|K|$, we say that $\mathcal{B}$ *refines* $\mathcal{A}$ if for each $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subset A$. If v is a vertex of K, the *closed star* $\overline{\mathrm{St}}(v, K)$ *of v in K* is defined as the closure of $\mathrm{St}(v, K)$ in $|K|$. Observe that $\overline{\mathrm{St}}(v, K)$ is the union of all simplices of K having v as a vertex. In particular, it is the realization of a simplicial subcomplex of K.

THEOREM 3.4. [M5, Thm. 16.4] *Let K be a locally finite simplicial complex and let $\mathcal{A}$ be an open covering of $|K|$. Then there exists a subdivision K' of K such that the collection of closed stars $\overline{\mathrm{St}}(v, K')$, where v ranges over the vertices of K', refines $\mathcal{A}$.*

If we define $\mathcal{A}$ as the collection of $F^{-1}(\mathrm{St}(w, L))$, where w ranges over the vertices of L, then there exists by Theorem 3.4 a subdivision K' of K whose closed stars refines $\mathcal{A}$. Consequently, F satisfies the star condition relative to K' and L and Lemma 3.3 implies:

THEOREM 3.5. (General simplicial approximation [M5, Thm. 16.5]) *Given a continuous map $F : |K| \to |L|$ between locally compact polyhedra, there exists a subdivision K' of K such that F has a simplicial approximation $F^\bullet : |K'| \to |L|$.*

As pointed out above, we need a suitable version of the preceding theorem that takes into account, not only simplicial approximation, but also strong $\mathscr{C}^0$ approximation: as Corollary 3.2 does with respect to the classical simplicial approximation Theorem 3.1. To do this, we introduce the notion of weakly simplicial map.

Definition 3.6. Let K and L be locally finite simplicial complexes and let $F: |K| \to |L|$ be a continuous map. Suppose $|K| \subset \mathbb{R}^p$ and $|L| \subset \mathbb{R}^q$. We say that F is *weakly simplicial* if, for each simplex $\sigma \in K$, there exist a simplex $\xi_\sigma \in L$ and an affine map $A_\sigma : \mathbb{R}^p \to \mathbb{R}^q$ such that $F(\sigma) \subset \xi_\sigma$ and $F(x) = A_\sigma(x)$ for each $x \in \sigma$.

Observe that each weakly simplicial map $F : |K| \to |L|$ is uniquely determined by their values on the vertices of K. Evidently, each simplicial map is weakly simplicial. On the contrary, the map $F : \{0\} \to [-1,1]$, $0 \mapsto 0$ is an easy example of a weakly simplicial map between polyhedra of $\mathbb{R}$ which is not simplicial, if $[-1,1]$ is the realization of the simplicial complex $K := \{\{-1\},\{1\},[-1,1]\}$.

Our variant of the general simplicial approximation Theorem 3.5 is the following.

THEOREM 3.7. (Weakly simplicial approximation) *Let K and L be locally finite simplicial complexes and let $F : |K| \to |L|$ be a continuous map. Assume $|L| \subset \mathbb{R}^q$. Then, for each strictly positive continuous function $\varepsilon : |K| \to \mathbb{R}^+$, there exist a subdivision K' of K and a weakly simplicial map $F^* : |K'| \to |L|$ such that*

$$\|F^*(w) - F(w)\|_q < \varepsilon(w) \quad \textit{for each } w \in |K'| = |K|.$$

In addition, if $w \in |K|$ and τ is the carrier of $F(w)$ in L, then $F^*(w) \in \tau$.

Proof. The proof is conducted in several steps:

Step I. Initial preparation. Assume the simplicial complex K is infinite, because if K is finite the result follows from the classical simplicial approximation Theorem 3.1.

Denote $P := |K| \subset \mathbb{R}^p$ the realization of K. It turns out that P is locally compact, but not compact. Choose a sequence $\{P_n\}_{n\in\mathbb{N}}$ of compact subsets of P such that for each $n \in \mathbb{N}^*$:

- $P_n := |K_n|$ is the realization of a finite subcomplex K_n of K.
- $\mathrm{Int}_P(P_n)$ is compatible with K_n, that is, it is the union of the interiors of some of the simplices of K_n.
- $P_{n-1} \subsetneq \mathrm{Int}_P(P_n)$, where $P_0 := \emptyset$.
- $\bigcup_{n\in\mathbb{N}} P_n = P$.

The compact sets P_n can be constructed as follows. Let $\theta : \mathbb{R}^p \to \mathbb{R}$ be a continuous function such that $\theta^{-1}(0) = \mathrm{Cl}_{\mathbb{R}^n}(P) \setminus P$ and consider the map $\theta^* : \mathbb{R}^p \setminus \theta^{-1}(0) \to \mathbb{R}^{p+1}$, $x \mapsto (x, 1/\theta(x))$, which is a homeomorphism onto its image and satisfies $\theta^*(P)$ is a closed subset of $\mathbb{R}^{p+1}$. For each $r > 0$ denote $\mathcal{B}(r) := P \cap (\theta^*)^{-1}(B_{p+1}(0,r))$ and $\overline{\mathcal{B}}(r) := P \cap (\theta^*)^{-1}(\overline{B}_{p+1}(0,r))$, where $B_{p+1}(0,r)$ is the open ball of $\mathbb{R}^{p+1}$ with center the origin and radius r and $\overline{B}_{p+1}(0,r)$ is its closure in $\mathbb{R}^{p+1}$. Take a strictly increasing sequence of natural numbers $\{m_n\}_{n\in\mathbb{N}^*}$ such that $\overline{\mathcal{B}}(m_1) \neq \emptyset$ and consider the collection of compact subsets $\{\overline{\mathcal{B}}(m_n)\}_{n\in\mathbb{N}^*}$ of P.

For each $n \in \mathbb{N}^*$ define $\mathcal{V}_n$ as the collection of the vertices v of K whose open stars $\mathrm{St}(v,K)$ meet $\overline{\mathcal{B}}(m_n)$. Let K_n be the subcomplex of K consisting of all simplices σ such that $\mathcal{V}_n$ contains a vertex of σ, and all their faces. Define $P_n := |K_n|$. Observe that $P_n = \bigcup_{v \in \mathcal{V}_n} \overline{\mathrm{St}}(v,K)$ and $\overline{\mathcal{B}}(m_n) \subset \mathrm{Int}_P(P_n)$. We may assume that $P_n \subset \mathcal{B}(m_{n+1})$ (changing m_{n+1} by a bigger integer if necessary). Set $K_0 := \emptyset$, $P_0 := |K_0| = \emptyset$ and $P_{-1} := \emptyset$. The compact sets P_n satisfy the required conditions. Only the second property requires some comments. Pick a point $x \in \mathrm{Int}_P(P_n)$ and let $\sigma \in K_n$ be the carrier of x. Let us check: $\sigma^0 \subset \mathrm{Int}_P(P_n)$. Once this is proved, $\mathrm{Int}_P(P_n)$ is the union of the interiors of some of the simplices of K_n.

The star $\mathrm{St}(\sigma,K)$ is the union of the interiors of all the simplices $\lambda \in K$ that have σ as one of their faces. As $x \in \mathrm{Int}_P(P_n)$, there exists an open ball $B_p(x,\delta)$ centered at x and radius $\delta > 0$ such that $B_p(x,\delta) \cap P \subset P_n$. Observe that $B_p(x,\delta) \cap P$ meets all the simplices $\lambda \in K$ that have σ as one of their faces. As P_n is the realization of the simplicial subcomplex K_n, the simplices $\lambda \in K$ that have σ as one of their faces belong to K_n. Consequently, $\mathrm{St}(\sigma,K) \subset P_n$ and as it is an open subset of P, we conclude $\sigma^0 \subset \mathrm{St}(\sigma,K) \subset \mathrm{Int}_P(P_n)$.

For each $n \in \mathbb{N}$ let U_n be an open subset of P such that

$$P_n \subsetneq U_n \subset \mathrm{Cl}_P(U_n) \subset \mathrm{Int}_P(P_{n+1}).$$

We set $U_{-1} = U_{-2} := \emptyset$. Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be the non-increasing sequence of positive real numbers

$$\epsilon_n := \min_{w \in P_n} \{\varepsilon(w)\} > 0 \quad \text{for each } n \in \mathbb{N}^*$$

and set $\epsilon_0 := \epsilon_1$.

Step II. Construction of a suitable covering. For each $n \in \mathbb{N}$ let $L_n := \mathrm{sd}^{\ell_n}(L)$ be an iterated barycentric subdivision of L such that $\ell_n \leq \ell_{n+1}$ and:

(3.1)
The diameters of all simplices ξ of L_n with $\xi \cap F(P_n \setminus \mathrm{Int}_P(P_{n-1})) \neq \emptyset$ are $< \epsilon_n$.

For each vertex v of L_n consider the open star $\mathrm{St}(v,L_n)$ of v in L_n and define the open subset $V_{n,v}$ of $|K|$ given by

$$V_{n,v} := F^{-1}\big(\mathrm{St}(v,L_n)\big) \cap \big(\mathrm{Int}_P(P_n) \setminus \mathrm{Cl}_P(U_{n-2})\big).$$

We claim: $\mathcal{A} := \{V_{n,v} : n \in \mathbb{N},\ v$ *is a vertex of* $L_n\}$ *is an open covering of* P.

Pick a point $x_0 \in P$. Then there exists $n \in \mathbb{N}$ such that

$$x_0 \in \mathrm{Int}(P_n) \setminus \mathrm{Int}(P_{n-1}) \subset \mathrm{Int}_P(P_n) \setminus \mathrm{Cl}_P(U_{n-2}).$$

In addition, $F(x_0) \in |L| = |L_n|$. As $\{\mathrm{St}(v,L_n) : v \text{ is a vertex of } L_n\}$ is an open covering of $|L|$, there exists a vertex v of L_n such that $F(x_0) \in \mathrm{St}(v,L_n)$, so $x_0 \in V_{n,v}$, as claimed.

By Theorem 3.4 there exists a subdivision K' of K such that the collection of closed stars $\overline{\mathrm{St}}(u,K')$, where u ranges over the vertices of K', refines $\mathcal{A}$. Observe that K' induces subdivisions K'_n of K_n for each $n\in\mathbb{N}^*$.

Step III. Construction of the weakly simplicial map. For each $n\in\mathbb{N}$, let us consider the finite subcomplex T_n of K defined by

$$T_n := \big\{\sigma\in K'_n : \sigma\subset P_n\setminus \mathrm{Int}_P(P_{n-1})\big\}.$$

We claim: $|T_n| = P_n\setminus\mathrm{Int}_P(P_{n-1})$ *and consequently* $|T_{n-1}|\cap|T_n| = P_{n-1}\setminus \mathrm{Int}_P(P_{n-1})$.

Let $x\in P_n\setminus\mathrm{Int}_P(P_{n-1})$, let σ be the carrier of x in K'_n and let τ be the carrier of x in K_n. It holds $\sigma\subset\tau$. It is enough to check: $\tau\subset P_n\setminus\mathrm{Int}_P(P_{n-1})$. As $P_n = |K_n|$, we have $\tau\in K_n$, so $\tau\subset P_n$. As $\mathrm{Int}_P(P_{n-1})$ is the union of the interiors of some of the simplices of K_{n-1}, either $\tau^0\subset\mathrm{Int}_P(P_{n-1})$ or $\tau\cap\mathrm{Int}_P(P_{n-1})=\varnothing$. As $x\in\tau^0\setminus\mathrm{Int}_P(P_{n-1})$, we conclude $\tau\cap\mathrm{Int}_P(P_{n-1})=\varnothing$ and the claim follows.

We construct inductively weakly simplicial maps $F_n^*:|T_n|\to|L_n|$ satisfying: $F_n^*|_{|T_n|\cap|T_{n-1}|} = F_{n-1}^*|_{|T_n|\cap|T_{n-1}|}$ *and for each vertex u of T_n it holds:*

- *If* $u\in\mathrm{Int}_P(P_n)\setminus\mathrm{Int}_P(P_{n-1})$, we have two possibilities: *either $F_n^*(u)$ is a vertex of L_n such that $F(\overline{\mathrm{St}}(u,K'))\subset\mathrm{St}(F_n^*(u),L_n)$ or it is a vertex of L_{n+1} such that $F(\overline{\mathrm{St}}(u,K'))\subset\mathrm{St}(F_n^*(u),L_{n+1})$.*
- *If* $u\in P_n\setminus\mathrm{Int}_P(P_n)$, we have only one possibility: *$F_n^*(u)$ is a vertex of L_{n+1} such that $F(\overline{\mathrm{St}}(u,K'))\subset\mathrm{St}(F_n^*(u),L_{n+1})$.*

Fix a vertex u of T_n and consider the following two cases:

Case 1. If $u\in P_n\setminus\mathrm{Int}_P(P_n)$, there exists a vertex v of L_{n+1} such that $F(\overline{\mathrm{St}}(u,K'))\subset\mathrm{St}(v,L_{n+1})$.

Case 2. If $u\in\mathrm{Int}_P(P_n)\setminus\mathrm{Int}_P(P_{n-1})$, then u is a point of $\mathrm{Int}_P(P_n)\setminus \mathrm{Cl}_P(U_{n-2})$. In addition, u can also belong to $\mathrm{Int}_P(P_{n+1})\setminus\mathrm{Cl}_P(U_{n-1})$ because $\mathrm{Int}_P(P_n)\setminus\mathrm{Cl}_P(U_{n-1})\neq\emptyset$. In any case, there exist:

- a vertex $v\in L_n$ such that $F(\overline{\mathrm{St}}(u,K'))\subset\mathrm{St}(v,L_n)$ and/or
- a vertex $v'\in L_{n+1}$ such that $F(\overline{\mathrm{St}}(u,K'))\subset\mathrm{St}(v',L_{n+1})$.

Consequently, with this procedure we cannot construct a priori a simplicial map, because the vertices v,v' could not belong to the same iterated barycentric subdivision of L. This is why we construct inductively a weakly simplicial map.

As $|T_0| = P_0=\varnothing$, in the first induction step we have nothing to do. Let $n\geq 1$ and assume we have already constructed the map F_{n-1}^* satisfying the required conditions. We construct next the map F_n^* and to that end we define first F_n^* on the vertices of T_n.

Fix u a vertex of T_n and suppose first $u\in P_{n-1}\setminus\mathrm{Int}_P(P_{n-1})$. As

$$P_{n-1}\setminus\mathrm{Int}_P(P_{n-1}) = |T_{n-1}|\cap|T_n|,$$

u is a vertex of the simplicial subcomplex $T_{n-1} \cap T_n$. We define $F_n^*(u) := F_{n-1}^*(u)$. By induction hypothesis $F_n^*(u) = F_{n-1}^*(u)$ is a vertex of L_n such that $F(\overline{\mathrm{St}}(u, K')) \subset \mathrm{St}(F_n^*(u), L_n)$.

Suppose next $u \in \mathrm{Int}_P(P_n) \setminus P_{n-1}$. As $\mathrm{Int}_P(P_n) \setminus P_{n-1} \subset \mathrm{Int}_P(P_n) \setminus \mathrm{Int}_P(P_{n-1})$, there exists a vertex $v \in L_n$ such that $F(\overline{\mathrm{St}}(u, K')) \subset \mathrm{St}(v, L_n)$ and/or there exists a vertex $v' \in L_{n+1}$ such that $F(\overline{\mathrm{St}}(u, K')) \subset \mathrm{St}(v', L_{n+1})$ (as pointed out above). Choose one of the mentioned vertices v or v' and define either $F_n^*(u) := v$ or $F_n^*(u) := v'$.

Finally, if $u \in P_n \setminus \mathrm{Int}_P(P_n)$, we choose a vertex $v \in L_{n+1}$ such that $F(\overline{\mathrm{St}}(u, K')) \subset \mathrm{St}(v, L_{n+1})$ and define $F_n^*(u) = v$.

Pick a simplex σ of T_n. We claim: *if σ has vertices $u_1, \ldots, u_r$, there exists a simplex $\xi \in L_n$ such that the points $v_1 := F_n^*(u_1), \ldots, v_r := F_n^*(u_r)$ belong to ξ.*

If v is a vertex of L_n, then $\mathrm{St}(v, L_{n+1}) \subset \mathrm{St}(v, L_n)$. After rearranging the indices if necessary, we assume that for some $s \in \{1, \ldots, r\}$ it holds:

- $\{v_\ell\} \in L_n$ and $F(\overline{\mathrm{St}}(u_\ell, K')) \subset \mathrm{St}(v_\ell, L_n)$ if $\ell \in \{1, \ldots, s\}$,
- $\{v_\ell\} \in L_{n+1} \setminus L_n$ and $F(\overline{\mathrm{St}}(u_\ell, K')) \subset \mathrm{St}(v_\ell, L_{n+1})$ if $\ell \in \{s+1, \ldots, r\}$,

where the latter case is omitted if $s = r$.

Pick a point $x \in \sigma^0$ and let ξ be the carrier of $F(x)$ in L_n. As $x \in \bigcap_{\ell=1}^r \overline{\mathrm{St}}(u_\ell, K')$, it holds

$$F(x) \in \bigcap_{\ell=1}^{r} F(\overline{\mathrm{St}}(u_\ell, K')) \subset \bigcap_{\ell=1}^{s} \mathrm{St}(v_\ell, L_n) \cap \bigcap_{\ell=s+1}^{r} \mathrm{St}(v_\ell, L_{n+1}).$$

Thus, $v_1, \ldots, v_s$ are vertices of ξ and $v_{s+1}, \ldots, v_r$ are vertices of the iterated barycentric subdivision $\mathrm{sd}^{\ell_{n+1}-\ell_n}(\widehat{\xi})$ of the simplicial complex $\widehat{\xi}$ constituted by the simplex ξ and all its faces. Consequently, $v_1, \ldots, v_r \in \xi$, as claimed.

We keep the notations already introduced and define $F_n^* : |T_n| \to |L_n|$ (simplex by simplex) as one can expect: *Let $\lambda_1, \ldots, \lambda_r > 0$ be such that $x = \sum_{i=1}^r \lambda_i u_i \in \sigma^0$ (where $u_1, \ldots, u_r$ are the vertices of σ) and $\sum_{i=1}^r \lambda_i = 1$. Then*

$$F_n^*(x) := \sum_{i=1}^{r} \lambda_i F_n^*(u_i) \in \xi.$$

Thus, F_n^* transforms (affinely) each simplex of T_n onto a convex polyhedron contained in a simplex of L_n (if $s < r$ we cannot assure that $F_n^*(\sigma)$ is a simplex because $v_{s+1}, \ldots, v_r$ are not vertices of ξ). In addition, *if τ is the carrier of $F(x)$ in L, then* $F_n^*(x) \in \xi \subset \tau$. Define

$$F^* : |K| \to |L|, \quad x \mapsto F_n^*(x) \quad \text{if } x \in T_n.$$

The previous map is well defined, continuous and weakly simplicial because $F_n^*|_{|T_n| \cap |T_{n-1}|} = F_{n-1}^*|_{|T_n| \cap |T_{n-1}|}$ and each F_n^* is continuous and weakly simplicial. By construction $F^*(x)$ belongs to the carrier τ of $F(x)$ in L for each $x \in |K|$.

Step IV. Approximation. Pick $x \in |T_n| = P_n \setminus \operatorname{Int}_P(P_{n-1})$. Let σ be the carrier of x in K' (or equivalently in K'_n) and let ξ be the carrier of $F(x)$ in L_n. The intersection $\xi \cap F(P_n \setminus \operatorname{Int}_P(P_{n-1}))$ is non-empty because it contains $F(x)$. By (3.1) the diameter of ξ is strictly smaller than ϵ_n. As $F^*(x) \in \xi$, we have $\|F^*(x) - F(x)\|_q < \epsilon_n \leq \varepsilon(x)$, as required. □

Remarks 3.8. (i) *If in the statement of Theorem 3.7 H is a subcomplex of K such that $F(|H|)$ is contained in the set of vertices of L, then $F^*|_{|H|} = F|_{|H|}$.*

Let C be a connected component of $|H|$ and let v be the vertex of L such that $F(C) = \{v\}$. If $x \in C$, then the carrier of $F(x)$ is v, so the last assertion in Theorem 3.7 implies $F^*(x) = v = F(x)$. Consequently $F^*|_{|H|} = F|_{|H|}$, as claimed.

(ii) *If F is proper in the statement of Theorem* 3.7, *then it is well known that F^* can be chosen simplicial.*

This result can be proven by combining the version of simplicial approximation theorem presented in [B, Ch. 5, p. 223] with the following elementary fact that follows from Lemma 2.1: *if $F : |K| \to |L|$ is a proper continuous map and $\varepsilon : |K| \to \mathbb{R}$ is a strictly positive function, then there exists a strictly positive function $\delta : |L| \to \mathbb{R}$ such that $\delta(F(x)) < \varepsilon(x)$ for each $x \in |K|$.*

3.B. The "shrink-widen" covering and approximation technique. Let σ be a simplex of $\mathbb{R}^p$, let $\operatorname{Bd}(\sigma)$ be the boundary of σ and let σ^0 be the interior of σ. Recall that $\operatorname{Bd}(\sigma)$ is the union of proper faces of σ and σ^0 is the open simplex of $\mathbb{R}^p$ such that $\sigma^0 = \sigma \setminus \operatorname{Bd}(\sigma)$. Let b_σ be the barycenter of σ. Given $\varepsilon \in (0,1)$ denote $h_\varepsilon : \mathbb{R}^p \to \mathbb{R}^p$, $x \mapsto b_\sigma + (1-\varepsilon)(x - b_\sigma)$ the homothety of $\mathbb{R}^p$ of center b_σ and ratio $1-\varepsilon$ and define the $(1-\varepsilon)$-*shrinking* σ^0_ε *of* σ^0 by $\sigma^0_\varepsilon := h_\varepsilon(\sigma^0)$. Note that $\operatorname{Cl}_{\mathbb{R}^p}(\sigma^0_\varepsilon) = h_\varepsilon(\sigma) \subset \sigma^0$ for each $\varepsilon \in (0,1)$ and σ^0_ε tends to σ^0 when $\varepsilon \to 0$. In addition, $\sigma^0 = \bigcup_{\varepsilon \in (0,1)} \sigma^0_\varepsilon$ and $\sigma^0_{\varepsilon_2} \subset \sigma^0_{\varepsilon_1}$ if $0 < \varepsilon_1 \leq \varepsilon_2 < 1$.

We fix the following notations for the rest of the subsection. Let $r \in \mathbb{N}^* \cup \{\infty\}$ and let $X \subset \mathbb{R}^m$ be a locally compact set. Suppose X is $\mathscr{C}^r$ *triangulable on open simplices*, that is, there exists a locally finite simplicial complex K of some $\mathbb{R}^p$ and a homeomorphism $\Phi : |K| \to X$ such that: *the set $\Phi(\sigma^0)$ is a $\mathscr{C}^r$ submanifold of $\mathbb{R}^m$ and the restriction $\Phi|_{\sigma^0} : \sigma^0 \to \Phi(\sigma^0)$ is a $\mathscr{C}^r$ diffeomorphism for each open simplex σ^0 of K*. Define $\mathcal{K}^0 := \{\Phi(\sigma^0)\}_{\sigma \in K}$ and $\mathcal{K} := \{\Phi(\sigma)\}_{\sigma \in K}$.

To lighten the notation the elements of $\mathcal{K}$ will be denoted with the letters $\mathsf{s}, \mathsf{t}, \dots$ while those of $\mathcal{K}^0$ with the letters $\mathsf{s}^0, \mathsf{t}^0, \dots$ in such a way that $\operatorname{Cl}_{\mathbb{R}^m}(\mathsf{s}^0) = \mathsf{s}$. In other words, if $\mathsf{s} = \Phi(\sigma)$, then $\mathsf{s}^0 = \Phi(\sigma^0)$. Moreover, we indicate s^0_ε the $(1-\varepsilon)$-shrinking of $\mathsf{s}^0 = \Phi(\sigma^0)$ corresponding to σ^0_ε via Φ, that is, $\mathsf{s}^0_\varepsilon := \Phi(\sigma^0_\varepsilon)$.

Consider a $\mathscr{C}^r$ tubular neighborhood $\rho_{\mathsf{s}^0} : T_{\mathsf{s}^0} \to \mathsf{s}^0$ of s^0 in $\mathbb{R}^m$ and for each $\eta > 0$ the open subset $T_{\mathsf{s}^0,\eta} := \{x \in T_{\mathsf{s}^0} : \|x - \rho_{\mathsf{s}^0}(x)\|_m < \eta\}$ of $\mathbb{R}^m$. We write $\mathsf{s}^0_{\varepsilon,\eta}$ to denote the η-*widening of* s^0_ε *with respect to* ρ_{s^0}, which is the open neighborhood $\mathsf{s}^0_{\varepsilon,\eta} := (\rho_{\mathsf{s}^0})^{-1}(\mathsf{s}^0_\varepsilon) \cap T_{\mathsf{s}^0,\eta}$ of s^0_ε in $\mathbb{R}^m$. If C is a closed subset of $\mathbb{R}^m$ such that $C \cap \operatorname{Cl}_{\mathbb{R}^m}(\mathsf{s}^0_\varepsilon) = \varnothing$, there exists $\eta > 0$ such that $C \cap \operatorname{Cl}_{\mathbb{R}^m}(\mathsf{s}^0_{\varepsilon,\eta}) = \varnothing$ (recall that

$\mathrm{Cl}_{\mathbb{R}^m}(\mathsf{s}^0_\varepsilon)$ is compact). Denote $\rho_{\mathsf{s}^0,\varepsilon,\eta} := \rho_{\mathsf{s}^0}|_{\mathsf{s}^0_{\varepsilon,\eta}} : \mathsf{s}^0_{\varepsilon,\eta} \cap X \to \mathsf{s}^0_\varepsilon$ the $\mathscr{C}^r$ retraction obtained restricting ρ_{s^0} from $\mathsf{s}^0_{\varepsilon,\eta} \cap X$ to s^0_ε.

LEMMA 3.9. *Fix a strictly positive function $\eta : \mathcal{K}^0 \to \mathbb{R}^+$. Then for each $\mathsf{s}^0 \in \mathcal{K}^0$ there exist a non-empty open subset V_{s^0} of s^0 (a "shrinking" of s^0), an open neighborhood U_{s^0} of V_{s^0} in X (a "widening" of V_{s^0}) satisfying $V_{\mathsf{s}^0} = U_{\mathsf{s}^0} \cap \mathsf{s}^0$ and a $\mathscr{C}^r$ retraction $r_{\mathsf{s}^0} : U_{\mathsf{s}^0} \to V_{\mathsf{s}^0}$ such that:*

(i) *$\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0 \in \mathcal{K}^0}$ is a locally finite open covering of X.*

(ii) *$\mathrm{Cl}_{\mathbb{R}^m}(U_{\mathsf{s}^0}) \cap \mathsf{t} = \varnothing$ for each pair $(\mathsf{s}^0, \mathsf{t}) \in \mathcal{K}^0 \times \mathcal{K}$ satisfying $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$.*

(iii) *$\sup_{x \in U_{\mathsf{s}^0}} \{\|x - r_{\mathsf{s}^0}(x)\|_m\} < \eta(\mathsf{s}^0)$ for each $\mathsf{s}^0 \in \mathcal{K}^0$.*

Proof. Define $d := \max\{\dim(\mathsf{s}^0) : \mathsf{s}^0 \in \mathcal{K}^0\} \le m$, where $\dim(\mathsf{s}^0)$ is the dimension of s^0 as a $\mathscr{C}^r$ submanifold of $\mathbb{R}^m$. Of course d coincides with the dimension of the semialgebraic set $|K|$, which is equal to $\max\{\dim(\sigma^0) : \sigma \in K\}$. Let $\mathcal{K}^0_e := \{\mathsf{s}^0 \in \mathcal{K}^0 : \dim(\mathsf{s}^0) \le e\}$ for $e \in \{0, 1, \dots, d\}$. Let us prove by induction on $e \in \{0, 1, \dots, d\}$ that: *For each $\mathsf{s}^0 \in \mathcal{K}^0_e$ there exist an open subset $U^e_{\mathsf{s}^0}$ of X and a $\mathscr{C}^r$ retraction $r^e_{\mathsf{s}^0} : U^e_{\mathsf{s}^0} \to V^e_{\mathsf{s}^0} := U^e_{\mathsf{s}^0} \cap \mathsf{s}^0 \neq \varnothing$ such that:*

(a) $\bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_e} \mathsf{s}^0 \subset \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_e} U^e_{\mathsf{s}^0}$.

(b) $\mathrm{Cl}_{\mathbb{R}^p}(U^e_{\mathsf{s}^0}) \cap \mathsf{t} = \varnothing$ *for each pair* $(\mathsf{s}^0, \mathsf{t}) \in \mathcal{K}^0_e \times \mathcal{K}$ *satisfying* $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$.

(c) $\sup_{x \in U^e_{\mathsf{s}^0}} \{\|x - r^e_{\mathsf{s}^0}(x)\|_m\} < \eta(\mathsf{s}^0)$ *for each* $\mathsf{s}^0 \in \mathcal{K}^0_e$.

Suppose first $e = 0$. Choose $\{v\} \in \mathcal{K}^0_0$. As the family $\mathcal{K}$ is locally finite in X, the union $\bigcup_{\mathsf{t} \in \mathcal{K}, v \notin \mathsf{t}} \mathsf{t}$ is closed in X and it does not contain v. Consequently, there exists $\eta'_v \in (0, \eta(\{v\}))$ such that the open ball $B(v, 2\eta'_v)$ of $\mathbb{R}^m$ of center v and radius $2\eta'_v$ does not meet $\bigcup_{\mathsf{t} \in \mathcal{K}, v \notin \mathsf{t}} \mathsf{t}$. Define $U^0_{\{v\}} := B(v, \eta'_v) \cap X$, $V^0_{\{v\}} := \{v\}$ and $r^0_{\{v\}} : U^0_{\{v\}} \to V^0_{\{v\}}$, $x \mapsto v$ the constant map for each $\{v\} \in \mathcal{K}^0_0$.

Fix $e \in \{0, \dots, d-1\}$ and suppose that the assertion is true for such an e. Pick $\sigma \in K$ of dimension $e+1$ and consider the compact subset

$$C_\sigma := \sigma \setminus \Phi^{-1}\Big(\bigcup_{\tau \in K, \tau \subset \mathrm{Bd}(\sigma)} U^e_{\Phi(\tau^0)}\Big)$$

of $\sigma^0 = \bigcup_{\varepsilon \in (0,1)} \sigma^0_\varepsilon$. Let $\varepsilon(\sigma^0) \in (0,1)$ be such that $C_\sigma \subset \sigma^0_{\varepsilon(\sigma^0)}$. If $\mathsf{s} = \Phi(\sigma)$, define $\varepsilon(\mathsf{s}^0) := \varepsilon(\sigma^0)$ and $\mathsf{s}^0_{\varepsilon(\mathsf{s}^0)} := \Phi(\sigma^0_{\varepsilon(\sigma^0)})$. We have

$$\bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_{e+1}} \mathsf{s}^0 \subset \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_e} U^e_{\mathsf{s}^0} \cup \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e} \mathsf{s}^0_{\varepsilon(\mathsf{s}^0)}.$$

If $(\mathsf{s}^0, \mathsf{t}) \in (\mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e) \times \mathcal{K}$ satisfies $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$, then $\mathrm{Cl}_{\mathbb{R}^m}(\mathsf{s}^0_{\varepsilon(\mathsf{s}^0)}) \cap \mathsf{t} = \varnothing$ because $\mathrm{Cl}_{\mathbb{R}^m}(\mathsf{s}^0_{\varepsilon(\mathsf{s}^0)}) \subset \mathsf{s}^0$. Let $\mathsf{s}^0 \in \mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e$. As the family $\mathcal{K}$ is locally finite in X, there exists $\eta'(\mathsf{s}^0) \in (0, \eta(\mathsf{s}^0))$ such that

$$\mathrm{Cl}_{\mathbb{R}^m}\big(\mathsf{s}^0_{\varepsilon(\mathsf{s}^0), \eta'(\mathsf{s}^0)} \cap X\big) \cap \mathsf{t} = \varnothing$$

for each pair $(\mathsf{s}^0,\mathsf{t}) \in (\mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e) \times \mathcal{K}$ satisfying $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$. For each $\mathsf{s}^0 \in \mathcal{K}^0_{e+1}$ define:

- $V^{e+1}_{\mathsf{s}^0} := V^e_{\mathsf{s}^0}$, $U^{e+1}_{\mathsf{s}^0} := U^e_{\mathsf{s}^0}$ and $r^{e+1}_{\mathsf{s}^0} := r^e_{\mathsf{s}^0}$ if $\mathsf{s}^0 \in \mathcal{K}^0_e$ and
- $V^{e+1}_{\mathsf{s}^0} := \mathsf{s}^0_{\varepsilon(\mathsf{s}^0)}$, $U^{e+1}_{\mathsf{s}^0} := \mathsf{s}^0_{\varepsilon(\mathsf{s}^0),\eta'(\mathsf{s}^0)} \cap X$ and $r^{e+1}_{\mathsf{s}^0} := \rho_{\mathsf{s}^0,\varepsilon(\mathsf{s}^0),\eta'(\mathsf{s}^0)}$ if $\mathsf{s}^0 \in \mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e$.

The open sets $U^{e+1}_{\mathsf{s}^0}$, the non-empty sets $V^{e+1}_{\mathsf{s}^0}$ and the retractions $r^{e+1}_{\mathsf{s}^0} : U^{e+1}_{\mathsf{s}^0} \to V^{e+1}_{\mathsf{s}^0}$ for $\mathsf{s}^0 \in \mathcal{K}^0_{e+1}$ satisfy conditions (a) to (c), as required.

Define the open subsets $U_{\mathsf{s}^0} := U^d_{\mathsf{s}^0}$ of X and the $\mathscr{C}^r$ retractions $r_{\mathsf{s}^0} := r^d_{\mathsf{s}^0}$ for each $\mathsf{s}^0 \in \mathcal{K}^0_d = \mathcal{K}^0$. Evidently properties (ii) and (iii) hold and the family $\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0 \in \mathcal{K}^0}$ is a covering of X. It remains to show that such a family is locally finite in X. Let $x \in X$ and let u^0 be the unique element of $\mathcal{K}^0$ such that $x \in \mathsf{u}^0$. For each $\mathsf{t} \in \mathcal{K}$ and each $\mathsf{s}^0 \in \mathcal{K}^0$ define the finite set $I_\mathsf{t} := \{\mathsf{s}^0 \in \mathcal{K}^0 : \mathsf{s}^0 \subset \mathsf{t}\}$ and the set $J_{\mathsf{s}^0} := \{\mathsf{t} \in \mathcal{K} : \mathsf{s}^0 \subset \mathsf{t}\}$. As the family $\mathcal{K}$ is locally finite in X, each set J_{s^0} is finite as well. If $\mathsf{t} \in \mathcal{K}$ and $\mathsf{s}^0 \in \mathcal{K}^0$ satisfies $U_{\mathsf{s}^0} \cap \mathsf{t} \neq \varnothing$, then $\mathsf{t} \in J_{\mathsf{s}^0}$ by property (ii). In particular $U_{\mathsf{s}_0} \subset \bigcup_{\mathsf{t} \in J_{\mathsf{s}^0}} \mathsf{t}$.

Define the finite set $M_{\mathsf{u}^0} := \bigcup_{\mathsf{t} \in J_{\mathsf{u}^0}} I_\mathsf{t}$ and the set $N_{\mathsf{u}^0} := \{\mathsf{s}^0 \in \mathcal{K}^0 : U_{\mathsf{u}^0} \cap U_{\mathsf{s}^0} \neq \varnothing\}$. Let us show: N_{u^0} *is finite* by showing that $N_{\mathsf{u}^0} \subset M_{\mathsf{u}^0}$. This will complete the proof. If $\mathsf{s}^0 \in N_{\mathsf{u}^0}$, then

$$\varnothing \neq U_{\mathsf{u}^0} \cap U_{\mathsf{s}^0} \subset \bigcup_{\mathsf{t} \in J_{\mathsf{s}^0}} \big(U_{\mathsf{u}^0} \cap \mathsf{t}\big).$$

Thus, there exists $\mathsf{t} \in J_{\mathsf{s}^0}$ such that $U_{\mathsf{u}^0} \cap \mathsf{t} \neq \varnothing$, so $\mathsf{s}^0 \in I_\mathsf{t}$ and $\mathsf{t} \in J_{\mathsf{u}^0}$, that is, $\mathsf{s}^0 \in M_{\mathsf{u}^0}$, as required. □

LEMMA 3.10. *Let L be a locally finite simplicial complex of $\mathbb{R}^q$ and let $g \in \mathscr{C}^0(X,|L|)$. Suppose that for each $\mathsf{t} \in \mathcal{K}$ the restriction $g|_{\mathsf{t}^0}$ belongs to $\mathscr{C}^r(\mathsf{t}^0,|L|)$ and there exists $\xi_\mathsf{t} \in L$ such that $g(\mathsf{t}) \subset \xi_\mathsf{t}$. Then for each strictly positive continuous function $\delta : X \to \mathbb{R}^+$, there exists $h \in \mathscr{C}^r(X,|L|)$ with the following properties:*

(i) *For each $\mathsf{t} \in \mathcal{K}$, there exists an open neighborhood W_t of t in X such that $h(W_\mathsf{t}) \subset \xi_\mathsf{t}$.*

(ii) *$\|h(x) - g(x)\|_q < \delta(x)$ for each $x \in X$.*

To prove this lemma and Proposition 4.1 below we need the following basic topological result that we borrow from [ABF, Lem. 2.4].

LEMMA 3.11. *Let T be a paracompact topological space, let $\{T_k\}_{k \in \mathbb{N}}$ be a locally finite family of subsets of T and for each $k \in \mathbb{N}$ let $V_k \subset T$ be an open neighborhood of T_k. Then there exist open neighborhoods $U_k \subset T$ of T_k such that $U_k \subset V_k$ for each $k \in \mathbb{N}$ and the family $\{U_k\}_{k \in \mathbb{N}}$ is locally finite in T.*

Proof. For each $x \in T$ let $B_x \subset T$ be an open neighborhood of x that meets only finitely many T_k. The family $\{B_x\}_{x \in T}$ is an open covering of T. As T is paracompact, there exists a locally finite open covering $\{W_\ell\}_{\ell \in L}$ of T, which is a refinement of $\{B_x\}_{x\in T}$. Observe that each W_ℓ meets only finitely many T_k. For each $k\in\mathbb{N}$ define $U'_k := \bigcup_{W_\ell\cap T_k\neq\varnothing} W_\ell$. Note that $T_k\subset U'_k$ for each $k\in\mathbb{N}$. We claim: *The family* $\{U'_k\}_{k\in\mathbb{N}}$ *is locally finite in* T.

Fix a point $x\in T$ and consider a neighborhood $V_x\subset T$ of x that meets finitely many W_ℓ, say $W_{\ell_1},\dots,W_{\ell_r}$. The union $\bigcup_{j=1}^r W_{\ell_j}$ meets only finitely many T_k, say $T_{k_1},\dots,T_{k_s}$. If $k\notin\{k_1,\dots,k_s\}$, the intersection $U'_k\cap V_x=\varnothing$. To finish it is enough to define $U_k := U'_k\cap V_k$ for each $k\in\mathbb{N}$. □

We are ready to prove Lemma 3.10.

Proof of Lemma 3.10. We will give the proof only in the case X is non-compact (because if X is compact, the proof is similar, but easier). Choose a sequence $\{X_n\}_{n\in\mathbb{N}}$ of compact subsets of X such that for each $n\in\mathbb{N}$:

- X_n is a finite union of elements of $\mathcal{K}$, say $X_n=\bigcup_{\mathsf{t}\in\mathcal{K}_n}\mathsf{t}$ for some finite set $\mathcal{K}_n\subset\mathcal{K}$.
- $X_{n-1}\subsetneq \mathrm{Int}_X(X_n)$, where $X_{-1}:=\varnothing$.
- $\bigcup_{n\in\mathbb{N}} X_n = X$.

Let $\{\delta_n\}_{n\in\mathbb{N}}$ be the decreasing sequence of positive real numbers defined by

$$\delta_n := \min_{x\in X_n}\{\delta(x)\}>0 \quad \text{for each } n\in\mathbb{N}.$$

Write $\varepsilon_{-1}:=1$. As the restriction $g|_{X_n}$ is uniformly continuous, for each $n\in\mathbb{N}$ there exists $\varepsilon_n\in(0,\varepsilon_{n-1})$ such that

$$\left\|g(x')-g(x)\right\|_q<\delta_n \quad \text{for each pair } x',x\in X_n \text{ with } \|x'-x\|_m<\varepsilon_n. \tag{3.2}$$

Fix $\mathsf{s}^0\in\mathcal{K}^0$. We claim: *there exists a unique integer* $n:=n(\mathsf{s}^0)\in\mathbb{N}$ *such that* $\mathsf{s}^0\subset X_n\setminus X_{n-1}$.

Pick $x\in\mathsf{s}^0$ and let $k\in\mathbb{N}$ be such that $x\in X_k=\bigcup_{\mathsf{t}\in\mathcal{K}_k}\mathsf{t}$. Thus, $x\in\mathsf{t}$ for some $\mathsf{t}\in\mathcal{K}_k$, so $x\in\mathsf{s}^0\cap\mathsf{t}$ and $\mathsf{s}^0\subset\mathsf{t}\subset X_k$. This proves that if $\mathsf{s}^0\cap X_k\neq\varnothing$, then $\mathsf{s}^0\subset X_k$. Note that $n=n(\mathsf{s}^0):=\min\{k\in\mathbb{N}:\mathsf{s}^0\subset X_k\}$ is the unique natural number such that $\mathsf{s}^0\subset X_n\setminus X_{n-1}$, as claimed.

Consider the strictly positive function $\eta:\mathcal{K}^0\to\mathbb{R}^+$, $\mathsf{s}^0\mapsto\varepsilon_{n(\mathsf{s}^0)+1}$. By Lemma 3.9 for each $\mathsf{s}^0\in\mathcal{K}^0$ there exist an open subset U_{s^0} of X with $V_{\mathsf{s}^0}:=U_{\mathsf{s}^0}\cap\mathsf{s}^0\neq\varnothing$ and a $\mathscr{C}^r$ retraction $r_{\mathsf{s}^0}:U_{\mathsf{s}^0}\to V_{\mathsf{s}^0}$ such that:

$$\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0\in\mathcal{K}^0} \text{ is a locally finite covering of } X,$$

$$\mathrm{Cl}_{\mathbb{R}^m}(U_{\mathsf{s}^0})\cap\mathsf{t}=\varnothing \quad \text{for each pair } (\mathsf{s}^0,\mathsf{t})\in\mathcal{K}^0\times\mathcal{K} \text{ such that } \mathsf{s}^0\cap\mathsf{t}=\varnothing, \tag{3.3}$$

$$\sup_{x\in U_{\mathsf{s}^0}}\left\{\|x-r_{\mathsf{s}^0}(x)\|_m\right\}<\varepsilon_{n(\mathsf{s}^0)+1} \quad \text{for each } \mathsf{s}^0\in\mathcal{K}^0. \tag{3.4}$$

Let $\{\theta_{\mathsf{s}^0}:X\to[0,1]\}_{\mathsf{s}^0\in\mathcal{K}^0}$ be a $\mathscr{C}^r$ partition of unity subordinated to the locally finite open covering $\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0\in\mathcal{K}^0}$ of X. To prove the existence of such $\mathscr{C}^r$ partition of unity one can proceed as follows. We may assume that X is a closed subset of $\mathbb{R}^m$.

The family $\{\mathrm{Cl}_{\mathbb{R}^m}(U_{\mathsf{s}^0})\}_{\mathsf{s}^0\in\mathcal{K}^0}$ is locally finite in $\mathbb{R}^m$. By Lemma 3.11 there exists a locally finite family $\{\Omega_{\mathsf{s}^0}\}_{\mathsf{s}^0\in\mathcal{K}^0}$ of open subsets of $\mathbb{R}^m$ such that $\mathrm{Cl}_{\mathbb{R}^m}(U_{\mathsf{s}^0})\subset\Omega_{\mathsf{s}^0}$. Let $\Omega'_{\mathsf{s}^0}\subset\Omega_{\mathsf{s}^0}$ be an open subset such that $U_{\mathsf{s}^0}=X\cap\Omega'_{\mathsf{s}^0}$. Let $\{\Theta_0\}\cup\{\Theta_{\mathsf{s}^0}: X\to[0,1]\}_{\mathsf{s}^0\in\mathcal{K}^0}$ be a $\mathscr{C}^r$ partition of unity subordinated to the locally finite open covering $\{\mathbb{R}^m\setminus X\}\cup\{\Omega'_{\mathsf{s}^0}\}_{\mathsf{s}^0\in\mathcal{K}^0}$ of $\mathbb{R}^m$. Now, it is enough to consider $\theta_{\mathsf{s}^0}:=\Theta_{\mathsf{s}^0}|_X$ for each $\mathsf{s}^0\in\mathcal{K}^0$ in order to have the desired $\mathscr{C}^r$ partition of unity subordinated to $\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0\in\mathcal{K}^0}$.

For each $\mathsf{s}^0\in\mathcal{K}^0$ the map

$$g\circ r_{\mathsf{s}^0}:U_{\mathsf{s}^0}\to V_{\mathsf{s}^0}\subset\mathsf{s}^0\subset\mathsf{s}\to\xi_{\mathsf{s}},\quad x\mapsto r_{\mathsf{s}^0}(x)\mapsto g\big(r_{\mathsf{s}^0}(x)\big)$$

is a $\mathscr{C}^r$ map, so also the map $H_{\mathsf{s}^0}:X\to\mathbb{R}^q$ defined by

$$H_{\mathsf{s}^0}(x):=\begin{cases}\theta_{\mathsf{s}^0}(x)\cdot g\big(r_{\mathsf{s}^0}(x)\big) & \text{if } x\in U_{\mathsf{s}^0},\\ 0 & \text{if } x\in X\setminus U_{\mathsf{s}^0},\end{cases}$$

belongs to $\mathscr{C}^r(X,\mathbb{R}^q)$. Consider the $\mathscr{C}^r$ map $H:=\sum_{\mathsf{s}^0\in\mathcal{K}^0}H_{\mathsf{s}^0}:X\to\mathbb{R}^q$.

Fix $\mathsf{t}\in\mathcal{K}$ and define $W_{\mathsf{t}}:=X\setminus\bigcup_{\mathsf{s}^0\in\mathcal{K}^0,\,\mathsf{s}^0\cap\mathsf{t}=\varnothing}\mathrm{Cl}_{\mathbb{R}^m}(U_{\mathsf{s}^0})$. As the family $\mathcal{K}^0$ is locally finite in X, we deduce that W_{t} is by (3.3) an open neighborhood of t in X. We claim: $H(W_{\mathsf{t}})\subset\xi_{\mathsf{t}}$.

Pick $x\in W_{\mathsf{t}}$. If $\mathsf{s}^0\in\mathcal{K}^0$ and $\mathsf{s}^0\cap\mathsf{t}=\varnothing$, then $\theta_{\mathsf{s}^0}(x)=0$ because the support of θ_{s^0} is contained in U_{s^0} and $x\notin\mathrm{Cl}_{\mathbb{R}^m}(U_{\mathsf{s}^0})$. If $\mathsf{s}^0\cap\mathsf{t}\neq\varnothing$, then $\mathsf{s}^0\subset\mathsf{t}$, so we conclude

$$\sum_{\substack{\mathsf{s}^0\in\mathcal{K}^0,\,\mathsf{s}^0\subset\mathsf{t},\\ x\in U_{\mathsf{s}^0}}}\theta_{\mathsf{s}^0}(x)=1 \tag{3.5}$$

and

$$H(x)=\sum_{\substack{\mathsf{s}^0\in\mathcal{K}^0,\,\mathsf{s}^0\subset\mathsf{t},\\ x\in U_{\mathsf{s}^0}}}\theta_{\mathsf{s}^0}(x)g\big(r_{\mathsf{s}^0}(x)\big). \tag{3.6}$$

If $\mathsf{s}^0\in\mathcal{K}^0$ satisfies $\mathsf{s}^0\subset\mathsf{t}$ and $x\in U_{\mathsf{s}^0}$, then $r_{\mathsf{s}^0}(x)\in V_{\mathsf{s}^0}\subset\mathsf{s}^0$, so $g(r_{\mathsf{s}^0}(x))\in\xi_{\mathsf{t}}$. As ξ_{t} is a convex subset of $\mathbb{R}^q$ and each $g(r_{\mathsf{s}^0}(x))\in\xi_{\mathsf{t}}$ if $\mathsf{s}^0\subset\mathsf{t}$ and $x\in U_{\mathsf{s}^0}$, we conclude by means of (3.5) and (3.6) that $H(x)\in\xi_{\mathsf{t}}$. Consequently, $H(W_{\mathsf{t}})\subset\xi_{\mathsf{t}}$, as claimed.

As $X=\bigcup_{\mathsf{t}\in\mathcal{K}}\mathsf{t}=\bigcup_{\mathsf{t}\in\mathcal{K}}W_{\mathsf{t}}$, we deduce $H(X)$ is contained in $|L|$ and $h:X\to|L|$, $x\mapsto H(x)$ is a $\mathscr{C}^r$ map that satisfies property (i).

It remains to prove (ii). Fix $x\in X_n\setminus X_{n-1}$ for some $n\in\mathbb{N}$. Denote u the unique element of $\mathcal{K}$ such that $x\in\mathsf{u}^0$. As $\mathsf{u}^0\cap X_n\neq\varnothing$, we have $\mathsf{u}^0\subset X_n$. Observe that $\mathsf{u}^0\cap X_{n-1}=\varnothing$, because otherwise $x\in\mathsf{u}^0\subset X_{n-1}$, which is a contradiction. Thus, $\mathsf{u}^0\subset X_n\setminus X_{n-1}$, so $n(\mathsf{u}^0)=n$. If $\mathsf{s}^0\in\mathcal{K}^0$ satisfies $x\in U_{\mathsf{s}^0}$, then $U_{\mathsf{s}^0}\cap\mathsf{u}\neq\varnothing$

and by (3.3) we have

$$\begin{aligned}\mathsf{s}^0 \subset \mathsf{u} \subset \mathrm{Cl}_X(X_n \setminus X_{n-1}) &\subset X_n \setminus \mathrm{Int}_X(X_{n-1})\\ &\subset X_n \setminus X_{n-2} = (X_n \setminus X_{n-1}) \sqcup (X_{n-1} \setminus X_{n-2}),\end{aligned}$$

where $X_{-2} := \varnothing$. Thus, $n(\mathsf{s}^0) \in \{n-1, n\}$. Consequently, $r_{\mathsf{s}^0}(x) \in \mathsf{s}^0 \subset \mathsf{u} \subset X_n$ and by (3.4) we have $\|x - r_{\mathsf{s}^0}(x)\|_m < \varepsilon_{n(\mathsf{s}^0)+1} \leq \varepsilon_n$. Now inequality (3.2) implies that

$$\begin{aligned}\big\|h(x) - g(x)\big\|_q &= \Bigg\| \sum_{\mathsf{s}^0 \in \mathcal{K}^0,\, x \in U_{\mathsf{s}^0}} \theta_{\mathsf{s}^0}(x)\big(g\big(r_{\mathsf{s}^0}(x)\big) - g(x)\big) \Bigg\|_q\\ &\leq \sum_{\mathsf{s}^0 \in \mathcal{K}^0,\, x \in U_{\mathsf{s}^0}} \theta_{\mathsf{s}^0}(x) \big\|g\big(r_{\mathsf{s}^0}(x)\big) - g(x)\big\|_q < \delta_n\\ &\leq \delta(x),\end{aligned}$$

as required. □

Remark 3.12. We keep the notations of the preceding proof. If there exist $\mathsf{t} \in \mathcal{K}$ and $w \in |L|$ such that g takes the constant value w on t, then by (3.5) and (3.6) the $\mathscr{C}^r$ map $h : X \to |L|$ is constant on the open neighborhood $W_{\mathsf{t}} \subset X$ of t and takes the constant value w. In particular, this is always true if $\mathsf{t} = \{v\}$ for any vertex v of K. As a consequence, if X has at least one accumulation point, then h is not injective.

3.C. Proof of Theorem 1.6. Let X be a locally compact subset of some $\mathbb{R}^m$. We assume X is non-compact. If X is compact the proof is similar, but easier. As $Y \subset \mathbb{R}^n$ is a weakly $\mathscr{C}^r$ triangulable set, there exist a locally finite simplicial complex L of some $\mathbb{R}^q$ and a homeomorphism $\Psi : |L| \to Y$ such that $\Psi|_\xi \in \mathscr{C}^r(\xi, Y)$ for each $\xi \in L$. In particular, Y is locally compact in $\mathbb{R}^n$. Consider a continuous map $f : X \to Y$ and a strictly positive continuous function $\varepsilon : X \to \mathbb{R}^+$. We will show: *There exists* $\mathcal{H} \in \mathscr{C}^r(X, Y)$ *such that* $\|\mathcal{H}(x) - f(x)\|_n < \varepsilon(x)$ *for each* $x \in X$.

By the first part of Corollary 2.2, we can assume X is closed in $\mathbb{R}^m$ and Y is closed in $\mathbb{R}^n$.

The proof is conducted in several steps:

Step I. Initial preparation. As X is closed in $\mathbb{R}^m$, Tietze's extension theorem guarantees the existence of a strictly positive continuous function $E : \mathbb{R}^m \to \mathbb{R}^+$ and a continuous map $\widehat{f} : \mathbb{R}^m \to \mathbb{R}^n$ such that $E(x) = \varepsilon(x)$ and $\widehat{f}(x) = f(x)$ for each $x \in X$. By [Ha, Cor. 3.5] Y is an absolute neighborhood retract. Consequently, as Y is closed in $\mathbb{R}^n$, there exists an open neighborhood $W \subset \mathbb{R}^n$ of Y and a continuous retraction $\rho : W \to Y$. Consider the open neighborhood $U := (\widehat{f})^{-1}(W)$

of X in $\mathbb{R}^m$ and the continuous extension $\widetilde{f}: U \to Y$, $x \mapsto \rho(\widehat{f}(x))$ of f. Renaming U as X, $E|_U$ as ε and $\widetilde{f}$ as f, we can assume that X is an open subset of $\mathbb{R}^m$, so in particular X is a $\mathscr{C}^\infty$ manifold.

By the Cairns-Whitehead triangulation theorem X is "$\mathscr{C}^\infty$ triangulable on open simplices", that is, there exist a locally finite simplicial complex K of some $\mathbb{R}^p$ and a homeomorphism $\Phi : |K| \to X$ such that $\Phi(\sigma^0)$ is a $\mathscr{C}^\infty$ submanifold of $\mathbb{R}^m$ and the restriction $\Phi|_{\sigma^0} : \sigma^0 \to \Phi(\sigma^0)$ is a $\mathscr{C}^\infty$ diffeomorphism for each open simplex σ^0 of K (see [Hu, Lemma 3.5 & p. 82]). Set $P := |K|$, $\mathcal{K} := \{\Phi(\sigma)\}_{\sigma\in K}$ and $\mathcal{K}^0 := \{\Phi(\sigma^0)\}_{\sigma\in K}$. Define $Q := |L| \subset \mathbb{R}^q$.

Step II. Reduction to the weakly simplicial case. Choose a sequence $\{X_k\}_{k\in\mathbb{N}}$ of compact subsets of X such that $\bigcup_{k\in\mathbb{N}} X_k = X$ and $X_{k-1} \subsetneq \mathrm{Int}_X(X_k)$ for each $k \in \mathbb{N}$, where $X_{-1} := \varnothing$. Note that the family $\{X_k \setminus X_{k-1}\}_{k\in\mathbb{N}}$ is locally finite in X.

Fix $k \in \mathbb{N}$ and consider the compact subsets $P_k := \Phi^{-1}(X_k)$ of P and $Q_k := \Psi^{-1}(f(X_k))$ of Q. Define $\epsilon_k := \min_{x\in X_k}\{\varepsilon(x)\} > 0$ and $\mu_k := \min\{1, \mathrm{dist}_{\mathbb{R}^q}(Q_k, \mathrm{Cl}_{\mathbb{R}^q}(Q) \setminus Q)\} > 0$, where $\mathrm{dist}_{\mathbb{R}^q}(Q_k, \varnothing) := +\infty$. Consider the compact subset V_k of Q defined by

$$V_k := \big\{z \in Q : \mathrm{dist}_{\mathbb{R}^q}(z, Q_k) \le \mu_k/2\big\}. \tag{3.7}$$

By the uniform continuity of Ψ on V_k there exists (for each $k \in \mathbb{N}$) $\delta_k > 0$ such that

$$\big\|\Psi(z') - \Psi(z)\big\|_n < \epsilon_k \quad \text{for each pair } z', z \in V_k \text{ with } \|z' - z\|_q < \delta_k. \tag{3.8}$$

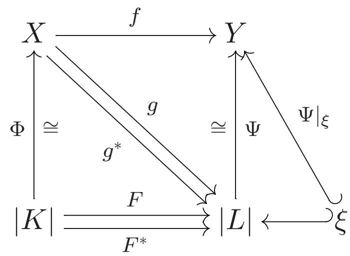
Consider the $\mathscr{C}^0$ map $F := \Psi^{-1} \circ f \circ \Phi : |K| = P \to Q = |L|$. Applying Theorem 3.7 to F we obtain, after replacing K by one of its subdivisions, that there exists a weakly simplicial map $F^* : |K| \to |L|$ such that

$$\begin{aligned}&\big\|F^*(w) - F(w)\big\|_q \\ &\quad < \min\{\mu_k/4, \delta_k/2\} \quad \text{for each } k \in \mathbb{N} \text{ and each } w \in P_k \setminus P_{k-1},\end{aligned} \tag{3.9}$$

where $P_{-1} := \varnothing$. Define the continuous maps $g := \Psi^{-1} \circ f = F \circ \Phi^{-1} : X \to |L|$ and $g^* := F^* \circ \Phi^{-1} : X \to |L|$. For each $\mathtt{t} \in \mathcal{K}$ the restriction $\Phi^{-1}|_{\mathtt{t}^0} : \mathtt{t}^0 \to \Phi^{-1}(\mathtt{t}^0)$ is a $\mathscr{C}^\infty$ diffeomorphism. Thus, as $F^*|_{\Phi^{-1}(\mathtt{t}^0)}$ is an affine map, $g^*|_{\mathtt{t}^0} \in \mathscr{C}^\infty(\mathtt{t}^0, |L|)$. As F^* is weakly simplicial and $\Phi^{-1}(\mathtt{t}) \in K$, there exists $\xi_{\mathtt{t}} \in L$ such that $g^*(\mathtt{t}) = F^*(\Phi^{-1}(\mathtt{t})) \subset \xi_{\mathtt{t}}$. By (3.9) we have:

$$\begin{aligned}&\big\|g^*(x) - g(x)\big\|_q \\ &\quad < \min\{\mu_k/4, \delta_k/2\} \quad \text{for each } k \in \mathbb{N} \text{ and each } x \in X_k \setminus X_{k-1}.\end{aligned} \tag{3.10}$$

The following commutative diagram summarizes the current situation.



Step III. Construction of the approximating map. By Lemma 3.10 there exist $h^*\in\mathscr{C}^\infty(X,|L|)$ and for each $\mathtt{t}\in\mathcal{K}$ an open neighborhood $W_{\mathtt{t}}\subset X$ of $\mathtt{t}$ satisfying:

$$h^*(W_{\mathtt{t}})\subset\xi_{\mathtt{t}}\quad\text{for each } \mathtt{t}\in\mathcal{K}, \tag{3.11}$$

$$\begin{aligned}&\big\|h^*(x)-g^*(x)\big\|_q\\ &\qquad<\min\{\mu_k/4,\delta_k/2\}\quad\text{for each } k\in\mathbb{N} \text{ and each } x\in X_k\setminus X_{k-1}.\end{aligned} \tag{3.12}$$

We define $\mathcal{H}:=\Psi\circ h^*:X\to Y$ and claim: $\mathcal{H}\in\mathscr{C}^r(X,Y)$.

Recall that $\{W_{\mathtt{t}}\}_{\mathtt{t}\in\mathcal{K}}$ is an open covering of X. By (3.11) the restriction $h^*|_{W_{\mathtt{t}}}:W_{\mathtt{t}}\to\xi_{\mathtt{t}}$ is a well-defined $\mathscr{C}^\infty$ map for each $\mathtt{t}\in\mathcal{K}$. In addition, $\mathcal{H}|_{W_{\mathtt{t}}}=\Psi|_{\xi_{\mathtt{t}}}\circ h^*|_{W_{\mathtt{t}}}$. As both $\Psi|_{\xi_{\mathtt{t}}}$ and $h^*|_{W_{\mathtt{t}}}$ are $\mathscr{C}^r$ maps, $\mathcal{H}|_{W_{\mathtt{t}}}$ is also a $\mathscr{C}^r$ map. Consequently, $\mathcal{H}\in\mathscr{C}^r(X,Y)$, as claimed.

Next, by (3.10) and (3.12) we have

$$\begin{aligned}&\big\|h^*(x)-g(x)\big\|_q\\ &\qquad<\min\{\mu_k/2,\delta_k\}\quad\text{for each } k\in\mathbb{N} \text{ and each } x\in X_k\setminus X_{k-1}.\end{aligned} \tag{3.13}$$

Recall that $g(X_k)=\Psi^{-1}(f(X_k))=Q_k$, so by (3.7) and (3.13) we have $h^*(x)\in V_k$ for each $x\in X_k\setminus X_{k-1}$. Thus, by (3.8) and (3.13) we conclude

$$\big\|\mathcal{H}(x)-f(x)\big\|_n=\big\|\Psi\big(h^*(x)\big)-\Psi\big(g(x)\big)\big\|_n<\epsilon_k\le\varepsilon(x),$$

for each $x\in X_k\setminus X_{k-1}$ and each $k\in\mathbb{N}$. Thus, $\|\mathcal{H}(x)-f(x)\|_n<\varepsilon(x)$ for each $x\in X$, as required. □

3.D. Proof of Theorems 1.24. As the pair (X,X') is weakly* $\mathscr{C}^r$ triangulable, there exists a locally finite simplicial complex K, a subcomplex K' of K and a homeomorphism $\Phi:|K|\to X$ such that $\Phi(|K'|)=X'$. We repeat the preceding proof of Theorem 1.6 with the following changes:

- We refine the triangulation $\Psi:|L|\to Y$ in such a way that each connected component of $f(X')$ is a vertex w_k of L. Denote $X'_k:=f^{-1}(\{w_k\})$ and observe that X'_k is a union of some connected components of X'. Let K'_k be the subcomplex of K' such that $\Phi(|K'_k|)=X'_k$.

• Replace Step I with the last two sentences of such step. Namely: "Set $P := |K|$, $\mathcal{K} := \{\Phi(\sigma)\}_{\sigma\in K}$ and $\mathcal{K}^0 := \{\Phi(\sigma^0)\}_{\sigma\in K}$. Define $Q := |L| \subset \mathbb{R}^q$".

• In Step II we apply Remark 3.8(i) to $F := \Psi^{-1}\circ f\circ\Phi$ with $H := K'$. The reader should have in mind that $F|_{|K'_k|}$ is constantly equal to the vertex w_k of L, so F is simplicial on K'_k. Thus, we obtain $g^* : X \to |L|$ such that $g^*(x) = g(x) = w_k$ for each $x \in X'_k$.

Finally, by Remark 3.12 the $\mathscr{C}^r$ map $h^* : X \to |L|$ that approximates g^* is constantly equal to w_k on X'_k. Consequently, the $\mathscr{C}^r$ map $\mathcal{H} : X \to Y$ that approximates f satisfies $\mathcal{H}(x) = \Psi(w_k) = f(x)$ for each $x \in X'_k$, so $\mathcal{H}|_{X'} = f|_{X'}$, as required. □

3.E. Proof of Theorem 1.21. Proceeding as in Step I of the proof of Theorem 1.6 we can assume that X is an open subset of $\mathbb{R}^m$. Thus, there exist a locally finite simplicial complex K of some $\mathbb{R}^p$ and a homeomorphism $\Phi : |K| \to X$ such that the restriction $\Phi|_{\sigma^0} : \sigma^0 \to X$ is a $\mathscr{C}^\infty$ map for each open simplex σ^0 of K. Next, we refine the triangulation Φ in such a way that there exists a subcomplex K' of K satisfying $\Phi(|K'|) = X'$. This proves that (X, X') is a weakly* $\mathscr{C}^\infty$ triangulable pair. Now, we apply Theorem 1.24 to complete the proof. □

3.F. Proof of Corollary 1.18. Let K and $P \subset \mathbb{R}^p$ satisfy the conditions in the statement and let $\varepsilon : P \to \mathbb{R}^+$ be a strictly positive continuous function. Apply Lemma 3.10 to $X := P$, $\Phi := \mathrm{id}_P$, $L := K$, $g := \mathrm{id}_P$ and $\delta := \varepsilon$. We obtain a map $\iota_\varepsilon \in \mathscr{C}^\infty(P,P)$ and for each $\sigma \in K$ an open neighborhood $W_\sigma \subset P$ of σ such that:

- $\iota_\varepsilon(W_\sigma) \subset \sigma$ for each $\sigma \in K$ and
- $\|\iota_\varepsilon(x) - x\|_p < \varepsilon(x)$ for each $x \in X$.

Thus, the net $\{\iota_\varepsilon\}_{\varepsilon\in\mathscr{C}^0(P,\mathbb{R}^+)}$ converges to the identity map in $\mathscr{C}^0(P,P)$. Consequently, by Lemma 2.1 if $f \in \mathscr{C}^0(P,Y)$, the net $\{f\circ\iota_\varepsilon\}_{\varepsilon\in\mathscr{C}^0(P,\mathbb{R}^+)}$ converges to f in $\mathscr{C}^0(P,Y)$. In addition, if $f|_\sigma \in \mathscr{C}^r(\sigma,Y)$ for each $\sigma \in K$, every composition $f\circ\iota_\varepsilon : P \to Y$ is a $\mathscr{C}^r$ map, because so is the restriction $(f\circ\iota_\varepsilon)|_{W_\sigma} = f|_\sigma\circ\iota_\varepsilon|_{W_\sigma}$ for each $\sigma \in K$. □

Remark 3.13. If K is compact, the family of strictly positive constant functions $\epsilon_n := 2^{-n}$ is cofinal in $\mathscr{C}^0(P,\mathbb{R}^+)$ and it is enough to construct (using again Lemma 3.10) for each $n \in \mathbb{N}$ a map $\iota_n \in \mathscr{C}^\infty(P,P)$ and for each $\sigma \in K$ an open neighborhood $W_\sigma \subset P$ of σ such that:

- $\iota_n(W_\sigma) \subset \sigma$ for each $\sigma \in K$ and
- $\|\iota_n(x) - x\|_p < \varepsilon_n(x) := \epsilon_n$ for each $x \in X$.

Once this is done one proceeds as above.

4. Proof of Theorem 1.15. In this section we develop first all the machinery we need to prove Theorem 1.15, which is inspired by some techniques contained in [BR]:

• the construction of $\mathscr{C}^\infty$ weak retractions for an analytic normal-crossings divisor X of a real analytic manifold M (that appears in Section 4.A),

• immersion of C-analytic sets as singular sets of coherent C-analytic sets homeomorphic to Euclidean spaces (that appears in Section 4.B),

and after we approach its proof (see Section 4.C).

A weaker and purely semialgebraic version of the arguments used in this section is contained in our manuscript [FG].

4.A. $\mathscr{C}^\infty$ weak retractions. In this subsection we construct $\mathscr{C}^\infty$ weak retractions $\rho : W \to X$ of open neighborhoods W of an analytic normal-crossings divisor X of a real analytic manifold M (Proposition 4.1). $\mathscr{C}^\infty$ weak retractions $\rho : W \to X$ are $\mathscr{C}^\infty$ maps that are arbitrarily close to the identity id_X on X in the strong $\mathscr{C}^0$ topology. As we have already commented, if X is not a $\mathscr{C}^\infty$ manifold, we cannot expect that ρ is a retraction onto X, that is, there is no hope to have $\rho|_X = \mathrm{id}_X$.

Let $M \subset \mathbb{R}^m$ be a d-dimensional real analytic manifold and let X be a C-analytic subset of M. We say that X is an *analytic normal-crossings divisor of* M if:

• for each point $x \in X$ there exists an open neighborhood $U \subset M$ of x and a real analytic diffeomorphism $\varphi : U \to \mathbb{R}^d$ such that $\varphi(x) = 0$ and $\varphi(X \cap U) = \{x_1 \cdots x_r = 0\}$ for some $r \in \{1, \dots, d\}$ and

• the (C-analytic) irreducible components [WB] of (the C-analytic set) X are non-singular analytic hypersurfaces of M.

In the next result we establish the existence of $\mathscr{C}^\infty$ weak retractions $\rho : W \to X$.

PROPOSITION 4.1. ($\mathscr{C}^\infty$ weak retractions) *Let X be an analytic normal-crossings divisor of a real analytic manifold M and let $\mathcal{U}$ be an open neighborhood of* id_X *in $\mathscr{C}^0(X, X)$. Then there exist an open neighborhood W of X in M and a $\mathscr{C}^\infty$ map $\rho : W \to X$ such that $\rho|_X \in \mathcal{U}$.*

Proof. Assume that M is a real analytic submanifold of some $\mathbb{R}^m$. Choose a strictly positive continuous function $\varepsilon : X \to \mathbb{R}^+$ such that

$$\big\{g \in \mathscr{C}^0(X, X) : \|g(x) - x\|_m < \varepsilon(x)\ \forall x \in X\big\} \subset \mathcal{U}. \tag{4.1}$$

As X is closed in M, we can extend by Tietze's extension theorem ε to a positive continuous function on M that we denote again ε.

Let $\{X_j\}_{j \in J}$ be the family of the irreducible components of X (see [WB]). Such a family is locally finite in M, so J is countable and we assume $J = \mathbb{N}$. If J is finite, the proof is similar but easier.

For each $j \in \mathbb{N}$ denote $\pi_j : \mathscr{E}_j \to X_j$ the normal bundle of X_j in M, where $\mathscr{E}_j \subset X_j \times \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$. Proceeding as the authors do in the proof of [BR, Lem. 2.5] one shows that there exists a $\mathscr{C}^\infty$ tubular neighborhood map

$\phi_j : \mathscr{E}_j \hookrightarrow M$ of X_j compatible with the other X_k in the following sense: *for each* $x \in \mathscr{E}_j$ *and each* $k \in \mathbb{N} \setminus \{j\}$ *the image* $\phi_j(x) \in X_k$ *if and only if* $\phi_j(\pi_j(x)) \in X_k$. Define $\Omega_j := \phi_j(\mathscr{E}_j)$ for each $j \in \mathbb{N}$. By Lemma 3.11 we may assume that the family $\{\Omega_j\}_{j\in\mathbb{N}}$ is locally finite in M.

Fix $j \in \mathbb{N}$ and let $\eta_j : X_j \to \mathbb{R}^+$ be a strictly positive $\mathscr{C}^\infty$ function. Choose a $\mathscr{C}^\infty$ function $f : \mathbb{R} \to [0,1]$ such that $f(t) = 0$ if $|t| \le 1/4$ and $f(t) = 1$ if $|t| \ge 1$. Consider the map

$$h_j : \mathscr{E}_j \to \mathscr{E}_j, \quad (x,w) \mapsto \big(x, f\big(\|w\|_m^2/\eta_j^2(x)\big)w\big)$$

and the composition $\psi_j := \phi_j \circ h_j \circ \phi_j^{-1} : \Omega_j \to \Omega_j$, which is a $\mathscr{C}^\infty$ map that extends by the identity to a $\mathscr{C}^\infty$ map $\Psi_j : M \to M$. Denote

$$W_j := \phi_j\big(\{(x,w) \in \mathscr{E}_j : \|w\|_m < \eta_j(x)\}\big) \subset \Omega_j,$$
$$W_j^* := \phi_j\big(\{(x,w) \in \mathscr{E}_j : \|w\|_m < \eta_j(x)/2\}\big) \subset W_j.$$

We have:

- $\Psi_j(W_j^*) = X_j$.
- $\Psi_j(y) = y$ for each $y \in M \setminus W_j$.
- $\Psi_j(X_k) \subset X_k$ for each k.
- Ψ_j is arbitrarily close to the identity on M if η_j is small enough.

Only the last assertion requires a further comment. As Ψ_j is the identity on $M \setminus W_j$ and ϕ_j is a real analytic embedding (in particular a proper map onto its image), by Lemma 2.1 it is enough to show that h_j can be chosen arbitrarily close to $\mathrm{id}_{\mathscr{E}_j}$. Pick $(x,w) \in \mathscr{E}_j$. We have:

$$h_j(x,w) - (x,w) = \big(0, \big(f\big(\|w\|_m^2/\eta_j^2(x)\big) - 1\big)w\big),$$
$$\big|f\big(\|w\|_m/\eta_j(x)\big) - 1\big| \begin{cases} \le 1 & \text{if } \|w\|_m < \eta_j(x), \\ = 0 & \text{if } \|w\|_m \ge \eta_j(x). \end{cases}$$

Consequently,

$$\big\|h_j(x,w) - (x,w)\big\|_{2m} = \big|f\big(\|w\|_m/\eta_j(x)\big) - 1\big|\|w\|_m < \eta_j(x),$$

so h_j is arbitrarily close to $\mathrm{id}_{\mathscr{E}_j}$ and Ψ_j is arbitrarily close to id_M, provided η_j is small enough. In particular, the restriction $\Psi_j|_X : X \to X$ is arbitrarily close to id_X.

We claim: *If the functions* η_j *are small enough, the countable composition* $\rho : M \to M$, $\rho := \cdots \circ \Psi_j \circ \cdots \circ \Psi_0$ *is a well-defined* $\mathscr{C}^\infty$ *map and the restriction* $\rho|_X : X \to X$ *belongs to* $\mathcal{U}$.

As M is Hausdorff, second countable and locally compact (hence also paracompact) and the family $\{\Omega_j\}_{j\in\mathbb{N}}$ is locally finite in M, there exists an open covering $\{U_\ell\}_{\ell\in\mathbb{N}}$ of M such that each closure $\mathrm{Cl}_M(U_\ell)$ of U_ℓ in M is compact and only meets finitely many Ω_j and the family $\{\mathrm{Cl}_M(U_\ell)\}_{\ell\in\mathbb{N}}$ is locally finite in M. Let $\{V_\ell\}_{\ell\in\mathbb{N}}$ be a shrinking of $\{U_\ell\}_{\ell\in\mathbb{N}}$ that is an open covering of M and satisfies $K_\ell := \mathrm{Cl}_M(V_\ell) \subset U_\ell$ for each $\ell\in\mathbb{N}$. For each $\ell\in\mathbb{N}$, denote $s_\ell\in\mathbb{N}$ the cardinality of the set of all $j\in\mathbb{N}$ such that $\Omega_j\cap\mathrm{Cl}_M(U_\ell)\neq\varnothing$. Note that $\mathrm{dist}_{\mathbb{R}^m}(K_\ell, M\setminus U_\ell)>0$ and pick $\varepsilon_\ell\in\mathbb{R}$ with $0<\varepsilon_\ell<\mathrm{dist}_{\mathbb{R}^m}(K_\ell, M\setminus U_\ell)$. Bearing in mind Remark 2.3, for each $j\in\mathbb{N}$ we choose η_j small enough to have

$$\big\|\Psi_j(x)-x\big\|_m<\frac{\varepsilon_\ell}{s_\ell+1}\quad\text{for each } \ell\in\mathbb{N} \text{ and each } x\in\mathrm{Cl}_M(U_\ell).$$

Fix $\ell\in\mathbb{N}$ with $s_\ell>0$. Write $\{j\in\mathbb{N}:\Omega_j\cap\mathrm{Cl}_M(U_\ell)\neq\varnothing\}=\{j_1,\dots,j_{s_\ell}\}$ and assume $j_1<\dots<j_{s_\ell}$. Let us check: $\|(\Psi_{j_k}\circ\dots\circ\Psi_{j_1})(y)-y\|_m<\frac{k\varepsilon_\ell}{s_\ell+1}$ *for each* $y\in K_\ell$ *and each* $k\in\{1,\dots,s_\ell\}$. *In particular,*

$$(\Psi_{j_k}\circ\dots\circ\Psi_{j_1})(y)\in U_\ell \tag{4.2}$$

for each $k\in\{1,\dots,s_\ell\}$.

We proceed by induction on k. If $k=1$ the result is true by construction. Assume the result true for $k-1$ and let us check that it is also true for k. Pick a point $y\in K_\ell$. As $\|(\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)-y\|_m<\frac{(k-1)\varepsilon_\ell}{s_\ell+1}<\varepsilon_\ell$, we have $(\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)\in U_\ell$, so

$$\big\|\Psi_{j_k}\big((\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)\big)-(\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)\big\|_m<\frac{\varepsilon_\ell}{s_\ell+1}.$$

Thus, we deduce

$$\begin{aligned}&\big\|(\Psi_{j_k}\circ\dots\circ\Psi_{j_1})(y)-y\big\|_m\\&\quad\le\big\|\Psi_{j_k}\big((\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)\big)-(\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)\big\|_m\\&\qquad+\big\|(\Psi_{j_{k-1}}\circ\dots\circ\Psi_{j_1})(y)-y\big\|_m<\frac{\varepsilon_\ell}{s_\ell+1}+\frac{(k-1)\varepsilon_\ell}{s_\ell+1}=\frac{k\varepsilon_\ell}{s_\ell+1}.\end{aligned}$$

By (4.2) and the fact that $\Psi_j|_{M\setminus\Omega_j}=\mathrm{id}_{M\setminus\Omega_j}$, we have

$$(\Psi_{j_s}\circ\dots\circ\Psi_0)(y)=(\Psi_{j_s}\circ\dots\circ\Psi_{j_1})(y)\in U_\ell\quad\text{for each } y\in K_\ell.$$

As $U_\ell\subset\bigcap_{j>j_s}(M\setminus\Omega_j)$, we conclude

$$\rho(y)=(\Psi_{j_s}\circ\dots\circ\Psi_0)(y)=(\Psi_{j_s}\circ\dots\circ\Psi_{j_1})(y).$$

It follows that

$$\|\rho(y)-y\|_m<\varepsilon_\ell\quad\text{for each } \ell\in\mathbb{N} \text{ and each } y\in K_\ell=\mathrm{Cl}_M(V_\ell).$$

In particular, as $\{V_\ell\}_{\ell\in\mathbb{N}}$ is an open covering of M, the composition ρ turns out to be a well-defined $\mathscr{C}^\infty$ map, which is arbitrarily close to id_M in $\mathscr{C}^0(M,M)$ if the values ε_ℓ are chosen small enough. We may assume in addition

$$\|\rho(y)-y\|_m<\varepsilon(y) \quad \text{for each } y\in M. \tag{4.3}$$

We claim: *ρ maps the open neighborhood*

$$W:=\bigcup_{j\in\mathbb{N}}(\Psi_{j-1}\circ\cdots\circ\Psi_0)^{-1}(W_j^*)\subset M$$

of X onto X, where $\Psi_{j-1}\circ\cdots\circ\Psi_0$ denotes id_M if $j=0$.

Indeed, pick $y\in W$ and let $j\in\mathbb{N}$ be such that $y\in(\Psi_{j-1}\circ\cdots\circ\Psi_0)^{-1}(W_j^*)$. Define $z\in W_j^*$ by $z:=(\Psi_{j-1}\circ\cdots\circ\Psi_0)(y)$. We have

$$\rho(y)=\big((\cdots\circ\Psi_{j+1})\circ\Psi_j\circ(\Psi_{j-1}\circ\cdots\circ\Psi_0)\big)(y)=(\cdots\circ\Psi_{j+1})\big(\Psi_j(z)\big).$$

As $\Psi_j(z)\in\Psi_j(W_j^*)=X_j$ and $\Psi_k(X_j)\subset X_j$ for each k, we have $\rho(y)=(\cdots\circ\Psi_{j+1})(\Psi_j(z))\in X_j\subset X$, so we conclude $\rho(W)\subset X$. Thus, the corresponding restriction $\rho:W\to X$ is a well-defined $\mathscr{C}^\infty$ map. By (4.1) and (4.3) the restriction $\rho|_X$ belongs to $\mathcal{U}$, as required. □

4.B. Immersions of C-analytic sets as singular sets. The following result is a C-analytic version of Lemma 2.2 in [BR], which is crucial for the proof of Theorem 1.15. Recall that a C-analytic set $Y\subset\mathbb{R}^n$ is *coherent* if the ideal $\mathcal{J}_y$ (that is, the stalk of the sheaf of ideals $\mathcal{J}:=\mathcal{J}(Y)\mathscr{C}^\omega_{\mathbb{R}^n}$ at y) coincides with the ideal of germs of real analytic functions on $\mathbb{R}^n$ whose zero sets contain the germ Y_y for each $y\in Y$.

LEMMA 4.2. (C-analytic sets as singular sets) *Let Y be a C-analytic subset of $\mathbb{R}^n$. Denote (x,y_1,y_2) the coordinates of $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}=\mathbb{R}^{n+2}$. Then there exists an irreducible coherent C-analytic subset Z of $\mathbb{R}^{n+2}$ such that* $\mathrm{Sing}(Z)=Y\times\{(0,0)\}$ *and the restriction to Z of the projection $\pi:\mathbb{R}^{n+2}\to\mathbb{R}^{n+1}$, $(x,y_1,y_2)\mapsto(x,y_1)$ is a homeomorphism. In addition, the restriction $\pi|_{Z\setminus(Y\times\{(0,0)\})}:Z\setminus(Y\times\{(0,0)\})\to\mathbb{R}^{n+1}\setminus(Y\times\{0\})$ is an analytic diffeomorphism.*

Proof. Let $f\in\mathscr{C}^\omega(\mathbb{R}^n)$ be a global analytic equation of Y and consider the real analytic function $g(x,y_1,y_2):=f(x)^2+y_1^2-y_2^3\in\mathscr{C}^\omega(\mathbb{R}^{n+2})$. Define $Z:=\{g=0\}$. Given $(x,y_1)\in\mathbb{R}^{n+1}$, the formula $y_2=(f(x)^2+y_1^2)^{1/3}$ provides the unique solution to the equation $g(x,y_1,y_2)=0$. Hence, $\pi|_Z:Z\to\mathbb{R}^{n+1}$ is a homeomorphism and the restriction $\pi|_{Z\setminus(Y\times\{(0,0)\})}:Z\setminus(Y\times\{(0,0)\})\to\mathbb{R}^{n+1}\setminus(Y\times\{0\})$ is an analytic diffeomorphism.

Let $p:=(p_0,p_1,p_2)\in Z$. If $p\notin Z\cap\{f(x)=0,y_1=0\}=Y\times\{(0,0)\}$, then it is a regular point of Z, because $\frac{\partial g}{\partial y_2}(p)\neq0$. As a consequence, the germ Z_p is irreducible and coherent.

Suppose now $p=(p_0,0,0)\in Y\times\{(0,0)\}$. Let us prove: *The ideal $\mathcal{J}(Z_p)$ of analytic germs vanishing identically on Z_p is generated by g*. Consequently, Z_p is coherent and

$$\operatorname{Sing}(Z)=Z\cap\{\nabla g(x,y_1,y_2)=0\}=\{f(x)=0,y_1=0,y_2=0\}=Y\times\{(0,0)\}.$$

After a translation we may assume $p=(0,0,0)$ and we change f by $f(x+p_0)$ (but keep the notation f to simplify notation). As $f(0)=0$, the convergent series g is a distinguished polynomial of degree 2 with respect to the variable y_1. Pick $h\in\mathcal{J}(Z_p)$ and divide it by g using Rückert division theorem. There exist analytic series $q\in\mathbb{R}\{x,y_1,y_2\}$ and $a,b\in\mathbb{R}\{x,y_2\}$ such that

$$h(x,y_1,y_2)=q(x,y_1,y_2)g(x,y_1,y_2)+a(x,y_2)y_1+b(x,y_2). \tag{4.4}$$

Changing y_1 by $-y_1$ we obtain

$$h(x,-y_1,y_2)=q(x,-y_1,y_2)g(x,y_1,y_2)-a(x,y_2)y_1+b(x,y_2). \tag{4.5}$$

Adding equations (4.4) and (4.5), we obtain

$$h(x,y_1,y_2)+h(x,-y_1,y_2)=\big(q(x,y_1,y_2)+q(x,-y_1,y_2)\big)g(x,y_1,y_2)+2b(x,y_2).$$

Observe that Z is symmetric with respect to the variable y_1, that is, $(x,y_1,y_2)\in Z$ if and only if $(x,-y_1,y_2)\in Z$. Consequently, $h(x,y_1,y_2)+h(x,-y_1,y_2)\in\mathcal{J}(Z_p)$ and we deduce that $b(x,y_2)\in\mathcal{J}(Z_p)$. Assume by contradiction that $b(x,y_2)\neq 0$. Then $b(x,y_2)\in(\mathcal{J}(Z_p)\cap\mathbb{R}\{x,y_2\})\setminus\{0\}$ and by [Rz, II.2.3] the ideal $\mathcal{J}(Z_p)$ has height ≥ 2. This means that the dimension of the germ Z_p is $\leq n+2-2=n$, which is a contradiction because, as Z is homeomorphic to $\mathbb{R}^{n+1}$, we have $\dim(Z_p)=n+1$. Thus, $b=0$ and

$$h(x,y_1,y_2)=q(x,y_1,y_2)g(x,y_1,y_2)+a(x,y_2)y_1.$$

As $h,g\in\mathcal{J}(Z_p)$, we have $a(x,y_2)y_1\in\mathcal{J}(Z_p)$. Assume by contradiction that $a(x,y_2)\neq 0$. Then

$$\begin{aligned}&a(x,y_2)^2f(x)^2-a(x,y_2)^2y_2^3\\&\qquad=a(x,y_2)^2g(x,y_1,y_2)-a(x,y_2)^2y_1^2\in\big(\mathcal{J}(Z_p)\cap\mathbb{R}\{x,y_2\}\big)\setminus\{0\}.\end{aligned}$$

Analogously to what we have inferred from the assumption $b\neq 0$, we also achieve a contradiction in this case. We conclude that $h=qg$. Thus, $\mathcal{J}(Z_p)=g\mathbb{R}\{x,y_1,y_2\}$, as claimed.

To finish we have to prove: *g is irreducible in* $\mathbb{R}\{x,y_1,y_2\}$. This means that the ideal $\mathcal{J}(Z_p)$ is prime and the analytic germ Z_p is irreducible. Note that local irreducibility (at each point $p\in Z$) together with the connectedness of Z imply global irreducibility.

As g is a distinguished polynomial with respect to y_1, it is enough to prove the irreducibility of g in $\mathbb{R}\{x,y_2\}[y_1]$. As g is a monic polynomial with respect to y_1, if it is reducible, there exists polynomials of degree one $y_1+a_1, y_1+a_2 \in \mathbb{R}\{x,y_2\}[y_1]$ such that $g=(y_1+a_1)(y_1+a_2)$. If we make $x=0$, we have

$$\begin{aligned} y_1^2-y_2^3 = g(0,y_1,y_2) &= \big(y_1+a_1(0,y_2)\big)\big(y_1+a_2(0,y_2)\big) \\ &= y_1^2+\big(a_1(0,y_2)+a_2(0,y_2)\big)y_1+a_1(0,y_2)a_2(0,y_2), \end{aligned}$$

so $a_2(0,y_2)=-a_1(0,y_2)$ and $y_2^3=a_1(0,y_2)^2$, which is a contradiction. Consequently, g is irreducible in $\mathbb{R}\{x,y_1,y_2\}$, as required. □

4.C. Proof of Theorem 1.15. Let $X\subset\mathbb{R}^m$ be a locally compact set and let Y be a C-analytic set. We must prove that $\mathscr{C}^\infty_*(X,Y)$ is dense in $\mathscr{C}^0_*(X,Y)$. By the second part of Corollary 2.2, we can assume X is closed in $\mathbb{R}^m$ and Y is a C-analytic subset of some $\mathbb{R}^n$. We assume X is non-compact. If X is compact the proof is similar, but easier. Denote $\pi:\mathbb{R}^{n+2}\to\mathbb{R}^{n+1}$ the projection onto the first $n+1$ coordinates. By Lemma 4.2 there exists an irreducible coherent C-analytic subset Z of $\mathbb{R}^{n+2}$ such that $\operatorname{Sing}(Z)=Y\times\{(0,0)\}\subset\{x_{n+1}=0,x_{n+2}=0\}$ and the restriction $\psi:=\pi|_Z:Z\to\mathbb{R}^{n+1}$ is a homeomorphism.

As Z is a coherent analytic subset of $\mathbb{R}^{n+2}$, then the pair $(Z,\mathscr{C}^\omega_{\mathbb{R}^{n+2}}|_Z)$ (with the analytic structure induced by the one of $\mathbb{R}^{n+2}$) is a (coherent) real analytic space. By [BM2, §13] there exist a real analytic manifold $Z'\subset\mathbb{R}^q$ and a proper real analytic map $\phi:Z'\to Z$ such that the restriction

$$\phi|_{Z'\setminus\phi^{-1}(\operatorname{Sing}(Z))}:Z'\setminus\phi^{-1}(\operatorname{Sing}(Z))\to Z\setminus\operatorname{Sing}(Z)$$

is a real analytic diffeomorphism and $Y':=\phi^{-1}(\operatorname{Sing}(Z))=\phi^{-1}(Y\times\{(0,0)\})$ is an analytic normal-crossings divisor of Z'.

Let $f\in\mathscr{C}^0_*(X,Y)$ and let $\varepsilon:X\to\mathbb{R}^+$ be a strictly positive continuous function. As X is non-compact and f is proper, $f(X)$ is unbounded in $\mathbb{R}^n$. In this way, there exists an exhaustion $\{L_\ell\}_{\ell\in\mathbb{N}}$ of $\mathbb{R}^{n+1}$ by compact sets such that $L_{\ell-1}\subsetneq\operatorname{Int}_{\mathbb{R}^{n+1}}(L_\ell)$ and $(L_\ell\setminus L_{\ell-1})\cap(f(X)\times\{0\})\neq\varnothing$ for each $\ell\in\mathbb{N}$, where $L_{-1}:=\varnothing$. As Y is closed in $\mathbb{R}^n$ and f is proper, $K_\ell:=(f,0)^{-1}(L_\ell\cap(Y\times\{0\}))$ is a compact subset of X for each $\ell\in\mathbb{N}$. Define the non-empty compact sets $N_\ell:=L_\ell\setminus\operatorname{Int}_{\mathbb{R}^{n+1}}(L_{\ell-1})$ and $H_\ell:=K_\ell\setminus\operatorname{Int}_X(K_{\ell-1})$ for each $\ell\in\mathbb{N}$, where $K_{-1}:=\varnothing$. Note that $X=\bigcup_{\ell\in\mathbb{N}}H_\ell$ and $(f,0)(H_\ell)\subset N_\ell$ for each $\ell\in\mathbb{N}$. Define $\delta_\ell:=\min_{H_\ell}(\varepsilon/6)>0$ and choose a strictly positive continuous function $\delta:\mathbb{R}^{n+1}\to\mathbb{R}^+$ such that $\max_{N_\ell}(\delta)\le\delta_\ell$ for each $\ell\in\mathbb{N}$ (see Remark 2.3). Observe that $\delta\circ(f,0):X\to\mathbb{R}$ satisfies

$$\delta\circ(f,0)\le\varepsilon/6 \text{ on } X.$$

Indeed, $\max_{H_\ell}(\delta\circ(f,0))\le\max_{N_\ell}(\delta)\le\delta_\ell=\min_{H_\ell}(\varepsilon/6)$ for each $\ell\in\mathbb{N}$.

By Lemma 2.1 the map

$$\mathscr{C}^0(X,\mathbb{R}^{n+1})\to\mathscr{C}^0(X,\mathbb{R}),\quad g\mapsto\delta\circ g$$

is continuous. Thus, there exists a strictly positive continuous function $\gamma : X \to \mathbb{R}^+$ such that: if $f_1 \in \mathscr{C}^0(X,\mathbb{R}^{n+1})$ satisfies $\|f_1 - (f,0)\|_{n+1} < \gamma$, then $|\delta \circ f_1 - \delta \circ (f,0)| < \varepsilon/6$. In particular,

$$\delta \circ f_1 < \varepsilon/3 \quad \text{on } X. \tag{4.6}$$

Consider the proper surjective map $\psi \circ \phi : Z' \to \mathbb{R}^{n+1}$, which satisfies $(\psi \circ \phi)(Y') = Y \times \{0\}$. Denote $(\psi \circ \phi)' : Y' \to Y \times \{0\}$ the restriction of $\psi \circ \phi$ from Y' to $Y \times \{0\}$. Using Lemma 2.1 again we deduce that the map

$$\mathscr{C}^0(Y',Y') \to \mathscr{C}^0\big(Y', Y \times \{0\}\big), \quad g \mapsto (\psi \circ \phi)' \circ g$$

is continuous. Let $\zeta : Y' \to \mathbb{R}^+$ be a strictly positive continuous function such that: if $g \in \mathscr{C}^0(Y',Y')$ satisfies $\|g - \mathrm{id}_{Y'}\|_q < \zeta$, then $\|(\psi \circ \phi)' \circ g - (\psi \circ \phi)' \circ \mathrm{id}_{Y'}\|_{n+1} < \delta \circ (\psi \circ \phi)'$. By Proposition 4.1 there exists an open neighborhood $W \subset Z'$ of Y' and a $\mathscr{C}^\infty$ weak retraction $\rho : W \to Y'$ such that $\|\rho|_{Y'} - \mathrm{id}_{Y'}\|_q < \zeta$, so $\|(\psi \circ \phi)' \circ \rho|_{Y'} - (\psi \circ \phi)' \circ \mathrm{id}_{Y'}\|_{n+1} < \delta \circ (\psi \circ \phi)'$. Define the open neighborhood $W' \subset W$ of Y' by setting

$$W' := \big\{ z \in W : \big\|(\psi \circ \phi)\big(\rho(z)\big) - (\psi \circ \phi)(z)\big\|_{n+1} < \delta\big((\psi \circ \phi)(z)\big) \big\}.$$

Consider the closed subset $C' := Z' \setminus W'$ of Z', which does not meet $Y' = (\psi \circ \phi)^{-1}(Y \times \{0\})$. As $\psi \circ \phi : Z' \to \mathbb{R}^{n+1}$ is proper, $C := (\psi \circ \phi)(C')$ is a closed subset of $\mathbb{R}^{n+1}$, which does not meet $Y \times \{0\}$. Let $\eta : Y \times \{0\} \to \mathbb{R}^+$ and $\eta' : X \to \mathbb{R}^+$ be the strictly positive continuous functions given by

$$\eta(y,0) := \mathrm{dist}_{\mathbb{R}^{n+1}}\big((y,0),C\big)/2 \text{ if } y \in Y \text{ and } \eta' := \eta \circ (f,0).$$

Define the strictly positive continuous function $\xi : X \to \mathbb{R}^+$ as $\xi := \min\{\gamma, \eta', \varepsilon/3\}/2$. The map $f' := (f,\xi) : X \to \mathbb{R}^{n+1}$ satisfies

$$\big\|f' - (f,0)\big\|_{n+1} = \xi < \min\{\gamma, \eta', \varepsilon/3\} \le \frac{\varepsilon}{3} \tag{4.7}$$

and $\delta \circ f' < \varepsilon/3$ on the whole X (see (4.6)). Consider the continuous function

$$\begin{aligned} f'' := (\phi|_{Z' \setminus Y'})^{-1} \circ \psi^{-1}|_{\mathbb{R}^{n+1} \setminus \{x_{n+1}=0\}} \circ f' :& \\ X \to \mathbb{R}^{n+1} \setminus \{x_{n+1} = 0\} \to Z \setminus \big(\{Y\} \times \{(0,0)\}\big) &\to Z' \setminus Y' \end{aligned}$$

and observe that $f'(x) = (\psi \circ \phi)(f''(x))$ for each $x \in X$.

We claim: $f''(X) \subset W'$.

If $x \in X$, then $(\psi \circ \phi)(f''(x)) = (f(x), \xi(x))$. Thus,

$$\big\|(\psi \circ \phi)\big(f''(x)\big) - \big(f(x),0\big)\big\|_{n+1} = \xi(x) \le \eta'(x) < \mathrm{dist}_{\mathbb{R}^{n+1}}\big(\big(f(x),0\big),C\big),$$

so $(\psi \circ \phi)(f''(x)) \notin C$. Consequently, $f''(x) \notin C' = Z' \setminus W'$, that is, $f''(x) \in W'$.

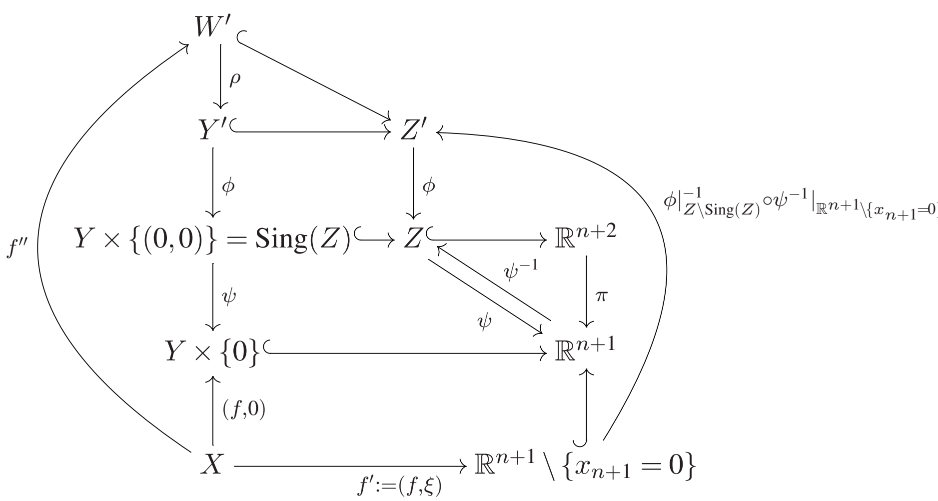
In addition, we have:

$$\big\|(\psi\circ\psi)\big(\rho\big(f''(x)\big)\big)-f'(x)\big\|_{n+1}<\frac{\varepsilon(x)}{3}\quad \textit{for each } x\in X. \tag{4.8}$$

Indeed, pick $x\in X$. As $f''(x)\in W'$ and $(\delta\circ f')(x)<\varepsilon(x)/3$, it holds

$$\begin{aligned}\big\|(\psi\circ\psi)\big(\rho\big(f''(x)\big)\big)-f'(x)\big\|_{n+1}&=\big\|(\psi\circ\psi)\big(\rho\big(f''(x)\big)\big)-(\psi\circ\psi)\big(f''(x)\big)\big\|_{n+1}\\&<\delta\big((\psi\circ\phi)\big(f''(x)\big)\big)=(\delta\circ f')(x)<\frac{\varepsilon(x)}{3}.\end{aligned}$$

The following commutative diagram summarizes the situation we have achieved until the moment.



Proceeding as in Step I of the proof of Theorem 1.6, we deduce that there exists an open neighborhood $U\subset\mathbb{R}^m$ of X and a continuous extension $F'':U\to Z'$ of f''. The inverse image $U_0:=F''^{-1}(W')$ is an open neighborhood of X in U and the restriction, $F''|_{U_0}:U_0\to W'$ is a continuous map between the real analytic manifolds U_0 and W'. We substitute U by U_0 and F'' by $F''|_{U_0}$, but we keep the original notation to ease the writing. Let $H_0:U\to W'$ be a real analytic map arbitrarily close to F'' in $\mathscr{C}^0(U,W')$, which exists by Whitney's approximation theorem. The restriction $h_0:=H_0|_X:X\to W'$ is a real analytic map arbitrarily close to f'' in $\mathscr{C}^0(X,W')$. Consider the $\mathscr{C}^\infty$ map $h:X\to Y$ such that $(h,0):=(\psi\circ\phi)'\circ\rho|_{W'}\circ h_0$. As the map

$$\mathscr{C}^0(X,W')\to\mathscr{C}^0(X,Y),\quad g\mapsto(\psi\circ\phi)'\circ\rho|_{W'}\circ g$$

is continuous, we may assume

$$\big\|\big(h(x),0\big)-(\psi\circ\phi)'\big(\rho\big(f''(x)\big)\big)\big\|_{n+1}<\frac{\varepsilon(x)}{3}\quad\text{for each } x\in X. \tag{4.9}$$

Given any $x\in X$ we deduce by (4.7), (4.8) and (4.9),

$$
\begin{aligned}
&\big\|h(x)-f(x)\big\|_n\\
&\quad=\big\|\big(h(x),0\big)-\big(f(x),0\big)\big\|_{n+1}\le\big\|\big(h(x),0\big)-(\psi\circ\phi)'\big(\rho\big(f''(x)\big)\big)\big\|_{n+1}\\
&\qquad+\big\|(\psi\circ\phi)'\big(\rho\big(f''(x)\big)\big)-f'(x)\big\|_{n+1}+\big\|f'(x)-\big(f(x),0\big)\big\|_{n+1}\\
&\quad<\frac{\varepsilon(x)}{3}+\frac{\varepsilon(x)}{3}+\frac{\varepsilon(x)}{3}=\varepsilon(x).
\end{aligned}
$$

Thus, we have found a $\mathscr{C}^\infty$ map $h:X\to Y$ that is arbitrarily close to f. In the same vein of [H3, Thm. II.1.5] one easily shows that $\mathscr{C}^0_*(X,Y)$ is an open subset of $\mathscr{C}^0(X,Y)$, so we can assume that h is in addition proper, as required. □

Remarks 4.3. (i) In the preceding proof we have used that the continuous map $f:X\to Y\subset\mathbb{R}^n$, we want to approximate, is proper exactly when we need to find for a strictly positive continuous function $\varepsilon:X\to\mathbb{R}^+$ another strictly positive continuous function $\delta:\mathbb{R}^{n+1}\to\mathbb{R}^+$ such that $\delta\circ(f,0)<\varepsilon/6$.

(ii) The techniques we have developed in this section cannot be adapted to guarantee relative approximation. Let us summarize the principal points of the proof of Theorem 1.15 (we keep all the notations already introduced there). We begin moving the image of the map f we want to approximate outside Y, but inside a small neighborhood of Y in $\mathbb{R}^{n+1}$ (we have constructed the map f'). *This movement is crucial to make the rest of our construction work but keeps off relative approximation.*

Recall that $Y\times\{(0,0)\}$ is the singular set of an irreducible coherent C-analytic subset Z of $\mathbb{R}^{n+2}$, which is in addition homeomorphic to $\mathbb{R}^{n+1}$ (via the projection of $\mathbb{R}^{n+2}$ to $\mathbb{R}^{n+1}$ that forgets the last coordinate). Once this is done, we lift the image of f' to $Z\setminus\operatorname{Sing}(Z)$. Next, we use resolution of singularities to change $Y\times\{(0,0)\}=\operatorname{Sing}(Z)$ by an analytic normal-crossings divisor Y' contained in a real analytic manifold Z'. We denote $\phi:Z'\to Z$ the (proper) resolution map and consider f'' the composition of the lift of f' with $(\phi|_{Z'\setminus Y'})^{-1}$. The image of f'' is contained in $Z'\setminus Y'$.

We extend f'' continuously to an open neighborhood $U_0\subset\mathbb{R}^m$ of X such that its image is contained in a small neighborhood $W'\subset Z'$ of Y' endowed with a $\mathscr{C}^\infty$ weak retraction $\rho:W\to Y$. The previous extension $F'':U_0\to W'$ is a continuous map between submanifolds of Euclidean spaces, so here Whitney's approximation theorem works and provides a $\mathscr{C}^\infty$ approximation $H_0:U_0\to W'$ close to F''. We can even assume that the images of H_0 and F'' are contained in $Y'\subset W'$. Now, we compose $h_0:=H_0|_X$ with $(\psi\circ\phi)'\circ\rho$ to obtain the $\mathscr{C}^\infty$ approximating map $h:X\to Y$ of the continuous map $f:X\to Y$.

There are some difficulties to achieve relative approximation results:

(1) We have moved the image of f off Y to construct F'' and we have lost the control of the restrictions of f to subsets X' of X.

(2) In case Y is an analytic normal-crossings divisor of a real analytic manifold Z, we can skip the first part of the proof and keep the image of f inside Y. Then, we extend f continuously to an open neighborhood $U_0 \subset \mathbb{R}^m$ of X such that its image is contained in a small neighborhood $W \subset Z$ of Y endowed with a $\mathscr{C}^\infty$ weak retraction $\rho : W \to Y$. But now we have to deal with ρ, which is not a true retraction and moves the points of Y. Thus, if $X' \subset X$ satisfies that $f(X')$ is not contained in the set of fixed points of ρ, then it seems difficult to assure that the restriction to X' of the $\mathscr{C}^\infty$ approximation behaves as f on X'.

(iii) The previous remark does not mean that in the framework of C-analytic sets relative approximation is not possible (see the example below). What we have pointed out is that our techniques are not a good tool to approach relative approximation and new ideas are needed.

Example 4.4. Let $X' := [-1,0] \subset X := [-1,1] \subset \mathbb{R}$ and let $Y := \{xy = 0\} \subset \mathbb{R}^2$. Consider the continuous map

$$f : X \to Y, \quad t \mapsto \begin{cases} (t,0) & \text{if } t \in [-1,0], \\ (0,t) & \text{if } t \in [0,1], \end{cases}$$

which is $\mathscr{C}^\infty$ on X'. Fix $\varepsilon \in (0,1)$ and let $\theta_1, \theta_2 : [-1,1] \to [0,1]$ be $\mathscr{C}^\infty$ bump functions such that:

- $\theta_1|_{[-1,0]} = 1$ and $\theta_1|_{[\frac{\varepsilon}{8},1]} = 0$.
- $\theta_2|_{[-1,\frac{\varepsilon}{2}]} = 0$ and $\theta_1|_{[\varepsilon,1]} = 1$.

Define

$$g : X \to Y, \quad t \mapsto \begin{cases} \big(t\theta_1(t), 0\big) & \text{if } t \in \Big[-1, \dfrac{\varepsilon}{4}\Big], \\ \big(0, t\theta_2(t)\big) & \text{if } t \in \Big[\dfrac{\varepsilon}{4}, 1\Big], \end{cases}$$

which is a $\mathscr{C}^\infty$ function. Observe that g coincides with f outside the interval $[0,\varepsilon]$. We have

$$\big\|f(t) - g(t)\big\|_2 = \begin{cases} \big\|(0,t) - \big(t\theta_1(t),0\big)\big\|_2 = |t|\sqrt{1+\theta_1^2(t)} < \varepsilon & \text{if } t \in \Big[0, \dfrac{\varepsilon}{2}\Big], \\ \big\|(0,t) - \big(0,t\theta_2(t)\big)\big\|_2 = |t||1-\theta_2(t)| < \varepsilon & \text{if } t \in \Big[\dfrac{\varepsilon}{2}, \varepsilon\Big], \end{cases}$$

so g is a $\mathscr{C}^\infty$ approximation of f such that $g|_{X'} = f|_{X'}$.

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