Discrete & Computational Geometry (2019) 62:292–347
https://doi.org/10.1007/s00454-019-00065-2

Complements of Unbounded Convex Polyhedra as Polynomial Images of $\mathbb{R}^n$

José F. Fernando[1] · Carlos Ueno[2]

Received: 8 February 2017 / Revised: 15 January 2019 / Accepted: 22 January 2019 /
Published online: 20 February 2019

Abstract

We prove constructively that: *The complement* $\mathbb{R}^n\setminus\mathcal{K}$ *of an* n*-dimensional unbounded convex polyhedron* $\mathcal{K}\subset\mathbb{R}^n$ *and the complement* $\mathbb{R}^n\setminus\operatorname{Int}(\mathcal{K})$ *of its interior are polynomial images of* $\mathbb{R}^n$ *whenever* $\mathcal{K}$ *does not disconnect* $\mathbb{R}^n$. The case of a compact convex polyhedron and the case of convex polyhedra of small dimension were approached by the authors in previous works. The techniques here are more sophisticated than those corresponding to the compact case and require rational separation results for tuples of variables, which have interest by their own and can be applied to separate certain types of (non-compact) semialgebraic sets.

Keywords Semialgebraic sets · Polynomial maps and images · Complement of a convex polyhedra · Rational separation of tuples of variables

Mathematics Subject Classification Primary 14P10, 14P05 · Secondary 52B99

1 Introduction and Statement of the Main Results

A map $\mathtt{f}:=(\mathtt{f}_1,\dots,\mathtt{f}_m)\colon\mathbb{R}^n\to\mathbb{R}^m$ is *polynomial* if its components $f_k\in\mathbb{R}[\mathtt{x}]:=\mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_n]$ are polynomials. Analogously, $\mathtt{f}$ is *regular* if its components can be

---

Editor in Charge: Kenneth Clarkson

The first author (J.F. Fernando) is supported by Spanish GRAYAS MTM2014-55565-P, Spanish STRANO MTM2017-82105-P and Grupos UCM 910444, whereas the second author (C. Ueno) is an external collaborator of the first two grants. This work was also partially supported by the "National Group for Algebraic and Geometric Structures and their Applications" (GNSAGA - INdAM). This article was partially written during a common one year research stay of the authors in the Dipartimento di Matematica of the Università di Pisa. The authors would like to thank the department for the invitation and the very pleasant working conditions. The one year research stay of the first author was partially supported by MECD grant PRX14/00016. The one year research stay of the second author was mainly supported by 'Scuola Galileo Galilei' Research Grant at the Dipartimento di Matematica of the Università di Pisa (D.R. n$^{\text{o}}$27519 of 29/07/2013, MAT/03).

Extended author information available on the last page of the article

represented as quotients $\mathtt{f}_k := \frac{\mathtt{g}_k}{\mathtt{h}_k}$ of two polynomials $\mathtt{g}_k, \mathtt{h}_k \in \mathbb{R}[\mathtt{x}]$ such that $\mathtt{h}_k$ never vanishes on $\mathbb{R}^n$. We will consider here the *degree* $\deg(\mathtt{f})$ of a polynomial map $\mathtt{f} = (\mathtt{f}_1, \ldots, \mathtt{f}_m)$ as the maximum degree of its components $\mathtt{f}_i$. The degree of a polynomial map is not affected by compositions with affine bijections. A subset $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* when it has a description by a finite boolean combination of polynomial equations and inequalities. The category of semialgebraic sets is closed under basic boolean operations but also under usual topological operations: taking closures (denoted by $\mathrm{Cl}(\cdot)$), taking interiors (denoted by $\mathrm{Int}(\cdot)$), considering connected components, etc. By Tarski–Seidenberg's principle [5, 1.4] the image of an either polynomial or regular map is a semialgebraic set. During the last decade we have approached the following question:

Problem 1.1 Characterize which (semialgebraic) subsets $\mathcal{S} \subset \mathbb{R}^m$ are polynomial or regular images of $\mathbb{R}^n$.

The first proposal for studying this problem and related ones, like the famous 'quadrant problem', goes back to [22] (see also [6, §3.IV, p. 69]). A related problem concerns the parameterization of semialgebraic sets of dimension d using continuous semialgebraic maps whose domains are semialgebraic subsets of $\mathbb{R}^d$ satisfying certain nice properties [23]. The approach proposed by Gamboa in [22] sacrifices injectivity but chooses the simplest possible domains (Euclidean spaces) and the simplest possible maps (polynomial and regular) to represent semialgebraic sets. The class of semialgebraic sets that can be represented as polynomial and regular images of Euclidean spaces (even sacrificing injectivity) is surely much smaller than the one consisting of the images under injective continuous semialgebraic maps of nice semialgebraic sets. Of course, more general domains than Euclidean spaces can be considered and compact semialgebraic sets deserve special attention: balls, spheres, compact convex polyhedra, … For instance, in [26] the authors develop a computational study of images under polynomial maps $\mathtt{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$ (and the corresponding convex hulls) of compact (principal) semialgebraic subsets $\{\mathtt{h} \geq 0\} \subset \mathbb{R}^3$, where $\mathtt{h} \in \mathbb{R}[\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3]$ (this includes for example the case of a three-dimensional ball). In addition, other types of maps (like Nash, continuous rational, etc.) have been already considered to represent semialgebraic sets as images of Euclidean spaces (see for instance [9,10]). Recall that an analytic function $\mathtt{f}\colon U \to \mathbb{R}$ on an open semialgebraic set $U \subset \mathbb{R}^n$ is a *Nash function* if there exists a non-zero polynomial $\mathtt{P} \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n, \mathtt{y}]$ such that $\mathtt{P}(\mathtt{x}, \mathtt{f}(\mathtt{x})) = 0$ for each $\mathtt{x} \in U$.

1.1 Brief State of the Art

We feel very far from solving Problem 1.1 as stated above in its full generality, but we have developed significant progress in three ways:

Obtention of general properties. We have found conditions [8,12,17,32] that a semialgebraic subset $\mathcal{S} \subset \mathbb{R}^m$ must satisfy in order to be an either polynomial or regular image of $\mathbb{R}^n$. The most remarkable one states that the set of points at infinity of a polynomial image of $\mathbb{R}^n$ is connected [17]. In addition, the one-dimensional case has been completely described in [8]. In [10] we have proved that the family of images of

$\mathbb{R}^2$ under regular maps and the family of images of $\mathbb{R}^2$ under continuous rational maps coincide. On the other hand, in [9] the first author has provided a full characterization of the semialgebraic subsets $\mathcal{S} \subset \mathbb{R}^m$ that are Nash images of $\mathbb{R}^n$.

Optimization of polynomial maps with a given image. Even in the simplest cases, it is difficult to determine which is the minimum degree for a polynomial map that has a given set as image (see Question 1.6 below). As a relevant example, we have been trying to find the least degree of a polynomial map $\mathtt{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose image is the open quadrant $\mathcal{Q} := \{\mathtt{x} > 0, \mathtt{y} > 0\}$. We already know that it is bounded above by 16 (see [11,14,20]), but we guess that this bound can still be lowered down (hopefully to 8). Similar values are expected for the complement $\mathcal{S} := \mathbb{R}^2 \setminus \overline{\mathcal{Q}}$ of the closed quadrant $\overline{\mathcal{Q}} := \{\mathtt{x} \geq 0, \mathtt{y} \geq 0\}$. We consider that our degree bounds for more complex polyhedra are still far from being optimal and, from a computational point of view, it would be interesting an improvement on these current bounds.

Explicit representation of families of semialgebraic sets as polynomial or regular images of $\mathbb{R}^n$. We have devised techniques to represent large families of significant semialgebraic sets as either polynomial or regular images of $\mathbb{R}^n$, with the already mentioned 'open quadrant problem' as a recurrent matter. In [8,11,13,15,16,18,19,33] we focused on semialgebraic sets with piecewise linear boundary, that is, semialgebraic sets that admit a semialgebraic description involving only linear equations. To be more precise, we analyzed the cases of convex polyhedra and their interiors, together with their respective complements. For these families of semialgebraic sets we already had full understanding concerning their representation as regular images [13,18] but we were lacking information when trying to represent them as polynomial (instead of regular) images.

We guess that in general there are 'few' polynomial maps that have the complement of a concrete convex polyhedron (or its interior) as a polynomial image and that there are even fewer for which it is affordable to show that their images actually correspond to the complement of our given convex polyhedron (or its interior). Let us be more explicit in this point. Since the complements of proper convex polyhedra (or of their interiors) are unbounded semialgebraic sets, it makes sense to wonder whether these sets are not only images of regular maps, but also of polynomial maps. This in fact provides a priori 'simpler' representations for this type of sets because polynomial representations do not involve denominators. Our initial purpose when writing [18] was to approach the previous problem in its full generality, but the techniques developed there required, in order to use polynomial maps, to assume that the involved convex polyhedra were compact when the dimension of $\mathcal{K}$ matched that of the ambient space and was greater than or equal to 4 (for further details see [19]). Therefore, the main results appearing in [18] that follow next referred to the compact case, together with the case of convex polyhedra of smaller dimension than their ambient space. From now on, we denote by $\operatorname{Int}(\mathcal{K})$ the relative interior of $\mathcal{K}$ as a topological manifold with boundary, which coincides with the topological interior of $\mathcal{K}$ in the affine subspace of $\mathbb{R}^n$ spanned by $\mathcal{K}$.

Theorem 1.2 ([18, Thm. 1.1 (i)]) *Let $\mathcal{K}$ be an n-dimensional compact convex polyhedron of $\mathbb{R}^n$. Then the semialgebraic sets $\mathcal{S} := \mathbb{R}^n \setminus \mathcal{K}$ and $\overline{\mathcal{S}} := \mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K})$ are polynomial images of $\mathbb{R}^n$.*

Proposition 1.3 ([18, Thm. 3.1]) *Let $\mathcal{K}$ be a d-dimensional convex polyhedron of $\mathbb{R}^n$ such that $d < n$ and it is not a hyperplane. Then the semialgebraic sets $\mathcal{S} := \mathbb{R}^n \setminus \mathcal{K}$ and $\overline{\mathcal{S}} := \mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K})$ are polynomial images of $\mathbb{R}^n$.*

Similar techniques to those developed to prove Theorem 1.2 can be adapted for unbounded two-dimensional and three-dimensional convex polyhedra [19,33], but unfortunately do not extend any further to higher dimensions [19]. The purpose of this work is to close this gap and provide a full answer to the representation of the complements of convex polyhedra $\mathcal{K}$ and their interiors as polynomial images of $\mathbb{R}^n$ (see Table 1), dropping the compactness assumption on $\mathcal{K}$ that appears in [12]. Since our previous methods did not work in this more general setting, we have developed new algorithms that use more sophisticated polynomial maps to achieve 'constructively' our goals. This requires a more technical approach than the one devised in [12], but reveals a better understanding on how polynomial maps can act on $\mathbb{R}^n$ to produce our desired image sets.

1.2 Main Results of this Article

A *layer* is a convex polyhedron of $\mathbb{R}^n$ affinely equivalent to $[-a, a] \times \mathbb{R}^{n-1}$ with $a > 0$. Our main results in this work, which complete the full picture in regard to the representation of complements of convex polyhedra and their interiors as polynomial images of Euclidean spaces, are the following:

Theorem 1.4 *Let $n \geq 1$ and let $\mathcal{K}$ be an n-dimensional unbounded convex polyhedron in $\mathbb{R}^n$ that is not a layer. Then the semialgebraic set $\overline{\mathcal{S}} := \mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K})$ is a polynomial image of $\mathbb{R}^n$.*

Theorem 1.5 *Let $n \geq 2$ and let $\mathcal{K}$ be an n-dimensional unbounded convex polyhedron in $\mathbb{R}^n$ that is not a layer. Then the semialgebraic set $\mathcal{S} := \mathbb{R}^n \setminus \mathcal{K}$ is a polynomial image of $\mathbb{R}^n$.*

Table 1 Full picture concerning the values of the invariants p and r

	$\mathcal{K}$ bounded		$\mathcal{K}$ unbounded	
	$n = 1$	$n \geq 2$	$n = 1$	$n \geq 2$
$\mathrm{r}(\mathcal{K})$	1	n	1	n
$\mathrm{r}(\mathrm{Int}(\mathcal{K}))$	2	n	2	n
$\mathrm{p}(\mathcal{K})$	$+\infty$	$+\infty$	1	$n, +\infty$ $(*)$
$\mathrm{p}(\mathrm{Int}(\mathcal{K}))$	$+\infty$	$+\infty$	2	$n, n+1, +\infty$ $(\star)$
$\mathrm{r}(\mathcal{S})$	$+\infty$	n	2	n
$\mathrm{r}(\overline{\mathcal{S}})$	$+\infty$	n	1	n
$\mathrm{p}(\mathcal{S})$	$+\infty$	n	2	n
$\mathrm{p}(\overline{\mathcal{S}})$	$+\infty$	n	1	n

1.3 Full Picture for PL Semialgebraic Sets

To summarize the presentation of the results concerning the representation of piecewise linear boundary semialgebraic sets as either polynomial or regular images [13,16–18,33] we introduce the following two invariants. Given a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$, we define

$$\mathrm{p}(\mathcal{S}) := \inf \{n \geq 1 : \exists\, \mathtt{f}\colon \mathbb{R}^n \to \mathbb{R}^m \text{ polynomial such that } \mathtt{f}(\mathbb{R}^n) = \mathcal{S}\},$$
$$\mathrm{r}(\mathcal{S}) := \inf \{n \geq 1 : \exists\, \mathtt{f}\colon \mathbb{R}^n \to \mathbb{R}^m \text{ regular such that } \mathtt{f}(\mathbb{R}^n) = \mathcal{S}\}.$$

The condition $\mathrm{p}(\mathcal{S}) := +\infty$ characterizes the non representability of $\mathcal{S}$ as a polynomial image of some $\mathbb{R}^n$ while $\mathrm{r}(\mathcal{S}) := +\infty$ has the analogous meaning for regular maps. Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and assume below that $\mathcal{S} := \mathbb{R}^n \setminus \mathcal{K}$ and $\overline{\mathcal{S}} := \mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K})$ are connected. Denote by $\vec{\mathfrak{C}}(\mathcal{K})$ the recession cone of $\mathcal{K}$ (see Sect. 2.3 for further details). Let us explain some (marked) cases in Table 1 developed in [16]:

- (∗) $(n, +\infty)$: An n-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ has $\mathrm{p}(\mathcal{K}) = +\infty$ if and only if its recession cone $\vec{\mathfrak{C}}(\mathcal{K})$ has dimension $< n$. Otherwise, $\mathrm{p}(\mathcal{K}) = n$.
- (⋆) $(n, n+1, +\infty)$: If the recession cone $\vec{\mathfrak{C}}(\mathcal{K})$ of an n-dimensional convex polyhedron $\mathcal{K}$ has dimension $< n$, then $\mathrm{p}(\mathrm{Int}(\mathcal{K})) = +\infty$. Otherwise, if $\mathcal{K}$ has bounded facets, $\mathrm{p}(\mathrm{Int}(\mathcal{K})) = n + 1$ and if $\mathcal{K}$ has no bounded facets, $\mathrm{p}(\mathrm{Int}(\mathcal{K})) = n$.

1.4 Related Problems and Open Questions

The effective representation of a semialgebraic subset $\mathcal{S} \subset \mathbb{R}^m$ as a polynomial or regular image of $\mathbb{R}^n$ may help the handling of certain classical problems in Real Geometry by reducing them to its study in $\mathbb{R}^n$. Let us comment some of them:

Positivstellensätze. A widespread studied problem is the algebraic characterization of those polynomial or regular functions $\mathtt{g}\colon \mathbb{R}^m \to \mathbb{R}$ which are either strictly positive or positive semidefinite on a semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$. When $\mathcal{S}$ is a basic closed semialgebraic set these problems were solved in [31] (see also [5, 4.4.3]). For the particular case of compact convex polyhedra $\mathcal{S} = \mathcal{K} \subset \mathbb{R}^n$ we refer the reader to [24], where stronger Positivstellensätze are obtained, specially for strictly positive polynomials on $\mathcal{K}$. In this case the obtained certificate of positiveness is the best possible one.

Let $\mathtt{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial and denote $\mathcal{S} := \mathtt{f}(\mathbb{R}^n)$. Note that $\mathtt{g}$ is strictly positive (respectively positive semidefinite) on $\mathcal{S}$ if and only if $\mathtt{g} \circ \mathtt{f}$ is strictly positive (respectively positive semidefinite) on $\mathbb{R}^n$ and both questions are decidable using for instance [31]. Thus, this provides an algebraic characterization of positiveness for polynomial and regular functions on semialgebraic sets that are either polynomial or regular images of $\mathbb{R}^n$. Observe that these semialgebraic sets need not to be neither closed, as is the case with the interior of a convex polyhedron, nor basic, as is the case with the complement of a convex polyhedron. Thus, our results in this article provide certificates of positivity for a large class of semialgebraic sets (neither closed nor basic) which cannot be approached by the classical Positivstellensätze.

Stochastic finance. In the pricing of assets in an arbitrage-free market the analysis of the interior of the convex hull of the support of a certain probability measure is crucial. Roughly speaking, recall that a market is *arbitrage-free* if there is no positive probability of having a positive economical position after a period of time, if the initial position is ≤ 0 (see for instance [21, §1.5] for further details). In many occasions, the market is constituted by n assets which are "independent" and all the others are derivative products of these "independent assets" and are described by measurable piecewise linear functions on this assets (*put options*, *call options*, *straddles*, *butterfly spreads*, etc.). This kind of assets provides a convex hull, which is an either bounded or unbounded open convex polyhedron. Our results allow to represent the set $\mathcal{S}$ of arbitrage-free prices as the image of $\mathbb{R}^n$ under a regular or a polynomial map [13,16], but also the set of arbitrage prices $\mathbb{R}^n \setminus \mathcal{S}$ (the complement of an either bounded or unbounded open convex polyhedron) can be described again as a polynomial image of $\mathbb{R}^n$ (Theorem 1.5). Although the involved polynomial or regular map in each case has a great complexity, we eliminate with this map the contour conditions and, at least from a theoretical point of view, we introduce a different approach to study the sets of arbitrage-free and arbitrage prices of this type of markets.

Optimization. Suppose that $\mathtt{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ is either a polynomial or a regular map and let $\mathcal{S} := \mathtt{f}(\mathbb{R}^n)$. Then the optimization of a given regular or polynomial function $\mathtt{g}\colon \mathcal{S} \to \mathbb{R}$ is equivalent to the optimization of the composition $\mathtt{g} \circ \mathtt{f}$ on $\mathbb{R}^n$. In this way one can avoid contour conditions and only apply elementary analysis to approach optimization (see for instance [27,28,30,34] for relevant tools concerning optimization of polynomial functions on $\mathbb{R}^n$). Of course, the user should evaluate in each case whether the increase of the complexity of the composition $\mathtt{g} \circ \mathtt{f}$ with respect to that of $\mathtt{f}$ is preferable to the existence of contour conditions.

Alternatively, let $\mathcal{T} \subset \mathbb{R}^n$ be a semialgebraic set and let $\mathtt{h}\colon \mathcal{T} \to \mathbb{R}$ be a continuous semialgebraic function (that is, a continuous function on $\mathcal{T}$ with semialgebraic graph). Then there exists a semialgebraic compactification $\mathcal{T}^* \subset \mathbb{R}^{n+1}$ of $\mathcal{T}$ and a continuous (semialgebraic) extension $\overline{\mathtt{h}}\colon \mathcal{T}^* \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \sqcup \{-\infty, +\infty\}$ is the (semialgebraic) compactification of $\mathbb{R}$ by two points. As it is well-known, compact semialgebraic sets are triangulable and by [5, Thm. 9.4.1] also continuous semialgebraic functions on compact semialgebraic sets can be 'triangulated'. Thus, a continuous semialgebraic function on a compact semialgebraic set could be assumed, up to a suitable triangulation, as a continuous function on a finite simplicial complex that is affine on each simplex of the complex. Of course, optimization problems for this type of functions are 'straightforwardly' approached.

However, the usual algorithms to triangulate a compact semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ (and continuous semialgebraic maps) [5, Chap. 9] are based in the use of cylindrical decompositions, which have doubly exponential complexity in the number n of variables involved in describing $\mathcal{S}$. More precisely, its complexity is in general $(\ell d)^{O(1)^n}$ where $O(1)$ represents a constant, ℓ is a bound on the number of polynomials need to describe $\mathcal{S}$ and d is a bound on the degrees of a family of polynomials describing $\mathcal{S}$, see [2, Chap. 11]. If $\mathcal{S}$ has piecewise linear boundary, the complexity of cylindrical decomposition is $\ell^{O(1)^n}$, which is still doubly exponential in the number n of variables involved in describing $\mathcal{S}$.

Questions regarding complexity. The algorithms developed in this work to show that certain semialgebraic sets with piecewise linear boundary are polynomial images of $\mathbb{R}^n$ are constructive without much control on the complexity of the construction. In particular, the degrees of the involved polynomial maps are very high. From a computational perspective, the efficiency of our constructions is at stake and perhaps at this moment the interest of our results relies more on their existential implications than on their practical applications. However, two natural questions arise when considering the issue of complexity:

Question 1.6 Which is the minimum (or a good bounding) degree of a polynomial map whose image is the complement of a convex polyhedron with m facets?

In Remarks 4.2 and 6.8 we provide bounds for the polynomial maps constructed in the proofs of Theorems 1.4 and 1.5 below.

Question 1.7 If we think of polynomial maps $\mathbb{R}^m \to \mathbb{R}^n$ with $m > n$, that is, if we allow an increase in the number of variables, is it possible to devise constructions which lower the complexity of the polynomial maps presented here?

We refer the reader to Examples 4.3 and 6.9 for a partial positive answer to this question in the two-dimensional case, which takes advantage of the techniques developed in this paper. This increase in the number of variables is somehow related with the *extension complexity*, which is the smallest integer k such that a compact convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ can be expressed as the linear image of a polytope with k facets. This invariant is used to 'simplify' the formulation of linear programming problems over polytopes increasing the number of variables (and providing new constrains). For further details we refer the reader to [7,35].

1.5 Rational Separation for Tuples of Variables

It is worthwhile to mention here in the Introduction that the proof of Theorem 1.5 involves a separation result for tuples of variables that has interest by its own. A *rational separator for the pair of positive integers* (r, s) is a rational function $\phi_{r,s} : \mathbb{R}^r \times \mathbb{R}^s \dashrightarrow \mathbb{R}$ that is regular on the interior of the polyhedron

$$\mathcal{Q}_{r,s} := \big\{(\mathtt{y}_1, \ldots, \mathtt{y}_r; \mathtt{z}_1, \ldots, \mathtt{z}_s) \in \mathbb{R}^r \times \mathbb{R}^s : \max\{\mathtt{y}_1, \ldots, \mathtt{y}_r\} \le \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}\big\},$$

extends to a continuous (semialgebraic) function on $\mathcal{Q}_{r,s}$ and satisfies

$$\max\{\mathtt{y}_1, \ldots, \mathtt{y}_r\} < \phi_{r,s}(\mathtt{y}; \mathtt{z}) < \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}$$

for each $(\mathtt{y}; \mathtt{z}) := (\mathtt{y}_1, \ldots, \mathtt{y}_r; \mathtt{z}_1, \ldots, \mathtt{z}_s) \in \operatorname{Int}(\mathcal{Q}_{r,s})$. In Proposition 5.4 we show that rational separators exist for each pair of positive integers (r, s). As a consequence we prove in Proposition 5.9 the following statement: *Given an n-dimensional convex polyhedron of $\mathbb{R}^n$ and the projection $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $(\mathtt{x}_1, \ldots, \mathtt{x}_n) \mapsto (\mathtt{x}_1, \ldots, \mathtt{x}_{n-1})$, the two connected components of the difference* $(\operatorname{Int}(\pi_n(\mathcal{K})) \times \mathbb{R}) \setminus \mathcal{K}$ *can be separated by a rational function that is regular on* $\operatorname{Int}(\pi_n(\mathcal{K}))$ *and extends*

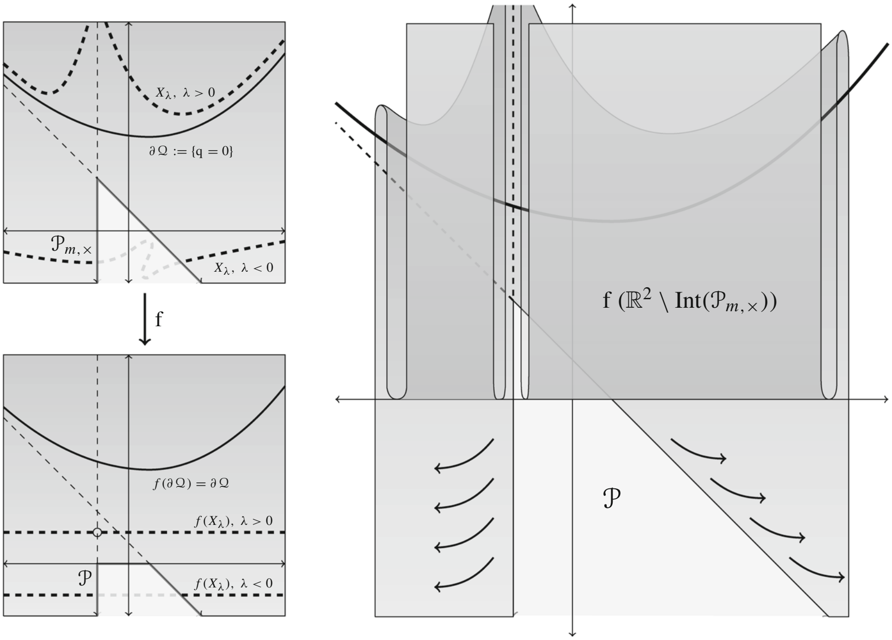


Fig. 1 A sketch of the behavior of the polynomial map $\mathtt{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$

to a continuous function on $\pi_n(\mathcal{K})$. As it is well-known, the separation of disjoint semialgebraic sets is a delicate issue and we refer the reader to [1] for further details.

1.6 Structure of the Article

All basic notions and (standard) notations appear in Sect. 2. In Sect. 3 we focus our attention in a special family of polynomial maps $\mathtt{f} := (\mathtt{f}_1, \mathtt{f}_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ and prove a key result (Theorem 3.3), which allows us to prove in Sect. 4 Theorem 1.4 by using induction on the dimension and reducing its proof to the two-dimensional scene, and has as a straightforward consequence Corollary 3.5, which is the two-dimensional version of Theorem 1.4 (see also [33, Thm. 1]). Figure 1 illustrates the behavior of these polynomial maps and should help understand how they work. In Sect. 5 we provide some rational separation results for certain types of (non-compact) semialgebraic sets. The separating polynomials arising from these results will be an important ingredient for constructing in Sect. 6 the polynomial maps needed to prove Theorem 1.5. A great deal of work here is devoted to comprehend how these maps act on 'vertical' lines in $\mathbb{R}^n$, that is, lines whose direction is generated by the 'vertical' vector $\vec{e}_n := (0, \ldots, 0, 1)$. An annoying difficulty which has to be dealt with is related to the intersections of the spans of facets of the target polyhedron. To circumvent this problem we construct a suitable *enveloping polyhedron* $\mathcal{K}_0$ of $\mathcal{K}$ (see Sect. 6.2), on whose complement we apply a sequence of our maps in order to obtain $\mathbb{R}^n \setminus \mathcal{K}$ as a polynomial image of $\mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K}_0)$. Figure 8 shows a two-dimensional sketch of how these maps act on complements of polyhedra and should give an idea on how we achieve our goal.

2 Preliminaries on Convex Polyhedra

We begin by introducing some preliminary terminology and notations concerning convex polyhedra. For a detailed study of the main properties of convex sets we refer the reader to [3,29,36]. An affine hyperplane of $\mathbb{R}^n$ will be written as $H := \{\mathtt{x} \in \mathbb{R}^n : \mathtt{h}(\mathtt{x}) = 0\} \equiv \{\mathtt{h} = 0\}$, where $\mathtt{h}$ is the corresponding linear equation. It determines two *closed half-spaces*

$$H^+ := \{\mathtt{x} \in \mathbb{R}^n : \mathtt{h}(\mathtt{x}) \geq 0\} \equiv \{\mathtt{h} \geq 0\} \quad \text{and}$$
$$H^- := \{\mathtt{x} \in \mathbb{R}^n : \mathtt{h}(\mathtt{x}) \leq 0\} \equiv \{\mathtt{h} \leq 0\}.$$

A linear subspace U in the vector space $\mathbb{R}^n$ is called *vertical* when it contains the vector $\vec{e}_n := (0, \dots, 0, 1) \in \mathbb{R}^n$. Otherwise, we say that U is *non-vertical*. Analogously, we say that an affine subspace $Z \subset \mathbb{R}^n$ is vertical (non-vertical) when its associated linear subspace $\vec{Z} := \{\overrightarrow{pq} : p, q \in Z\}$ is vertical (non-vertical). Notice that Z is vertical if and only if Z can be defined by a finite set of implicit linear equations that do not involve the variable $\mathtt{x}_n$, that is,

$$Z := \begin{cases} a_{11}\mathtt{x}_1 + \cdots + a_{1,n-1}\mathtt{x}_{n-1} = b_1, \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}\mathtt{x}_1 + \cdots + a_{m,n-1}\mathtt{x}_{n-1} = b_m, \end{cases} \qquad \text{where } a_{ij}, b_i \in \mathbb{R}.$$

In particular, *vertical vectors* are the non-zero multiples of $\vec{e}_n$ and *vertical lines* ℓ are those whose direction $\vec{\ell}$ is generated by $\vec{e}_n$. In general, given an affine object we will use an overlying arrow $\vec{\cdot}$ to denote its corresponding vectorial counterpart, whenever its meaning is clear.

The *vertical projection* is the linear projection

$$\pi_n \colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \; \mathtt{x} := (\mathtt{x}_1, \dots, \mathtt{x}_n) \mapsto \mathtt{x}' := (\mathtt{x}_1, \dots, \mathtt{x}_{n-1})$$

and the *vertical projection of a set* $S \subset \mathbb{R}^n$ is its image under π_n. We introduce the previous nomenclature with the aim of lightening the statements and the proofs in the sequel.

2.1 Generalities on Convex Polyhedra

A subset $\mathcal{K} \subset \mathbb{R}^n$ is a *convex polyhedron* if it can be described as the finite intersection $\mathcal{K} := \bigcap_{i=1}^r H_i^+$ of closed half-spaces H_i^+. The dimension $\dim(\mathcal{K})$ of $\mathcal{K}$ is the dimension of the smallest affine subspace of $\mathbb{R}^n$ that contains $\mathcal{K}$. If $\mathcal{K}$ has non-empty interior there exists by [3, 12.1.5] a unique minimal family $\{H_1, \dots, H_m\}$ of affine hyperplanes in $\mathbb{R}^n$ such that $\mathcal{K} = \bigcap_{i=1}^m H_i^+$. This family is the *minimal presentation* of $\mathcal{K}$. We assume that we choose the linear equation $\mathtt{h}_i$ of each H_i so that $\mathcal{K} \subset H_i^+$. For inductive processes we will write $\mathcal{K}_{i,\times} := \bigcap_{j \neq i} H_j^+$, which is a convex polyhedron that strictly contains $\mathcal{K}$, satisfies $\mathcal{K} = \mathcal{K}_{i,\times} \cap H_i^+$ and has one facet less than $\mathcal{K}$.

2.1.1. The *facets* or $(n-1)$-*faces* of $\mathcal{K}$ are the intersections $\mathcal{F}_i := H_i \cap \mathcal{K}$ for $1 \leq i \leq m$. Only the convex polyhedron $\mathbb{R}^n$ has no facets. Each facet $\mathcal{F}_i := H_i^- \cap \bigcap_{j=1}^m H_j^+$ is a convex polyhedron contained in H_i. The convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a topological manifold with boundary, whose interior is $\operatorname{Int}(\mathcal{K}) = \bigcap_{i=1}^m (H_i^+ \setminus H_i)$ and its boundary is $\partial\mathcal{K} = \bigcup_{i=1}^m \mathcal{F}_i$. For $0 \leq j \leq n-2$ we define inductively the j-*faces* of $\mathcal{K}$ as the facets of the $(j+1)$-faces of $\mathcal{K}$, which are again convex polyhedra. The 0-faces are the *vertices* of $\mathcal{K}$ and the 1-faces are the *edges* of $\mathcal{K}$. A face $\mathcal{E}$ of $\mathcal{K}$ is *vertical* if the affine subspace of $\mathbb{R}^n$ generated by $\mathcal{E}$ is vertical, that is, if its linear implicit equations do not involve the variable x_n. Otherwise, we say that $\mathcal{E}$ is *non-vertical*. In particular, if $\mathcal{E} := \mathcal{F}$ is a facet of $\mathcal{K}$, it is vertical if and only if an implicit linear equation of the affine hyperplane H spanned by $\mathcal{F}$ is of the type $a_0 + a_1\mathrm{x}_1 + \cdots + a_{n-1}\mathrm{x}_{n-1} = 0$. Thus, non vertical facets generate affine hyperplanes that admit implicit linear equations of the type $a_0 + a_1\mathrm{x}_1 + \cdots + a_{n-1}\mathrm{x}_{n-1} + \mathrm{x}_n = 0$.

Obviously, if $\mathcal{K}$ has a vertex, then $m \geq n$. A convex polyhedron of $\mathbb{R}^n$ is *non-degenerate* if it has at least one vertex. Otherwise, we say that the convex polyhedron is *degenerate*.

2.1.2. A *supporting hyperplane* of a convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ is a hyperplane H of $\mathbb{R}^n$ that meets $\mathcal{K}$ and satisfies $\mathcal{K} \subset H^+$ or $\mathcal{K} \subset H^-$. This is equivalent to have $\varnothing \neq \mathcal{K} \cap H \subset \partial\mathcal{K}$. The intersection of $\mathcal{K}$ with a supporting hyperplane H is a face of $\mathcal{K}$ and conversely each face of $\mathcal{K}$ is the intersection of $\mathcal{K}$ with some supporting hyperplane. In particular, the vertices of a convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ are those points $p \in \mathcal{K}$ for which there exists a (supporting) hyperplane $H \subset \mathbb{R}^n$ such that $\mathcal{K} \cap H = \{p\}$.

2.2 Projections of Convex Polyhedra

Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1},\ \mathrm{x} := (\mathrm{x}', \mathrm{x}_n) \to \mathrm{x}'$ be the projection onto the first $n-1$ coordinates. We denote the origin of $\mathbb{R}^n$ with $\mathbf{0}$ and that of $\mathbb{R}^{n-1}$ with $\mathbf{0}'$. Let $\mathcal{P} := \pi_n(\mathcal{K})$ and let $\vec{\ell}_n$ be the line generated by $\vec{e}_n := (0, \ldots, 0, 1) = (\mathbf{0}', 1)$. By [29, II, Thm. 6.6] we have $\pi_n(\operatorname{Int}(\mathcal{K})) = \operatorname{Int}(\mathcal{P})$ and consequently $\pi_n^{-1}(\partial\mathcal{P}) \cap \mathcal{K} \subset \partial\mathcal{K}$. Thus, if ℓ is a vertical line and $\pi_n(\ell) \subset \partial\mathcal{P}$, then

$$\ell \cap \mathcal{K} = \pi_n^{-1}(\pi_n(\ell)) \cap \mathcal{K} \subset \pi_n^{-1}(\partial\mathcal{P}) \cap \mathcal{K} \subset \partial\mathcal{K}.$$

In fact, $\pi_n^{-1}(\partial\mathcal{P}) \cap \mathcal{K}$ *is a union of faces of* $\mathcal{K}$.

Indeed, let $\mathcal{F}'$ be a facet of $\mathcal{P}$ and let H' be the hyperplane of $\mathbb{R}^{n-1}$ generated by $\mathcal{F}'$. Notice that $H := \pi_n^{-1}(H') = (H' \times \{0\}) + \vec{\ell}_n$ is a hyperplane of $\mathbb{R}^n$ that meets $\mathcal{K}$ but does not meet $\operatorname{Int}(\mathcal{K})$. Thus, H is a supporting hyperplane of $\mathcal{K}$ and $H \cap \mathcal{K} =: \mathcal{E}$ is a face of $\mathcal{K}$. Therefore

$$\pi_n^{-1}(\mathcal{F}') \cap \mathcal{K} = \pi_n^{-1}(H' \cap \mathcal{P}) \cap \mathcal{K} = \pi_n^{-1}(H') \cap \pi_n^{-1}(\mathcal{P}) \cap \mathcal{K} = H \cap \mathcal{K} = \mathcal{E}$$

is a face of $\mathcal{K}$.

2.3 Recession Cone of a Convex Polyhedron

We associate to each convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ its *recession cone* (see [36, Chap. 1] or [29, II, §8]), defined as

$$\vec{\mathfrak{C}}(\mathcal{K}) := \{\vec{v} \in \mathbb{R}^n : \forall p \in \mathcal{K}, \forall \lambda > 0,\ p + \lambda\vec{v} \in \mathcal{K}\},$$

which is a polyhedral convex cone. If $\mathcal{K} := \bigcap_{i=1}^r H_i^+$, then $\vec{\mathfrak{C}}(\mathcal{K}) := \bigcap_{i=1}^r \vec{\mathfrak{C}}(H_i^+) = \bigcap_{i=1}^r \vec{H_i}^+$. Clearly, $\vec{\mathfrak{C}}(\mathcal{K}) = \{\mathbf{0}\}$ if and only if $\mathcal{K}$ is bounded. In addition, if $\mathcal{P} \subset \mathbb{R}^n$ is a non-degenerate convex polyhedron and $k \geq 1$, then $\vec{\mathfrak{C}}(\mathbb{R}^k \times \mathcal{P}) = \mathbb{R}^k \times \vec{\mathfrak{C}}(\mathcal{P})$. Recall that each degenerate convex polyhedron can be written as the product of a non-degenerate convex polyhedron times an Euclidean space. Besides, a convex polyhedron is degenerate if and only if it contains a line or, equivalently, if its recession cone contains a line. Consequently, a convex polyhedron is non-degenerate if and only if all its faces are non-degenerate polyhedra. We recall the following interpretation of the recession cone for the sake of intuition.

Remark 2.1 (*Projective interpretation of the recession cone*) Let us embed $\mathbb{R}^n$ inside the real projective space $\mathbb{RP}^n$ by means of the usual embedding

$$\mathbb{R}^n \hookrightarrow \mathbb{RP}^n,\ x := (x_1, \ldots, x_n) \mapsto [1 : x] := [1 : x_1 : \ldots : x_n].$$

Consider also the map

$$\varphi_0 \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{RP}^n,\ \vec{v} := (v_1, \ldots, v_n) \mapsto [0 : \vec{v}] := [0 : v_1 : \ldots : v_n].$$

Denote by $\mathsf{H}_\infty := \{\mathrm{x}_0 = 0\} = \mathbb{RP}^n \setminus \mathbb{R}^n$ the hyperplane of $\mathbb{RP}^n$ at infinity (with respect to the embedding of $\mathbb{R}^n$ in $\mathbb{RP}^n$ described above), which is the image of φ_0. Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron, let $\overline{\mathcal{K}}$ be the closure of $\mathcal{K}$ in $\mathbb{RP}^n$ and denote $\mathcal{K}_\infty := \overline{\mathcal{K}} \cap \mathsf{H}_\infty$. We claim:

$$\mathcal{K}_\infty = \varphi_0(\vec{\mathfrak{C}}(\mathcal{K})) = \varphi_0(\{\vec{v} \in \mathbb{R}^n : \forall \lambda > 0,\ p + \lambda\vec{v} \in \mathcal{K}\}) \tag{B.1}$$

for each $p \in \mathcal{K}$.

Fix any point $p \in \mathcal{K}$ and let $[0 : \vec{v}] \in \mathcal{K}_\infty$. Consider the projective line L that passes through p and $[0 : \vec{v}]$. Observe that $L \cap \mathsf{H}_\infty = \{[0 : \vec{v}]\}$. If $L \subset \overline{\mathcal{K}}$, then the half-line $T' := \{p + \lambda\vec{v} : \lambda \geq 0\} \subset \mathcal{K}$. Otherwise, we pick a point $p' \in L \setminus \overline{\mathcal{K}}$ and let H be a hyperplane through p' such that $\mathcal{K} \cap H^- = \varnothing$. To construct H, write $\mathcal{K} := \bigcap_{i=1}^r \{h_i \geq 0\}$ where each $h_i \in \mathbb{R}[\mathrm{x}]$ is a polynomial of degree one. Assume that $h_1(p') < 0$ and let $\alpha, \beta > 0$ be such that $(\alpha h_1 + h_2 + \cdots + h_r)(p') < 0$ and $(\alpha h_1 + h_2 + \cdots + h_r + \beta)(p') = 0$. Define $h := \alpha h_1 + h_2 + \cdots + h_r + \beta$ and $H := \{h = 0\}$. The hyperplane H satisfies the requirements. Let $\mathsf{H} \subset \mathbb{RP}^n$ be the projective completion of H. It holds that $\overline{\mathcal{K}} \setminus \mathsf{H}$ is a convex polyhedron of the affine space $\mathbb{RP}^n \setminus \mathsf{H}$. Consequently, the segment that connects p and $[0 : \vec{v}]$ in $\mathbb{RP}^n \setminus \mathsf{H}$ is contained in $\overline{\mathcal{K}} \setminus \mathsf{H}$. This means that (after changing $\vec{v}$ by $-\vec{v}$ if necessary) the half-line $T' := \{p + \lambda\vec{v} : \lambda \geq 0\} \subset \mathcal{K}$. As $h(p + \lambda\vec{v}) > 0$ for each $\lambda > 0$ and $h(p') = 0$, we deduce that $\vec{h}(\vec{v}) > 0$ (where $\vec{h} := h - h(0)$).

Let $q \in \mathcal{K}$ be another point and consider the line $\ell := \{q + t\vec{v} : t \in \mathbb{R}\}$. We know that one of the half-lines $T_1 := \{q + \lambda\vec{v} : \lambda \geq 0\}$ or $T_2 := \{q - \lambda\vec{v} : \lambda \geq 0\}$ is contained in $\mathcal{K}$ (follow the same argument we have done for p). As $\vec{h}(\vec{v}) > 0$ and $\mathcal{K} \cap H^- = \varnothing$, we conclude that $T_2 \not\subset \mathcal{K}$, so $T_1 \subset \mathcal{K}$. Consequently,

$$\mathcal{K}_\infty \subset \varphi_0(\vec{\mathfrak{C}}(\mathcal{K})) \subset \varphi_0(\{\vec{v} \in \mathbb{R}^n : \forall\lambda > 0,\ p + \lambda\vec{v} \in \mathcal{K}\}).$$

Conversely, fix $p \in \mathcal{K}$ and let $\vec{v} \in \mathbb{R}^n$ be a vector such that the half-line $T := \{p + \lambda\vec{v} : \lambda \geq 0\} \subset \mathcal{K}$. Then its closure $\overline{T}$ meets H_∞ in exactly one point, which is $[0 : \vec{v}]$ and belongs to $\overline{\mathcal{K}}$. In particular, $[0 : \vec{v}] \in \overline{\mathcal{K}} \cap \mathsf{H}_\infty = \mathcal{K}_\infty$. We conclude $\varphi_0(\{\vec{v} \in \mathbb{R}^n : \forall\lambda > 0,\ p + \lambda\vec{v} \in \mathcal{K}\}) \subset \mathcal{K}_\infty$, so (B.1) holds.

Now, it is a straightforward exercise left to the reader to check that given any point $p \in \mathcal{K}$ the recession cone $\vec{\mathfrak{C}}(\mathcal{K})$ coincides with the set $\{\vec{v} \in \mathbb{R}^n : \forall\lambda > 0,\ p+\lambda\vec{v} \in \mathcal{K}\}$. We will use this fact freely along this work. □

We will frequently use a convenient way to place an unbounded n-dimensional polyhedron in its ambient space. Consider the linear projection $\pi' : \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto x_n$.

Lemma 2.2 *Let $\mathcal{K} \subset \mathbb{R}^n$ be an unbounded n-dimensional convex polyhedron with $m \geq 3$ facets $\mathcal{F}_1,\dots,\mathcal{F}_m$. After an affine change of coordinates and reindexing the facets we may assume that $\mathcal{K}$ satisfies the following properties:*

(i) *The facet $\mathcal{F}_m$ spans the hyperplane $\{\mathrm{x}_n = 0\}$ and $\mathcal{K} \subset \{\mathrm{x}_n \leq 0\}$.*
(ii) *The origin $\mathbf{0} := (0, \dots, 0) \in \mathrm{Int}(\mathcal{F}_m)$.*
(iii) $\pi'(\mathcal{K}) = \pi'(\mathcal{K} \cap \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_{n-1} = 0\})$.
(iv) *The vector $\vec{e}_n \notin \mathcal{K}_{m,\times}$.*

Proof Suppose first that there exists a facet $\mathcal{F}$ of $\mathcal{K}$ such that $\dim(\vec{\mathfrak{C}}(\mathcal{F})) = \dim(\vec{\mathfrak{C}}(\mathcal{K}))$. After an affine change of coordinates and reindexing the facets we may assume that $\mathcal{F}_m := \mathcal{F}$ spans the hyperplane $\{\mathrm{x}_n = 0\}$ and $\mathcal{K} \subset \{\mathrm{x}_n \leq 0\}$. As the convex cones $\vec{\mathfrak{C}}(\mathcal{F}) \subset \vec{\mathfrak{C}}(\mathcal{K})$ and both have the same dimension, we conclude $\vec{\mathfrak{C}}(\mathcal{K}) \subset \{\mathrm{x}_n = 0\}$. As $\dim(\mathcal{K}) = n$, we can choose a finite set $\mathcal{W}$ that contains all the vertices of $\mathcal{K}$ and spans $\mathbb{R}^n$. Let $\mathcal{K}_0$ be the convex hull of $\mathcal{W}$. It holds $\mathcal{K} = \mathcal{K}_0 + \vec{\mathfrak{C}}(\mathcal{K})$. As $\vec{\mathfrak{C}}(\mathcal{K}) \subset \{\mathrm{x}_n = 0\}$, we have $\pi'(\mathcal{K}) = \pi'(\mathcal{K}_0)$, which is a compact interval $[-M, 0]$ of $\mathbb{R}$ (for some $M > 0$). As $m \geq 3$, then $\mathcal{K}$ is not a layer and there exists a facet $\mathcal{F}'$ of $\mathcal{K}$ that spans a hyperplane $H' := \{\mathrm{h}' = 0\}$ of $\mathbb{R}^n$ that is not parallel to $\{\mathrm{x}_n = 0\}$. Pick a vector $\vec{v} \in \vec{\mathfrak{C}}(\mathcal{K}) \setminus \vec{H}'$. Pick points $p \in \mathrm{Int}(\mathcal{F}_m)$ and $q \in \partial\mathcal{K}$ such that $\pi'(p) = -M$. If we set $\mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda \geq 0\}$, the ray $q + \mathbb{R}^+\vec{v} \subset \mathcal{K}$ and for each point $x \in q + \mathbb{R}^+\vec{v}$ we have $\pi'(x) = -M$, so we may assume $\vec{\mathrm{h}}'(\vec{pq}) > 0$. After an affine change of coordinates that keeps the hyperplane $\{\mathrm{x}_n = 0\}$ invariant, we may assume that p is the origin and $\vec{pq} = -M\vec{e}_n$. Thus, (i), (ii) and (iv) hold. Also (iii) holds: If $(p', p_n) \in \mathcal{K}$, then $-M \leq p_n \leq 0$, so $(\mathbf{0}', p_n) \in \mathcal{K}$ (because $\mathcal{K}$ is convex and both the origin and the point $(\mathbf{0}', -M)$ belong to $\mathcal{K}$).

Assume next that $\dim(\vec{\mathfrak{C}}(\mathcal{F})) < \dim(\vec{\mathfrak{C}}(\mathcal{K}))$ for each facet $\mathcal{F}$ of $\mathcal{K}$. Pick a facet $\mathcal{F}$ of $\mathcal{K}$ such that $\dim(\vec{\mathfrak{C}}(\mathcal{F}_i)) \leq \dim(\vec{\mathfrak{C}}(\mathcal{F}))$ for $i = 1, \dots, m$. We may assume $\mathcal{F} = \mathcal{F}_m$ spans the hyperplane $\{\mathrm{x}_n = 0\}$ and $\mathcal{K} \subset \{\mathrm{x}_n = 0\}$. As $m \geq 3$, then $\mathcal{K}$ is not a layer and there exists a facet $\mathcal{F}'$ of $\mathcal{K}$ that spans a hyperplane $H' := \{\mathtt{h}' = 0\}$ of $\mathbb{R}^n$ and that is not parallel to $\{\mathtt{x}_n = 0\}$. Pick a vector $\vec{v} \in \vec{\mathfrak{C}}(\mathcal{K})\setminus(\vec{H}' \cup \{\mathtt{x}_n = 0\})$ and a point $p \in \operatorname{Int}(\mathcal{F}_m)$. After an affine change of coordinates that keeps invariant the hyperplane $\{\mathtt{x}_n = 0\}$ we may assume that p is the origin, $\vec{v} = -\vec{e}_n$ and $\vec{\mathtt{h}}'(\vec{v}) > 0$. Thus, (i), (ii) and (iv) hold. Let us check that it also holds (iii). If $(p', p_n) \in \mathcal{K}$, then $p_n \leq 0$, so $(\mathbf{0}', p_n) \in \mathcal{K}$ (because the origin belongs to $\mathcal{K}$ and $-\vec{e}_n \in \vec{\mathfrak{C}}(\mathcal{K})$), as required. □

3 Complements of Interiors of Convex Polygons

In this section we construct, for a convex polygon $\mathcal{P} \subset \mathbb{R}^2$ defined by a convenient set of m linear inequalities described in 3.1, a polynomial map $\mathtt{f} := (\mathtt{f}_1, \mathtt{f}_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathtt{f}(\mathbb{R}^2\setminus \operatorname{Int}(\mathcal{P}_{m,\times})) = \mathbb{R}^2\setminus \operatorname{Int}(\mathcal{P})$. Here, the polygon $\mathcal{P}_{m,\times}$ is defined by the first $m-1$ linear inequalities defining $\mathcal{P}$ (see Theorem 3.3), and has one facet less than $\mathcal{P}$. Figure 1 illustrates the behaviour of the map $\mathtt{f}$. This polynomial map $\mathtt{f}$ is the key to prove Theorem 1.4. To fully understand the behaviour of $\mathtt{f}$, we study carefully in Sect. 3.4 the level curves $X_\lambda := \{\mathtt{f}_2(\mathtt{y}, \mathtt{z}) = \lambda\}$ $(\lambda \in \mathbb{R})$, so that later we can determine precisely the set $\mathtt{f}(\mathbb{R}^2\setminus \operatorname{Int}(\mathcal{K}_{m,\times})) = \bigcup_{\lambda\in\mathbb{R}} \mathtt{f}(X_\lambda\setminus \operatorname{Int}(\mathcal{K}_{m,\times}))$. As a consequence of Theorem 3.3 we provide in Corollary 3.5 a straightforward proof of Theorem 1.4 for $n = 2$. The reader can compare this proof with the one in [33, Thm. 1].

3.1 Choice of Suitable Linear Equations

Let us consider $m \geq 2$ linear equations $\{\bar{\mathtt{l}}_k(\mathtt{y}, \mathtt{z}), k = 1, \dots, m\}$ that can be expressed as follows:

$$\bar{\mathtt{l}}_k(\mathtt{y},\mathtt{z}) := \begin{cases} \mathtt{l}_{0k}(\mathtt{y}) := c_k(\mathtt{y} - b_k) & \text{if } k = 1, \dots, r, \\ \epsilon_k \mathtt{l}_k(\mathtt{y},\mathtt{z}) := \epsilon_k(\mathtt{l}_{0k}(\mathtt{y}) - \mathtt{z}) & \text{if } k = r+1, \dots, m-1, \\ \qquad := \epsilon_k(a_k\mathtt{y} + b_k - \mathtt{z}) & \\ -\mathtt{z} & \text{if } k = m. \end{cases} \tag{C.1}$$

Here, $r < m - 1$ (so that there is at least one equation of the second type), $b_k c_k \neq 0$ for $k = 1, \dots, r$ and $\epsilon_k \in \{-1, 1\}$ for $k = r+1, \dots, m-1$. We allow redundancy in the collection of equations but we ask that at least one of the ϵ_k is equal to $+1$. Notice that $\bar{\mathtt{l}}_k(\mathtt{y}, \mathtt{z}) = 0$ corresponds to a vertical line for $k = 1, \dots, r$ and to a non-vertical line for $k = r+1, \dots, m$, which is the horizontal axis when $k = m$. We define now the polygons

$$\mathcal{P} := \bigcap_{k=1}^{m} \{\bar{\mathtt{l}}_k \geq 0\} \subset \mathbb{R}^2 \quad \text{and} \quad \mathcal{P}_{m,\times} := \bigcap_{k=1}^{m-1} \{\bar{\mathtt{l}}_k \geq 0\} \subset \mathbb{R}^2, \tag{C.2}$$

and consider the projection $\pi'\colon \mathbb{R}^2 \to \mathbb{R},\ (y, z) \to z$. Besides, we assume two extra conditions:

- $(0, 0) \in \operatorname{Int}(\mathcal{P}_{m,\times})$,
- $\pi'(\mathcal{P}) = \pi'(\mathcal{P} \cap \{\mathtt{y} = 0\})$.

The conditions above imply that the unbounded two-dimensional convex polygon $\mathcal{P}$ satisfies properties (i)–(iv) in Lemma 2.2. Namely,

(i) The line $\mathtt{l}_m(\mathtt{y}, \mathtt{z}) = -\mathtt{z} = 0$ is spanned by one of the edges, say $\mathcal{E}$, of $\mathcal{P}$ and $\mathcal{P} \subset \{\mathtt{z} \leq 0\}$.
(ii) The origin $(0, 0) \in \mathrm{Int}(\mathcal{E})$.
(iii) $\pi'(\mathcal{P}) = \pi'(\mathcal{P} \cap \{\mathtt{y} = 0\})$.
(iv) The vector $\vec{e}_2 \notin \mathcal{P}_{m,\times}$ (this follows from the fact that $\mathcal{P}_{m,\times}$ is contained in a half-plane of the form $\mathtt{l}_{0k}(\mathtt{y}) - \mathtt{z} \geq 0$).

3.2 Auxiliary Polynomials

Now we introduce the auxiliary polynomials

$$\begin{aligned}
\mathtt{q}(\mathtt{y}, \mathtt{z}) &:= \mathtt{z} - \mathtt{y}^2 - 1 - \sum_{i=r+1}^{m-1} \frac{\mathtt{l}_{0i}{}^2(\mathtt{y}) + 1}{2}, \\
\mathtt{g}(\mathtt{y}, \mathtt{z}) &:= \Big(\prod_{j=1}^{r} \mathtt{l}_{0j}(\mathtt{y}) \Big)^2 \cdot \Big(\prod_{i=r+1}^{m-1} \mathtt{l}_i(\mathtt{y}, \mathtt{z}) \Big), \\
\mathtt{p}(\mathtt{y}, \mathtt{z}) &:= 1 - \mathtt{q}(\mathtt{y}, \mathtt{z})\, \mathtt{g}^2(\mathtt{y}, \mathtt{z}).
\end{aligned} \tag{C.3}$$

Some properties of these polynomials will be relevant to us.

Lemma 3.1 *The region* $\mathcal{Q} := \{\mathtt{q} > 0\}$ *satisfies:*

(i) $\mathcal{Q} \subset \{\mathtt{z} - \mathtt{y}^2 - 1 > 0\} \subset \{\mathtt{z} - |\mathtt{y}| > 0\} \subset \{\mathtt{z} > 0\} = \{\mathtt{l}_m < 0\}$. *Besides,* $\mathcal{Q}$ *is connected and its vertical projection covers* $\{\mathtt{y} = 0\}$.
(ii) $\mathcal{Q} \subset \bigcap_{i=r+1}^{m} \{\mathtt{l}_i < 0\}$ *and* $\mathcal{Q} \subset \{\bar{\mathtt{l}}_k < 0\}$ *for some* $k \in \{r + 1, \dots, m - 1\}$. *In particular,* $\mathcal{P} \cap \mathcal{Q} = \varnothing$ *and* $\mathcal{P}_{m,\times} \cap \mathcal{Q} = \varnothing$.
(iii) *Let* $M > 0$ *be such that* $1 + |a_i| + |b_i| < M$ *for* $i = r + 1, \dots, m - 1$. *Then*

$$\mathcal{Q} \subset \bigcap_{i=r+1}^{m-1} \{M\mathtt{z} - |\mathtt{l}_{0i}(\mathtt{y}) - \mathtt{z}| > 0\}.$$

Proof (i) This is trivial.

(ii) Since $r < m - 1$, fix $i_0 \in \{r + 1, \dots, m - 1\}$ and $(y_0, z_0) \in \mathcal{Q}$. We have $\mathtt{q}(y_0, z_0) > 0$, so

$$z_0 > y_0^2 + 1 + \sum_{i=r+1}^{m-1} \left(\frac{\mathtt{l}_{0i}^2(y_0) + 1}{2} \right) \geq \frac{\mathtt{l}_{0i_0}^2(y_0) + 1}{2} \geq \mathtt{l}_{0i_0}(y_0).$$

Therefore, $\mathtt{l}_{i_0}(y_0, z_0) = \mathtt{l}_{0i_0}(y_0) - z_0 < 0$ and $(y_0, z_0) \in \{\mathtt{l}_{i_0} < 0\}$.

As $\epsilon_k = +1$ for at least one index $k \in \{r+1, \dots, m-1\}$, we have for such k that $\bar{\mathtt{l}}_k = \epsilon_k \mathtt{l}_k = \mathtt{l}_k$. Consequently,

$$\mathcal{Q} \subset \bigcap_{i=r+1}^{m} \{\mathtt{l}_i < 0\} \subset \{\mathtt{l}_k < 0\} = \{\bar{\mathtt{l}}_k < 0\},$$

so $\mathcal{P} \cap \mathcal{Q} = \varnothing$ and $\mathcal{P}_{m,\times} \cap \mathcal{Q} = \varnothing$.

(iii) Fix $i \in \{r+1, \dots, m-1\}$ and $(y_0, z_0) \in \mathcal{Q}$. By (i) we have $|y_0| < z_0 = |z_0|$ and $z_0 > 1$. Thus,

$$\begin{aligned} |\mathtt{l}_{0i}(y_0) - z_0| = |a_i y_0 + b_i - z_0| &\le |a_i||y_0| + |b_i| + |z_0| \\ &\le |a_i||z_0| + |b_i||z_0| + |z_0| < M z_0, \end{aligned}$$

as required. □

Lemma 3.2 *For each $y_0 \in U := \mathbb{R} \backslash \{b_1, \dots, b_r\}$ there exists*

$$z_1 > z_0 := y_0^2 + 1 + \sum_{i=r+1}^{m-1} \frac{\mathtt{l}_{0i}^2(y_0) + 1}{2}$$

such that $\mathtt{p}(y_0, z_1) = 0$. In particular, $(y_0, z_1) \in \mathcal{Q}$.

Proof Let us consider the odd degree polynomial

$$\mathtt{p}_{y_0}(\mathtt{z}) := \mathtt{p}(y_0, \mathtt{z}) = 1 - \mathtt{q}(y_0, \mathtt{z})\, \mathtt{g}^2(y_0, \mathtt{z}).$$

and observe that $\mathtt{q}(y_0, z_0) = 0$. As $\lim_{z \to +\infty} \mathtt{p}_{y_0}(\mathtt{z}) = -\infty$ and $\mathtt{p}_{y_0}(z_0) = 1$, there exists $z_1 > z_0$ such that $\mathtt{p}_{y_0}(z_1) = 0$, as required. □

3.3 'Winning' Polynomial Map

Let us consider now the polynomial map $\mathtt{f} := (\mathtt{f}_1, \mathtt{f}_2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{aligned} \mathtt{f}_1(\mathtt{y}, \mathtt{z}) &:= \mathtt{y}((\mathtt{p}(\mathtt{y}, \mathtt{z}) - 1)^2 + \mathtt{p}^2(\mathtt{y}, \mathtt{z})), \\ \mathtt{f}_2(\mathtt{y}, \mathtt{z}) &:= \mathtt{z}\mathtt{p}^2(\mathtt{y}, \mathtt{z}), \end{aligned} \tag{C.4}$$

where $\mathtt{p} \in \mathbb{R}[\mathtt{y}, \mathtt{z}]$ is the polynomial introduced in (C.3).

Our main result in this section is the following.

Theorem 3.3 *For a non-degenerate unbounded convex polygon $\mathcal{P}$ described as in Sect. 3.1 the polynomial map $\mathtt{f} := (\mathtt{f}_1, \mathtt{f}_2) \colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies*

$$\mathbb{R}^2 \backslash \operatorname{Int}(\mathcal{P}) = \mathtt{f}(\mathbb{R}^2 \backslash \operatorname{Int}(\mathcal{P}_{m,\times})).$$

Let us apply Theorem 3.3 to show Theorem 1.4 when $n = 2$. We recall first how one can represent as a polynomial image of $\mathbb{R}^2$ the complement of the interior of an (unbounded) convex polygon $\mathcal{P} \subset \mathbb{R}^2$ with one or two edges.

Examples 3.4 (i) If $\mathcal{P} \subset \mathbb{R}^2$ has one edge, we may assume $\mathcal{P} := \{\mathtt{z} \leq 0\}$, so $\mathcal{S} := \mathbb{R}^2 \setminus \mathcal{P} = \{\mathtt{z} \geq 0\}$. The image of the polynomial map $\mathtt{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(\mathtt{x}_1, \mathtt{x}_2) \mapsto (\mathtt{x}_1, \mathtt{x}_2^2)$ is $\mathcal{S}$ and $\deg(\mathtt{f}) = 2$.

(ii) If $\mathcal{P} \subset \mathbb{R}^2$ has two edges, we may assume $\mathcal{P} := \{\mathtt{y} \geq 0, \mathtt{z} \geq 0\}$. Define

$$\begin{aligned} \mathtt{f}\colon \mathbb{R}^2 &\to \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{C} \equiv \mathbb{R}^2, \\ (\mathtt{x}_1, \mathtt{x}_2) &\mapsto (\mathtt{x}_1^2, \mathtt{x}_2^2) \equiv \mathtt{x}_1^2 + \sqrt{-1}\,\mathtt{x}_2^2 =: \omega \mapsto \overline{\omega}^3 = (\mathtt{x}_1^2 - \sqrt{-1}\,\mathtt{x}_2^2)^3 \\ &\qquad = (\mathtt{x}_1^6 - 3\mathtt{x}_1^2\mathtt{x}_2^4) - \sqrt{-1}\,(3\mathtt{x}_1^4\mathtt{x}_2^2 - \mathtt{x}_2^6) \\ &\qquad \equiv (\mathtt{x}_1^6 - 3\mathtt{x}_1^2\mathtt{x}_2^4, \mathtt{x}_2^6 - 3\mathtt{x}_1^4\mathtt{x}_2^2). \end{aligned}$$

The image of $\mathtt{f}$ is $\mathcal{S}$, as it maps first $\mathbb{R}^2$ to the closed quadrant $\{\mathtt{x}_1 \geq 0, \mathtt{x}_2 \geq 0\}$ and then this one to $\mathcal{S}$ using the complex operation $\omega \mapsto \overline{\omega}^3$. In addition, $\deg(\mathtt{f}) = 6$.

Corollary 3.5 *Let $\mathcal{P} \subset \mathbb{R}^2$ be an unbounded convex polyhedron that does not disconnect $\mathbb{R}^2$. Then $\mathcal{S} := \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$ is a polynomial image of $\mathbb{R}^2$.*

Proof We proceed by induction on the number m of edges of $\mathcal{P}$.

First cases. The cases $m = 1$ and $m = 2$ were provided in Example 3.4.

Induction step. Let $\mathcal{P} := \bigcap_{i=1}^m \{\bar{\mathtt{l}}_i(\mathtt{y}, \mathtt{z}) \geq 0\}$ be a convex polygon with $m \geq 3$ edges. After an affine change of coordinates we may assume that properties (i)–(iv) from Lemma 2.2 are satisfied. This in turn implies that, ordering adequately the edges of the polygon, we can express the equations of the $\bar{\mathtt{l}}_i$ as in (C.1), so they satisfy all the required conditions in Sect. 3.1.

Now, set $\mathcal{P}_{m,\times} := \bigcap_{i=1}^{m-1} \{\bar{\mathtt{l}}_i(\mathtt{y}, \mathtt{z}) \geq 0\}$. By the induction hypothesis there exists a polynomial map $\mathtt{h}_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathtt{h}_0(\mathbb{R}^2) = \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P}_{m,\times})$. By Theorem 3.3 we have $\mathtt{f}(\mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P}_{m,\times})) = \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$. Thus, the image of the polynomial map $\mathtt{F} := \mathtt{f} \circ \mathtt{h}_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathcal{S} := \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$, as required. □

Before proving Theorem 3.3 we need to study carefully the level curves of $\mathtt{f}_2$.

3.4 On the Level Curves of $\mathtt{f}_2$

For each $\lambda \in \mathbb{R}$ consider the plane algebraic curve

$$X_\lambda := \{\mathtt{f}_2 = \lambda\}.$$

The properties of these algebraic curves will help us prove later the equality $\mathtt{f}(X_\lambda \setminus \operatorname{Int}(\mathcal{P}_{m,\times})) = \{\mathtt{z} = \lambda\} \setminus \operatorname{Int}(\mathcal{P})$, which essentially represents the core of the proof. Propositions 3.6 and 3.7 provide semialgebraic parameterizations of (a part of) X_λ. The properties of these parameterizations depend strongly on the sign of λ.

The reader should have in mind Fig. 1, which sketches the behavior of the polynomial map $\mathtt{f}$ and helps understand how it acts on $\mathbb{R}^2$. In the sequel we make use of Nash functions on open semialgebraic subsets $U \subset \mathbb{R}$. Recall that an analytic function $\mathtt{f}\colon U \to \mathbb{R}$ is *Nash* if there exists a non-zero polynomial $\mathtt{P} \in \mathbb{R}[\mathtt{x}, \mathtt{y}]$ such that $\mathtt{P}(\mathtt{t}, \mathtt{f}(\mathtt{t})) = 0$ for each $\mathtt{t} \in U$.

Proposition 3.6 *Assume* $\lambda > 0$ *and define* $\varphi_\lambda(\mathtt{y}) := \max\{\mathtt{z} \in \mathbb{R} : (\mathtt{y}, \mathtt{z}) \in X_\lambda\}$ *for each* $\mathtt{y} \in U := \mathbb{R}\setminus\{b_1, \dots, b_r\}$. *Then*

(i) $\varphi_\lambda\colon U \to \mathbb{R}$ *is a Nash function and its graph* Γ_λ *is contained in* $X_\lambda \cap \mathcal{Q}$.
(ii) $\lim_{\mathtt{y}\to b_j} \varphi_\lambda(\mathtt{y}) = +\infty$ *for all* $j = 1, \dots, r$.

Proof We note first that: *the function* φ_λ *is well-defined on* U.

Fix $y_0 \in U$ and observe that $\mathtt{q}(y_0, \mathtt{z})$ is a polynomial of degree one. Thus, $\mathtt{p}(y_0, \mathtt{z})$ is a polynomial of degree ≥ 1 and $\mathtt{f}_2(y_0, \mathtt{z})$ is a polynomial of odd degree. Consequently, the set $\{\mathtt{f}_2(y_0, \mathtt{z}) = b\} \subset \mathbb{R}$ is non-empty and finite, so the value $\varphi_\lambda(y_0)$ exists.

3.6.1. The proof of (i) is conducted in several steps. We first claim: *The partial derivative* $\frac{\partial(\mathtt{q}\mathtt{g}^2)}{\partial \mathtt{z}}$ *is strictly positive on* $\mathcal{Q}_0 := \{\mathtt{q} > 0\} \cap (U \times \mathbb{R})$.

We have

$$\begin{aligned}\frac{\partial(\mathtt{q}\mathtt{g}^2)}{\partial \mathtt{z}} &= \frac{\partial \mathtt{q}}{\partial \mathtt{z}}\mathtt{g}^2 + 2\mathtt{q}\mathtt{g}\frac{\partial \mathtt{g}}{\partial \mathtt{z}}\\ &= \mathtt{g}^2 + 2\mathtt{q}\Bigg(\prod_{j=1}^{r} \mathtt{l}_j(\mathtt{y}, \mathtt{z})\Bigg)^4 \Bigg(\prod_{i=r+1}^{m-1} \mathtt{l}_i(\mathtt{y}, \mathtt{z})\Bigg)\Bigg(-\sum_{k=r+1}^{m-1}\prod_{i\neq k} \mathtt{l}_i(\mathtt{y}, \mathtt{z})\Bigg)\\ &= \mathtt{g}^2 + 2\mathtt{q}\Bigg(\prod_{j=1}^{r} \mathtt{l}_j(\mathtt{y}, \mathtt{z})\Bigg)^4 \Bigg(\sum_{k=r+1}^{m-1} (\mathtt{z} - \mathtt{l}_{0i}(\mathtt{y}))\prod_{i\neq k} \mathtt{l}_i(\mathtt{y}, \mathtt{z})^2\Bigg),\end{aligned}$$

and by Lemma 3.1 (ii) this last expression is strictly positive on $\mathcal{Q}_0$.

3.6.2. Fix $(y_0, \varphi_\lambda(y_0)) \in \Gamma_\lambda$ and denote

$$z_0 := y_0^2 + 1 + \sum_{i=r+1}^{m-1} \left(\frac{\mathtt{l}_{0i}^2(y_0) + 1}{2}\right).$$

We claim: *There exists* $z_0 < z_1 < z_2$ *such that* $\mathtt{p}(y_0, z_1) = 0$ *and* $\mathtt{f}_2(y_0, z_2) = \lambda$. *Besides, the graph* Γ_λ *is contained in* $\mathcal{Q}_0$ *and* $\mathtt{p}$ *is strictly negative on* Γ_λ.

Write $\mathcal{Q} \cap \{\mathtt{y} = y_0\} = \{(y_0, \mathtt{z}) : \mathtt{z} > z_0\}$. By Lemma 3.2 there exists $z_1 > z_0$ such that $\mathtt{p}(y_0, z_1) = 0$, so $\mathtt{f}_2(y_0, z_1) = 0$. As $\lambda > 0$ and $\lim_{\mathtt{z}\to+\infty} \mathtt{f}_2(y_0, \mathtt{z}) = +\infty$ (because it is an odd degree polynomial with positive leading coefficient), there exists $z_2 > z_1 > z_0$ such that $\mathtt{f}_2(y_0, z_2) = \lambda$. Thus,

$$\varphi_\lambda(y_0) \geq z_2 > z_1 > z_0, \tag{C.5}$$

so $(y_0, \varphi_\lambda(y_0)) \in \mathcal{Q} \cap \{\mathtt{y} = y_0\} \subset \mathcal{Q}_0$.

As $\frac{\partial(\mathtt{qg}^2)}{\partial \mathtt{z}} > 0$ on $\mathcal{Q}_0$, the polynomial function $\mathtt{p}_{y_0}(\mathtt{z}) := \mathtt{p}(y_0, \mathtt{z})$ is strictly decreasing on the interval $]z_0, +\infty)$. As $\mathtt{p}_{y_0}(z_1) = 0$ and $\varphi_\lambda(y_0) > z_1$, we deduce $\mathtt{p}_{y_0}(\varphi_\lambda(y_0)) < 0$.

3.6.3. Finally we prove: *The function φ_λ is Nash on U.*

Pick $y_0 \in U$. We know that $(y_0, \varphi_\lambda(y_0)) \in \mathcal{Q}_0$, so $\frac{\partial(\mathtt{qg}^2)}{\partial \mathtt{z}}(y_0, \varphi_\lambda(y_0)) > 0$. In addition, we know that $\mathtt{p}(y_0, \varphi_\lambda(y_0)) < 0$. It follows that

$$\frac{\partial \mathtt{f}_2}{\partial \mathtt{z}}(y_0, \varphi_\lambda(y_0)) = \mathtt{p}^2(y_0, \varphi_\lambda(y_0)) + \left(2\mathtt{zp} \cdot \left(-\frac{\partial(\mathtt{qg}^2)}{\partial \mathtt{z}}\right)\right)(y_0, \varphi_\lambda(y_0)) > 0.$$

By the Implicit Function Theorem [5, 2.9.8] there exist

- open bounded intervals $I, J \subset \mathbb{R}$ such that $y_0 \in I$ and $\varphi_\lambda(y_0) \in J$,
- $I \times J \subset \mathcal{Q}_0 \cap \{\mathtt{p} < 0\}$ and
- a Nash function $\phi\colon I \to J$ such that $X_\lambda \cap (I \times J) = \{\mathtt{z} = \phi(\mathtt{y})\}$.

Let us check: *After shrinking I, we have $\varphi_\lambda|_I = \phi|_I$.*

Indeed, suppose by contradiction that there exists a sequence $\{y_k\}_{k\geq 1} \subset I$ that converges to y_0 and $\varphi_\lambda(y_k) > \sup(J)$ for all $k \geq 1$. As $I \times J \subset \mathcal{Q}_0 \cap \{\mathtt{p} < 0\}$, we deduce

$$I \times]\inf(J), +\infty[\subset \mathcal{Q}_0 \cap \{\mathtt{p} < 0\}$$

because $\mathtt{p}$ is decreasing on the line $\mathcal{Q} \cap \{\mathtt{y} = y\}$ for all $y \in U$ (see the end of the proof of Sect. 3.6.2). In particular, $\mathtt{p}^2(y, \sup(J)) > 0$ for each $y \in I$. By Sect. 3.6.1

$$\frac{\partial \mathtt{p}^2}{\partial \mathtt{z}}(y, z) = -2\mathtt{p}(y, z)\,\frac{\partial(\mathtt{qg}^2)}{\partial \mathtt{z}}(y, z) > 0$$

for all $(y, z) \in I \times]\inf(J), +\infty[$. Thus, $\mathtt{p}^2(y_k, \varphi_\lambda(y_k)) > \mathtt{p}^2(y_k, \sup(J)) > 0$ for all $k \geq 1$. As $\mathtt{f}_2(y_k, \varphi_\lambda(y_k)) = \lambda$,

$$\sup(J) \leq \varphi_\lambda(y_k) = \frac{\lambda}{\mathtt{p}^2(y_k, \varphi_\lambda(y_k))} \leq \frac{\lambda}{\mathtt{p}^2(y_k, \sup(J))}.$$

As $\{y_k\}_{k\geq 1} \cup \{y_0\} \subset I$ is compact and the rational function $\frac{\lambda}{\mathtt{p}^2(\mathtt{y}, \sup(J))}$ is continuous on I, there exists $M > 0$ such that

$$\sup(J) < \varphi_\lambda(y_k) \leq \frac{\lambda}{\mathtt{p}^2(\mathtt{y}, \sup(J))} < M.$$

As $K := X_\lambda \cap (\mathrm{Cl}(I) \times [\sup(J), M])$ is a compact set, we may suppose that the sequence $\{(y_k, \varphi_\lambda(y_k))\}_{k\geq 1}$ converges to $(y_0, t_0) \in K$, so

$$\varphi_\lambda(y_0) < \sup(J) \leq t_0 \leq \varphi_\lambda(y_0),$$

which is a contradiction. Thus, after shrinking I, it holds $(y, \varphi_\lambda(y)) \in I \times J$ for all $y \in I$. Consequently, $\varphi_\lambda|_I = \phi|_I$.

Therefore, φ_λ is a Nash function because it is locally Nash and by its definition semialgebraic.

3.6.4. To prove (ii) we start by fixing b_{j_0} and taking y_0 close to b_{j_0}. Consider the algebraic curve

$$Y := \{\mathtt{p} := 1 - \mathtt{q}\mathtt{g}^2 = 0\}.$$

By Sect. 3.6.2 there exists $(y_0, z_1) \in Y \cap \mathcal{Q}_0$ such that $z_1 < \varphi_\lambda(y_0)$. By Lemma 3.1 (iii) there exists $M > 0$ such that

$$\begin{aligned} 1 = \mathtt{q}(y_0, z_1)\,\mathtt{g}^2(y_0, z_1) &\leq (z_1 - \mathtt{q}_0(y_0))(M z_1)^{2(m-1-r)} \prod_{j=1}^{r} c_j^4 (y_0 - b_j)^4 \\ &\leq z_1 (M z_1)^{2(m-1-r)} \prod_{j=1}^{r} c_j^4 (y_0 - b_j)^4. \end{aligned}$$

Therefore,

$$\frac{1}{\sqrt[2m-2r-1]{M^{2(m-1-r)} \prod_{j=1}^{r} (y_0 - b_j)^4}} \leq z_1 \leq \varphi_\lambda(y_0),$$

and so $\lim_{\mathtt{y} \to b_{j_0}} \varphi_\lambda(\mathtt{y}) = +\infty$, as required. □

Proposition 3.7 *Assume* $\lambda < 0$ *and let* $\pi : \mathbb{R}^2 \to \mathbb{R},\ (\mathtt{y}, \mathtt{z}) \mapsto \mathtt{y}$. *Then* $X_\lambda \subset \mathbb{R} \times]\lambda - 1, 0[$ *and there exists a continuous semialgebraic map* $\psi_\lambda := (\psi_{1,\lambda}, \psi_{2,\lambda}) : \mathbb{R} \to \mathbb{R}^2$ *such that* $\mathrm{Im}(\psi_\lambda) \subset X_\lambda$ *and* $\lim_{\mathtt{t} \to \pm\infty} \psi_{1,\lambda}(\mathtt{t}) = \pm\infty$. *In particular,* $\pi(\mathrm{Im}(\psi_\lambda)) = \mathbb{R}$.

Proof The proof is conducted in several steps:

3.7.1. We show first: *the algebraic curve* X_λ *when* $\lambda < 0$ *lies in the band* $\{\lambda - 1 < \mathtt{z} < 0\}$.

Pick $(y_0, z_0) \in X_\lambda$. As $\mathtt{q} < 0$ on the half-plane $\{\mathtt{z} < 0\}$ and $\lambda < 0$, we have

$$0 > z_0 = \frac{\lambda}{(1 - \mathtt{q}(y_0, z_0)\,\mathtt{g}^2(y_0, z_0))^2} \geq \lambda > \lambda - 1.$$

3.7.2. Next we prove: $\pi(X_\lambda) = \mathbb{R}$.

Fix $y_0 \in \mathbb{R}$ and consider the univariate polynomial $\mathtt{f}_{2,y_0}(\mathtt{z}) := \mathtt{f}_2(y_0, \mathtt{z})$. Then $\mathtt{f}_{2,y_0}(0) = 0 > \lambda$ and using again the fact that $\mathtt{q} < 0$ on $\{\mathtt{z} < 0\}$ we deduce that

$$\mathtt{f}_{2,y_0}(\lambda - 1) = (\lambda - 1)(1 - \mathtt{q}(y_0, \lambda - 1)\,\mathtt{g}^2(y_0, \lambda - 1))^2 \leq \lambda - 1 < \lambda. \quad \text{(C.6)}$$

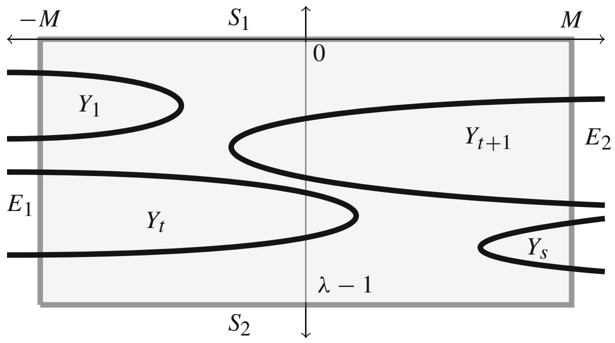


Fig. 2 Description of the situation

By continuity there exists $z_0 \in]\lambda - 1, 0[$ such that $\mathtt{f}_2(y_0, z_0) = \mathtt{f}_{2,y_0}(z_0) = \lambda$, so $(y_0, z_0) \in X_\lambda$. Hence, $y_0 = \pi(y_0, z_0) \in \pi(X_\lambda)$ as claimed.

3.7.3. By [5, 2.9.10] the curve X_λ is the disjoint union of a finite set of points $\mathfrak{F}$ and finitely many affine Nash manifolds $N_1, \ldots, N_p$, each Nash diffeomorphic to the open interval $]0, 1[$. As X_λ contains no vertical lines, we may assume that $\pi : N_i \to \mathbb{R}$ is a Nash diffeomorphism onto its image for $i = 1, \ldots, p$.

3.7.4. Let $M > 0$ be such that $\pi(\mathfrak{F}) \subset]-M, M[$. Let $Y_1, \ldots, Y_s$ be the connected components of X_λ. As $X_\lambda \subset \mathbb{R} \times]\lambda - 1, 0[$, the same happens for each Y_ℓ. Of course each Y_ℓ is a closed subset of $\mathbb{R}^2$. We claim: *Some Y_ℓ connects the vertical edges $E_1 := \{-M\} \times [\lambda - 1, 0]$ and $E_2 := \{M\} \times [\lambda - 1, 0]$ of the rectangle $\mathcal{R} := [-M, M] \times [\lambda - 1, 0]$.*

To prove this claim we will make use of Janiszewski's Theorem (see [25] or [4, Thm. A]): *If K_1 and K_2 are compact subsets of the plane $\mathbb{R}^2$ whose intersection is connected, a pair of points that is separated by neither K_1 nor K_2 is neither separated by their union $K_1 \cup K_2$.*

Suppose by contradiction that no Y_ℓ connects E_1 with E_2. As $\pi(X_\lambda) = \mathbb{R}$, we may assume that the first $t < s$ connected components of X_λ meet E_1, whereas the rest of them do not (Fig. 2). Define the compact sets

$$K_1 := \bigcup_{\ell=1}^{t} (Y_\ell \cap \mathcal{R}) \cup E_1, \quad L := \bigcup_{\ell=t+1}^{s} (Y_\ell \cap \mathcal{R}) \quad \text{and} \quad K_2 := L \cup \partial\mathcal{R},$$

and note that the intersection $K_1 \cap K_2 = E_1$ is connected.

The horizontal segments $S_1 := [-M, M] \times \{0\}$ and $S_2 := [-M, M] \times \{\lambda - 1\}$ satisfy $\mathtt{f}_2(S_1) = \{0\}$ and $\mathtt{f}_2(S_2) \subset]-\infty, \lambda[$ (see (C.6)), that is, $S_1 \subset \mathtt{f}_2^{-1}(]\lambda, +\infty[)$ and $S_2 \subset \mathtt{f}_2^{-1}(]-\infty, \lambda[)$. Consequently, the algebraic curve $X_\lambda = \mathtt{f}_2^{-1}(\lambda)$ separates the horizontal segments S_1 and S_2. Notice that if we restrict $\mathtt{f}_2$ to $\mathcal{R}$ the set $X_\lambda \cap \mathcal{R}$ still separates these segments on $\mathcal{R}$. Observe that $\partial\mathcal{R} = S_1 \cup S_2 \cup E_1 \cup E_2$. As

$$K_1 \cap \left(\partial\mathcal{R} \cap \left\{\mathtt{y} \geq -\frac{M}{2}\right\}\right) = \varnothing \quad \text{and} \quad L \cap \left(\partial\mathcal{R} \cap \left\{\mathtt{y} \leq \frac{M}{2}\right\}\right) = \varnothing,$$

Fig. 3 Positions of K_1 and W_2

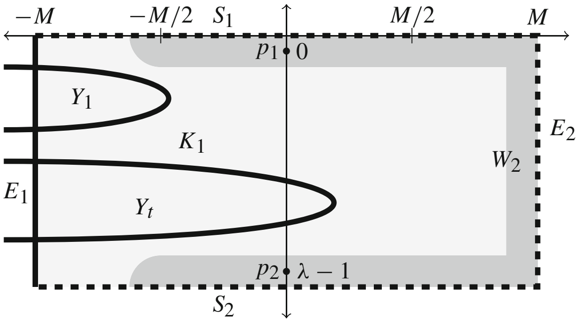
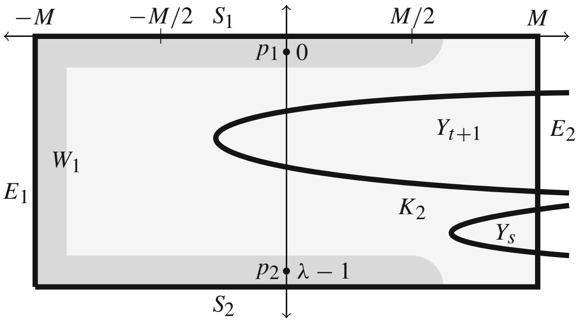


Fig. 4 Positions of K_2 and W_1



the real number

$$\varepsilon := \min\left\{\operatorname{dist}\left(K_1, \left(\partial\mathcal{R} \cap \left\{\mathtt{y} \geq -\frac{M}{2}\right\}\right)\right), \operatorname{dist}\left(L, \left(\partial\mathcal{R} \cap \left\{\mathtt{y} \leq \frac{M}{2}\right\}\right)\right)\right\}$$

is strictly positive and does not exceed $\frac{M}{2}$ (Figs. 3, 4). Let $0 < \delta < \frac{\varepsilon}{2}$ be such that $p_1 := (0, -\delta) \in f_2^{-1}(]\lambda, +\infty[)$ and $p_2 := (0, \lambda - 1 + \delta) \subset f_2^{-1}(]-\infty, \lambda[)$. It holds that $K_1 \cup K_2 = (X_\lambda \cap \mathcal{R}) \cup \partial\mathcal{R}$ separates the points p_1 and p_2. Note that both p_1, p_2 belong to the open connected subsets

$$W_1 := \left\{p \in \operatorname{Int}(\mathcal{R}) : 0 < \operatorname{dist}\left(p, \left(\partial\mathcal{R} \cap \left\{\mathtt{y} \leq \frac{M}{2}\right\}\right)\right) < \frac{\varepsilon}{2}\right\},$$
$$W_2 := \left\{p \in \operatorname{Int}(\mathcal{R}) : 0 < \operatorname{dist}\left(p, \left(\partial\mathcal{R} \cap \left\{\mathtt{y} \geq -\frac{M}{2}\right\}\right)\right) < \frac{\varepsilon}{2}\right\}$$

of $\operatorname{Int}(\mathcal{R})$, while $K_1 \cap W_2 = \varnothing$ and $K_2 \cap W_1 = \varnothing$. Thus, p_1, p_2 are separated neither by K_1 nor by K_2, which contradicts Janiszewski's Theorem. The claim follows.

3.7.5. Let $\alpha : [-M, M] \to X_\lambda$ be a continuous semialgebraic path such that $\alpha(-M) \in X_\lambda \cap E_1$ and $\alpha(M) \in X_\lambda \cap E_2$. As $\pi(\mathfrak{F}) \subset]-M, M[$, we may assume that

- $\alpha(-M) \in N_1$ and $\alpha(M) \in N_2$,
- $\pi(N_1) =]-\infty, -M[$ and $\pi(N_2) =]M, \infty[$, where $\pi|_{N_1}$ and $\pi|_{N_2}$ are homeomorphisms.

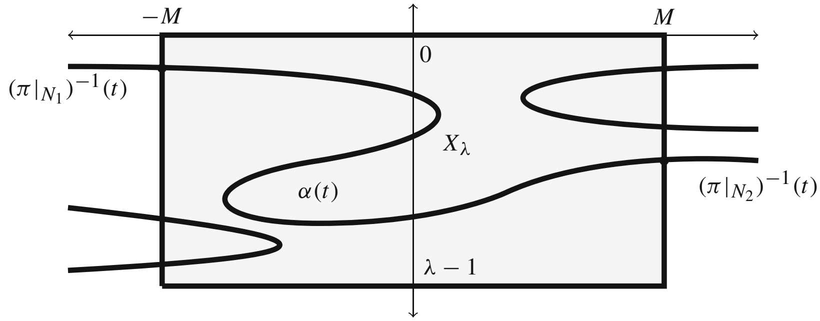


Fig. 5 Construction of the parameterization ψ_λ

Finally, the continuous semialgebraic map (Fig. 5)

$$\psi_\lambda(\mathtt{t}) = (\psi_{1,\lambda}(\mathtt{t}), \psi_{2,\lambda}(\mathtt{t})) := \begin{cases} (\pi|_{N_1})^{-1}(\mathtt{t}) & \text{it } \ \mathtt{t} < -M, \\ \alpha(\mathtt{t}) & \text{if } \ -M \le \mathtt{t} \le M, \\ (\pi|_{N_2})^{-1}(\mathtt{t}) & \text{it } \ \mathtt{t} > M, \end{cases}$$

satisfies $\pi(\mathrm{Im}(\psi_\lambda)) = \mathbb{R}$ and $\lim_{\mathtt{t}\to\pm\infty}(\psi_{1,\lambda}(\mathtt{t})) = \pm\infty$, as required. □

3.5 Proof of Theorem 3.3

Now we are ready to prove the main theorem in this section, which is key in order to prove later Theorem 1.4.

3.3.1. Denote $\overline{\mathcal{S}} := \mathbb{R}^2 \setminus \mathrm{Int}(\mathcal{P})$ and $\overline{\mathcal{T}} := \mathbb{R}^2 \setminus \mathrm{Int}(\mathcal{P}_{m,\times})$. We have to prove that $\mathtt{f}(\overline{\mathcal{T}}) = \overline{\mathcal{S}}$. Write $\overline{\mathcal{T}} = \overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2 \cup \overline{\mathcal{T}}_3$ and $\overline{\mathcal{S}} = \overline{\mathcal{S}}_1 \cup \overline{\mathcal{S}}_2 \cup \overline{\mathcal{S}}_3$ where

$$\begin{aligned} \overline{\mathcal{T}}_1 &:= \overline{\mathcal{T}} \cap \{\mathtt{z} > 0\}, & \overline{\mathcal{S}}_1 &:= \overline{\mathcal{S}} \cap \{\mathtt{z} > 0\}, \\ \overline{\mathcal{T}}_2 &:= \overline{\mathcal{T}} \cap \{\mathtt{z} = 0\}, & \overline{\mathcal{S}}_2 &:= \overline{\mathcal{S}} \cap \{\mathtt{z} = 0\}, \\ \overline{\mathcal{T}}_3 &:= \overline{\mathcal{T}} \cap \{\mathtt{z} < 0\}, & \overline{\mathcal{S}}_3 &:= \overline{\mathcal{S}} \cap \{\mathtt{z} < 0\}. \end{aligned}$$

Note that

$$\overline{\mathcal{S}}_1 \sqcup \overline{\mathcal{S}}_2 = \overline{\mathcal{S}} \cap \{\mathtt{z} \ge 0\} = \{\mathtt{z} \ge 0\} \quad \text{and} \quad \overline{\mathcal{T}}_3 = \overline{\mathcal{S}}_3 = \{\mathtt{z} < 0\} \setminus \mathrm{Int}(\mathcal{P}).$$

It is enough to show

$$\mathtt{f}(\overline{\mathcal{T}}_1 \sqcup \overline{\mathcal{T}}_2) = \overline{\mathcal{S}}_1 \sqcup \overline{\mathcal{S}}_2, \qquad \mathtt{f}(\overline{\mathcal{T}}_3) = \overline{\mathcal{S}}_3.$$

The inclusion $\mathtt{f}(\overline{\mathcal{T}}_1 \cup \overline{\mathcal{T}}_2) \subset \overline{\mathcal{S}}_1 \sqcup \overline{\mathcal{S}}_2$ is straightforward. Therefore, we are left to show

$$\overline{\mathcal{S}}_1 \sqcup \overline{\mathcal{S}}_2 \subset \mathtt{f}(\overline{\mathcal{T}}_1 \sqcup \overline{\mathcal{T}}_2), \tag{C.7}$$

$$\overline{\mathcal{S}}_3 \subset \mathtt{f}(\overline{\mathcal{T}}_3) \subset \overline{\mathcal{S}}_3. \tag{C.8}$$

To prove (C.7) it is enough to check:

$$\begin{aligned}\{(b_1,0),\dots,(b_r,0)\}&\subset \mathtt{f}(\overline{\mathcal{T}}_2),\\ \{\mathtt{z}=0\}\setminus\{(b_1,0),\dots,(b_r,0)\}&\subset \mathtt{f}(\overline{\mathcal{T}}_1),\\ \text{and } \{\mathtt{z}>0\}&\subset \mathtt{f}(\overline{\mathcal{T}}_1).\end{aligned}$$

3.3.2. We check first $\{(b_1,0),\dots,(b_r,0)\}\subset \mathtt{f}(\overline{\mathcal{T}}_2)$. Note that $\{b_j\}\times\mathbb{R}\subset\overline{\mathcal{T}}$ because

$$\mathbb{R}^2\setminus\overline{\mathcal{T}}=\operatorname{Int}(\mathcal{P}_{m,\times})\subset\bigcap_{j=1}^r\{\mathtt{l}_j(\mathtt{y},\mathtt{z})=c_j(\mathtt{y}-b_j)>0\}.$$

As $\mathtt{f}(b_j,\lambda)=(b_j,\lambda)$, we have $\{b_j\}\times\mathbb{R}\subset\mathtt{f}(\overline{\mathcal{T}})$. Therefore $\{(b_1,0),\dots,(b_r,0)\}\subset \mathtt{f}(\overline{\mathcal{T}}_2)$.

3.3.3. Next we show: $U\times\{0\}=\{\mathtt{z}=0\}\setminus\{(b_1,0),\dots,(b_r,0)\}\subset\mathtt{f}(\overline{\mathcal{T}}_1)$.

By Lemma 3.2, for each $y_0\in U:=\mathbb{R}\setminus\{b_1,\dots,b_r\}$ there exists $z_1\in\mathbb{R}$ such that $(y_0,z_1)\in\mathcal{Q}$ and $\mathtt{p}(y_0,z_1)=0$, so $\mathtt{f}_2(y_0,z_1)=0$ and $\mathtt{f}_1(y_0,z_1)=y_0$. Now, by Lemma 3.1

$$\mathcal{Q}\subset\{\mathtt{z}>0\}\setminus\mathcal{P}_{m,\times}\subset\{\mathtt{z}>0\}\cap\overline{\mathcal{T}}=\overline{\mathcal{T}}_1. \tag{C.9}$$

Thus $U\times\{0\}\subset\mathtt{f}(\overline{\mathcal{T}}_1)$.

3.3.4. Let us prove: $\{\mathtt{z}>0\}\subset\mathtt{f}(\overline{\mathcal{T}}_1)$. To that end we show: *If* $\lambda>0$, *the line* $\{\mathtt{z}=\lambda\}$ *is contained in* $\mathtt{f}(\overline{\mathcal{T}}_1)$.

Consider the curve $X_\lambda:=\{\mathtt{f}_2(\mathtt{y},\mathtt{z})=\lambda\}$. By Proposition 3.6 there exists a Nash function $\varphi_\lambda\colon U:=\mathbb{R}\setminus\{b_1,\dots,b_r\}\to\mathbb{R}$ such that $\lim_{\mathtt{y}\to b_j}\varphi_\lambda(\mathtt{y})=+\infty$ for $j=1,\dots,r$ and its graph $\Gamma_\lambda\subset X_\lambda\cap\mathcal{Q}$. The latter condition means in particular that $\lim_{\mathtt{y}\to\pm\infty}\varphi_\lambda(\mathtt{y})=+\infty$.

Consider the function

$$\Phi_\lambda\colon\mathbb{R}\to\mathbb{R},\ \mathtt{y}\mapsto\begin{cases}\mathtt{f}_1(\mathtt{y},\varphi_\lambda(\mathtt{y})) & \text{if } \mathtt{y}\in U,\\ b_j & \text{if } \mathtt{y}=b_j.\end{cases}$$

Let us check: $\operatorname{Im}(\Phi_\lambda)=\mathbb{R}$. It is enough to prove: Φ_λ *is continuous and* $\lim_{\mathtt{y}\to\pm\infty}\Phi_\lambda(\mathtt{y})=\pm\infty$.

Indeed, since $(\mathtt{y},\varphi_\lambda(\mathtt{y}))\in X_\lambda$ we have

$$\varphi_\lambda(\mathtt{y})\,\mathtt{p}^2(\mathtt{y},\varphi_\lambda(\mathtt{y}))=\lambda\quad\rightsquigarrow\quad\mathtt{p}(\mathtt{y},\varphi_\lambda(\mathtt{y}))=\sqrt{\frac{\lambda}{\varphi_\lambda(\mathtt{y})}}.$$

Thus, for each $\mathtt{y}\in U$ we have by (C.11)

$$\Phi_\lambda(\mathtt{y})=\mathtt{f}_1(\mathtt{y},\varphi_\lambda(\mathtt{y}))=\mathtt{y}\biggl(2\,\frac{\lambda}{\varphi_\lambda(\mathtt{y})}-2\sqrt{\frac{\lambda}{\varphi_\lambda(\mathtt{y})}}+1\biggr), \tag{C.10}$$

so $\lim_{\mathtt{y}\to b_j}\Phi_\lambda(\mathtt{y})=b_j$ for $j=1,\dots,r$ and Φ_λ is continuous. Also from (C.10) follows that $\lim_{\mathtt{y}\to\pm\infty}\Phi_\lambda(\mathtt{y})=\pm\infty$. Thus, the map $\mathbb{R}\to\mathbb{R}\times\lambda=\{\mathtt{z}=\lambda\}$, $\mathtt{y}\mapsto(\Phi_\lambda(\mathtt{y}),\lambda)$ is surjective. As the graph Γ_λ of φ_λ is contained in $\mathcal{Q}\subset\{\mathtt{z}>0\}\cap\overline{\mathcal{T}}$ (see inclusions (C.9)) and the points $(b_j,\lambda)\in\overline{\mathcal{T}}$ (see Sect. 3.3.2), we deduce $\{\mathtt{z}=\lambda\}\subset\mathtt{f}(\overline{\mathcal{T}}\cap\{\mathtt{z}>0\})=\mathtt{f}(\overline{\mathcal{T}}_1)$.

This concludes the proof of (C.7) and we proceed now with (C.8).

3.3.5. Let us check: $\overline{\mathcal{S}}_3\subset\mathtt{f}(\overline{\mathcal{T}}_3)$. To that end we show: *If* $\lambda<0$, *the difference* $\{\mathtt{z}=\lambda\}\setminus\operatorname{Int}(\mathcal{P})$ *is contained in* $\mathtt{f}(\overline{\mathcal{T}}_3)=\mathtt{f}(\overline{\mathcal{T}}\cap\{\mathtt{z}<0\})$.

Let $\psi_\lambda:\mathbb{R}\to X_\lambda$ be the continuous semialgebraic map constructed in Proposition 3.7. Write $\mathbb{R}=C\cup A$ where $C:=\psi_\lambda^{-1}(\operatorname{Im}(\psi_\lambda)\setminus\operatorname{Int}(\mathcal{P}))$ and $A:=\psi_\lambda^{-1}(\operatorname{Im}(\psi_\lambda)\cap\operatorname{Int}(\mathcal{P}))$. We distinguish two cases here:

Case 1: $A=\varnothing$. Consider the map $\mathtt{t}\mapsto\mathtt{f}(\psi_\lambda(\mathtt{t}))=(\mathtt{f}_1(\psi_\lambda(\mathtt{t})),\mathtt{f}_2(\psi_\lambda(\mathtt{t}))$. We know that $\mathtt{f}_2(\psi_\lambda(\mathtt{t}))=\lambda$. Rewrite $\mathtt{f}_1$ as follows:

$$\mathtt{f}_1=\mathtt{y}((\mathtt{p}-1)^2+\mathtt{p}^2)=\mathtt{y}(2\mathtt{p}^2-2\mathtt{p}+1)=\mathtt{y}\left(2\left(\mathtt{p}^2-\frac{1}{2}\right)^2+\frac{1}{2}\right). \tag{C.11}$$

We deduce

$$\mathtt{f}_1(\psi_\lambda(\mathtt{t}))=\psi_{1,\lambda}(\mathtt{t})\left(2\left(\mathtt{p}^2(\psi_\lambda(\mathtt{t}))-\frac{1}{2}\right)^2+\frac{1}{2}\right). \tag{C.12}$$

As $\lim_{\mathtt{t}\to\pm\infty}\psi_{1,\lambda}(\mathtt{t})=\pm\infty$, we have $\lim_{\mathtt{t}\to\pm\infty}\mathtt{f}_1(\psi_\lambda(\mathtt{t}))=\pm\infty$. As $\mathtt{f}_1\circ\psi_\lambda$ is continuous, we conclude $\mathtt{f}(\psi_\lambda(\mathbb{R}))=\mathbb{R}\times\{\lambda\}\subset\mathtt{f}(\overline{\mathcal{T}}_3)$.

Case 2: $A\neq\varnothing$. If $t_0\in\partial C$, there exist points $t_1\in C$ and $t_2\in A$ close to t_0. As ψ_λ is continuous, $\psi_\lambda(t_1)\in\operatorname{Im}(\psi_\lambda)\setminus\operatorname{Int}(\mathcal{P})$ and $\psi_\lambda(t_2)\in\operatorname{Int}(\mathcal{P})$ are close to $\psi_\lambda(t_0)$, so $\psi_\lambda(t_0)\in X_\lambda\cap\partial\mathcal{P}$.

As $\psi_\lambda(t_0)\in\partial\mathcal{P}$ we have $\mathtt{g}(\psi_\lambda(t_0))=0$, so $\mathtt{p}(\psi_\lambda(t_0))=1$ and $\mathtt{f}(\psi_\lambda(t_0))=\psi_\lambda(t_0)$. In addition, as $\psi_\lambda(t_0)\in X_\lambda$ we have $\mathtt{f}_2(\psi_\lambda(t_0))=\lambda$, so $\psi_\lambda(t_0)=\mathtt{f}(\psi_\lambda(t_0))=(\psi_{1,\lambda}(t_0),\lambda)$. Thus,

$$\psi_\lambda(t_0)\in\{\mathtt{z}=\lambda\}\cap\partial\mathcal{P}=\partial(\{\mathtt{z}=\lambda\}\setminus\operatorname{Int}(\mathcal{P})).$$

Notice that $\{\mathtt{z}=\lambda\}\setminus\operatorname{Int}(\mathcal{P})=(S_1\times\{\lambda\})\sqcup(S_2\times\{\lambda\})$ where

$$S_1:=\begin{cases}]-\infty,c_\lambda] & \text{or}\\ \varnothing\end{cases}\quad\text{and}\quad S_2:=\begin{cases}[d_\lambda,+\infty[& \text{or}\\ \varnothing.\end{cases}$$

As $\pi(\operatorname{Im}(\psi_\lambda))=\mathbb{R}$ and $\lim_{\mathtt{t}\to\pm\infty}\psi_{1,\lambda}(\mathtt{t})=\pm\infty$, there exists two intervals C_1 and C_2 of C (in case they are non-empty) such that

$$C_1:=\begin{cases}]-\infty,c'_\lambda] & \text{if } S_1=]-\infty,c_\lambda],\\ \varnothing & \text{if } S_1=\varnothing\end{cases}\quad\text{and}$$

$$C_2 := \begin{cases} [d'_\lambda, +\infty[& \text{if } S_2 = [d_\lambda, +\infty[, \\ \varnothing & \text{if } S_2 = \varnothing \end{cases}$$

where $\psi_{1,\lambda}(c'_\lambda) = c_\lambda$ and $\psi_{1,\lambda}(d'_\lambda) = d_\lambda$ if the corresponding $S_i \neq \varnothing$. Let us show: $\mathtt{f}(\psi_\lambda(C_i)) = S_i \times \{\lambda\}$ *for* $i = 1, 2$. As $\psi_\lambda(C) \subset \{\mathtt{z} < 0\} \setminus \operatorname{Int}(\mathcal{P}) = \{\mathtt{z} < 0\} \cap \overline{\mathcal{T}} = \overline{\mathcal{T}}_3$, we conclude

$$\{\mathtt{z} = \lambda\} \setminus \operatorname{Int}(\mathcal{P}) \subset \mathtt{f}(\psi_\lambda(C)) \subset \mathtt{f}(\overline{\mathcal{T}} \cap \{\mathtt{z} < 0\}) = \mathtt{f}(\overline{\mathcal{T}}_3).$$

Now, as in the previous case, from (C.11) and (C.12) we deduce that, since $\lim_{\mathtt{t}\to\pm\infty} \psi_{1,\lambda}(\mathtt{t}) = \pm\infty$, then $\lim_{\mathtt{t}\to\pm\infty} \mathtt{f}_1(\psi_\lambda(\mathtt{t})) = \pm\infty$. In addition,

$$\left.\begin{aligned} \psi_\lambda(c'_\lambda) = (c_\lambda, \lambda) \\ \psi_\lambda(d'_\lambda) = (d_\lambda, \lambda) \end{aligned}\right\} \in \partial\mathcal{P},$$

so $\mathtt{f}_1(\psi_\lambda(c'_\lambda)) = c_\lambda$ and $\mathtt{f}_1(\psi_\lambda(d'_\lambda)) = d_\lambda$. Consequently, $\mathtt{f}_1(\psi_\lambda(C_i)) = S_i$ for $i = 1, 2$, as required.

3.3.6. Finally we show: $\mathtt{f}(\overline{\mathcal{T}}_3) \subset \overline{\mathcal{S}}_3 = \{\mathtt{z} < 0\} \setminus \operatorname{Int}(\mathcal{P})$.

Note first that $\mathtt{f}(\overline{\mathcal{S}}_3) \subset \{\mathtt{z} < 0\}$. Thus, we only have to check: $\mathtt{f}(\overline{\mathcal{S}}_3) \cap \operatorname{Int}(\mathcal{P})$ *is the empty set*.

Let $(y_0, z_0) \in \overline{\mathcal{T}}_3 = \overline{\mathcal{T}} \cap \{\mathtt{z} < 0\} = \overline{\mathcal{S}} \cap \{\mathtt{z} < 0\} = \{\mathtt{z} < 0\} \setminus \operatorname{Int}(\mathcal{P})$ and choose an edge of $\mathcal{P}$ (different from $\mathcal{E}_m$) and a linear equation $\mathtt{l} := a\mathtt{y} + b\mathtt{z} + c$ of the line containing it that satisfies $\mathcal{P} \subset \{\mathtt{l} \geq 0\}$ and

$$\mathtt{l}(y_0, z_0) = ay_0 + bz_0 + c \leq 0. \tag{C.13}$$

We also have $c > 0$, because $\mathtt{l}(0, 0) > 0$. We distinguish two cases:

Case 1: $b \leq 0$. Let us check: $\mathtt{l}(\mathtt{f}(y_0, z_0)) \leq 0$.

Indeed,

$$\begin{aligned} \mathtt{l}(\mathtt{f}(y_0, z_0)) = {} & ay_0((1-\mathtt{p})^2(y_0, z_0) + \mathtt{p}^2(y_0, z_0) + bz_0\mathtt{p}^2(y_0, z_0) + c \\ & = (ay_0 + bz_0 + c)((1-\mathtt{p})^2(y_0, z_0) + \mathtt{p}^2(y_0, z_0)) \\ & \quad - bz_0(1-\mathtt{p})^2(y_0, z_0) + c(1 - (1-\mathtt{p})^2(y_0, z_0) - \mathtt{p}^2(y_0, z_0)) \leq 0 \end{aligned}$$

because $ay_0 + bz_0 + \gamma^2 \leq 0$, $z_0 < 0$ and $(1-\mathtt{p})^2(y_0, z_0) + \mathtt{p}^2(y_0, z_0) > 1$. Hence, $\mathtt{f}(y_0, z_0) \notin \operatorname{Int}(\mathcal{P})$.

Case 2: $b > 0$. Let $-M := \inf\{z : (y, z) \in \mathcal{P}\}$. If $M = \infty$, then $-\vec{e}_n \in \vec{\mathfrak{C}}(\mathcal{P})$, so $b \leq 0$. Thus, $M \in \mathbb{R}$. As $(0, -M) \in \mathcal{P}$, we have $-bM + c \geq 0$, so $-M \geq -\frac{c}{b}$. We consider two subcases. If $ay_0 \geq 0$, we have

$$ay_0 + bz_0 + c \leq 0 \quad \rightsquigarrow \quad z_0 \leq -\frac{ay_0}{b} - \frac{c}{b} \leq -\frac{ay_0}{b} - M \leq -M.$$

Consequently, $z_0\mathtt{p}^2(y_0, z_0) \leq M$, because $\mathtt{p}(y_0, z_0) \geq 1$ as $z_0 < 0$. Therefore, $\mathtt{f}(y_0, z_0) \in \operatorname{Int}(\mathcal{P})$ and $\mathtt{f}(y_0, z_0) \in \overline{\mathcal{S}}_3$. If $ay_0 \leq 0$, we have

$$\begin{aligned}\mathtt{l}(\mathtt{f}(y_0, z_0)) &= ay_0((1-\mathtt{p})^2(y_0, z_0) + \mathtt{p}^2(y_0, z_0)) + bz_0\mathtt{p}^2(y_0, z_0) + c\\ &= \mathtt{p}^2(y_0, z_0)(ay_0 + bz_0 + c) + ay_0(1-\mathtt{p})^2(y_0, z_0)\\ &\quad + c(1-\mathtt{p}^2(y_0, z_0)) \leq 0,\end{aligned}$$

because all the addends in the last expression are non-positive (recall that as $z_0 < 0$, it holds $\mathtt{p}(y_0, z_0) \geq 1$). Again, we conclude $\mathtt{f}(y_0, z_0) \in \overline{\mathcal{S}}_3$, as required. □

4 Complements of Interiors of Convex Polyhedra

The purpose of this section is to provide a constructive proof of Theorem 1.4. The proof can be schematized as follows:

- We place the polyhedron $\mathcal{K}$ so that the vector $-\vec{e}_n := (0, \ldots, 0, -1)$ lies in its recession cone and one of its facets $\mathcal{F}_i$ lies in the hyperplane $\{\mathtt{x}_n = 0\}$.
- We proceed by double induction on the dimension and the number of facets of $\mathcal{K}$.
- By the induction hypothesis, the complement $\mathcal{T}_i$ of the interior of the unbounded convex polyhedron $\mathcal{K}_{i,\times}$ (which has one facet less than $\mathcal{K}$) is a polynomial image of $\mathbb{R}^n$.
- The main task now is to construct a polynomial map $\mathtt{F}_{\mathcal{K}}$ that sends this complement $\mathcal{T}_i$ onto the complement of the original polyhedron $\mathcal{K}$.
- We show that the previous map satisfies our requirements by reducing the problem to a two-dimensional setting, so that we are able to apply Theorem 3.3.

The reader could follow together the proof of Theorem 1.4 and the simple working Example 4.1 in order to get a better idea on how the construction works.

4.1 Proof of Theorem 1.4

The proof is conducted in several steps.

4.1.1 Setting Up the Scenario

We proceed by double induction on the pair (n, m), where n denotes the dimension of $\mathcal{K}$ and m its number of facets. The result is trivial for $n = 1$ because in this case layers correspond precisely to bounded closed intervals that disconnect $\mathbb{R}$, whereas $\mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K})$ for unbounded $\mathcal{K}$ is affinely equivalent to $[0, +\infty[$, which is the image of the polynomial map $\mathtt{f}\colon \mathbb{R} \to \mathbb{R},\ \mathtt{x} \mapsto \mathtt{x}^2$. Assume $n \geq 2$ and the result true for all convex polyhedra that have either dimension $\leq n-1$, or dimension n and less than m facets.

Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron with m facets, which is not a layer. If $\mathcal{K}$ is degenerate, we can assume $\mathcal{K} = \mathcal{P} \times \mathbb{R}$ where $\mathcal{P} \subset \mathbb{R}^{n-1}$ is a convex polyhedron different from a layer. By the induction hypothesis there exists a polynomial map $\mathtt{f}_0\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ whose image is $\mathbb{R}^{n-1} \setminus \operatorname{Int}(\mathcal{P})$. The image of the polynomial map $\mathtt{f}\colon \mathbb{R}^n \to \mathbb{R}^n,\ \mathtt{x} := (\mathtt{x}', \mathtt{x}_n) \mapsto (\mathtt{f}_0(\mathtt{x}'), \mathtt{x}_n)$ is $\mathbb{R}^n \setminus \operatorname{Int}(\mathcal{K})$ and we are done.

Assume next that $\mathcal{K}$ is non-degenerate. We can assume that $\mathcal{K}$ satisfies properties (i)–(iv) in Lemma 2.2. Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be the vertical facets of $\mathcal{K}$ and let $\mathcal{F}_{r+1}, \ldots, \mathcal{F}_m$ be the non-vertical ones, with $\mathcal{F}_m$ lying in the hyperplane $\{\mathtt{x}_n = 0\}$ as usual.

4.1.2 Construction of the Polynomial Map $\mathbf{F}_{\mathcal{K}}$

As before, set $\mathtt{x}' := (\mathtt{x}_1, \ldots, x_{n-1})$ and $\mathtt{x} := (\mathtt{x}_1, \ldots, \mathtt{x}_n) = (\mathtt{x}', \mathtt{x}_n)$. Let us express the equations for the spans of the facets of $\mathcal{K}$ as follows:

$$\bar{\mathtt{h}}_i(\mathtt{x}) := \begin{cases} \mathtt{h}_{0i}(\mathtt{x}') & \text{for } \ i = 1, \ldots, r, \\ \epsilon_i \mathtt{h}_i(\mathtt{x}) := \epsilon_i(\mathtt{h}_{0i}(\mathtt{x}') - \mathtt{x}_n) & \text{for } \ i = r+1, \ldots, m-1, \\ \mathtt{h}_m(\mathtt{x}_n) := -\mathtt{x}_n & \text{for } \ i = m, \end{cases}$$

where $\epsilon_i \in \{-1, +1\}$ for $i = r+1, \ldots, m-1$. The hyperplanes H_i for $i = 1, \ldots, r$ are vertical, while those corresponding to $i = r+1, \ldots, m$ are non-vertical. Here, $H_i^+ = \{\bar{\mathtt{h}}_i \geq 0\}$ and $\bar{\mathtt{h}}_i(\mathbf{0}', 0) > 0$ for $i = 1, \ldots, m-1$ because the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{F}_m$. We also have for some index $k \in \{r+1, \ldots, m-1\}$ that $\epsilon_k = +1$, because otherwise the vector $\vec{e}_n$ would belong to $\vec{\mathfrak{C}}(\mathcal{K}_{m,\times})$ as all non-vertical half-spaces H_i^+ would be of the form $\{-\mathtt{h}_{0i}(\mathtt{x}') + \mathtt{x}_n \geq 0\}$. We introduce now the n-dimensional versions of the auxiliary polynomials that we introduced in Sects. 3.1 and 3.2. Consider first

$$\mathtt{Q}(\mathtt{x}) := \mathtt{x}_n - \|\mathtt{x}'\|^2 - 1 - \sum_{i=r+1}^{m-1} \left(\frac{\mathtt{h}_{0i}^2(\mathtt{x}') + 1}{2} \right), \tag{D.1}$$

where the last addend becomes 0 when $r = m-1$. The polynomial $\mathtt{Q}(\mathtt{x})$ has two properties of interest to us. First, *the region* $\mathtt{Q}(\mathtt{x}) \geq 0$ *lies 'above' all the hyperplanes containing non-vertical facets of* $\mathcal{K}$ *and 'above' the hypersurface* $\mathtt{x}_n = \|\mathtt{x}'\|^2 + 1$. In addition, this region is connected and projects onto $\{\mathtt{x}_n = 0\}$. Second, $\mathtt{Q}(\mathtt{x})$ *is always negative on* $\{\mathtt{x}_n \leq 0\}$. Next, we introduce

$$\mathtt{G}(\mathtt{x}) := \left(\prod_{j=1}^{r} \mathtt{h}_{0j}(\mathtt{x}') \right)^2 \cdot \left(\prod_{i=r+1}^{m-1} \mathtt{h}_i(\mathtt{x}) \right). \tag{D.2}$$

Notice that *this polynomial function vanishes on the hyperplanes containing the facets of* $\mathcal{K}$ *and is positive on its interior*. In addition, *vertical facets of* $\mathcal{K}$ *do not change the sign of* $\mathtt{G}(\mathtt{x})$. Finally, consider the polynomial

$$\mathtt{P}(\mathtt{x}) := 1 - \mathtt{Q}(\mathtt{x})\,\mathtt{G}^2(\mathtt{x}), \tag{D.3}$$

which is ≥ 1 whenever $\mathtt{Q} \leq 0$ and $= 1$ when $\mathtt{G} = 0$ (that is, on the hyperplanes spanning the facets of $\mathcal{K}$).

Define with the aid of (D.3)

$$\begin{aligned} \mathtt{F}_1(\mathtt{x}) &:= \mathtt{x}'((\mathtt{P}(\mathtt{x}) - 1)^2 + \mathtt{P}^2(\mathtt{x})) \in \mathbb{R}[\mathtt{x}]^{n-1} \quad \text{and} \\ \mathtt{F}_2(\mathtt{x}) &= \mathtt{x}_n \mathtt{P}^2(\mathtt{x}) \in \mathbb{R}[\mathtt{x}]. \end{aligned} \tag{D.4}$$

4.1.3.

We claim: *The map* $\mathtt{F}_{\mathcal{K}} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R},\ \mathtt{x} \mapsto (\mathtt{F}_1(\mathtt{x}), \mathtt{F}_2(\mathtt{x}))$ *satisfies*

$$\mathtt{F}_{\mathcal{K}}(\mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K}_{m,\times})) = \mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K}).$$

Assume the previous claim true for a while. As $\mathcal{K}_{m,\times}$ has one facet less than $\mathcal{K}$, by the induction hypothesis the complement $\mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K}_{m,\times})$ is the image of a polynomial map $\mathtt{F}_0 : \mathbb{R}^n \to \mathbb{R}^n$. Therefore, the composition $\mathtt{F}_{\mathcal{K}} \circ \mathtt{F}_0$ satisfies $\mathtt{F}_{\mathcal{K}} \circ \mathtt{F}_0(\mathbb{R}^n) = \mathtt{F}_{\mathcal{K}}(\mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K}_{m,\times})) = \mathbb{R}^n \setminus \mathcal{K}$, as required. Hence, it only remains to prove this claim.

4.1.4 Reduction to the Two-Dimensional Case

Consider the family of vertical hyperplanes through the origin. This family can be parameterized as follows: *for each* $\vec{u} \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$ *define* $\pi_{\vec{u}} := \{(\vec{u}\mathtt{y}, \mathtt{z}) : (\mathtt{y}, \mathtt{z}) \in \mathbb{R}^2\}$, *which is the plane through the origin generated by the vectors* $\vec{u}$ *and* $\vec{e}_n$. Obviously $\mathbb{R}^n = \bigcup_{\vec{u} \in \mathbb{S}^{n-2}} \pi_{\vec{u}}$ and $\mathtt{F}_{\mathcal{K}}(\pi_{\vec{u}}) \subset \pi_{\vec{u}}$. Therefore it is enough to check that

$$\mathtt{F}_{\mathcal{K}}(\pi_{\vec{u}} \setminus \mathrm{Int}(\mathcal{K}_{m,\times})) = \pi_{\vec{u}} \setminus \mathrm{Int}(\mathcal{K})$$

for all $\vec{u} \in \mathbb{S}^{n-2}$. As $\mathrm{Int}(\mathcal{K}_{m,\times})$ and $\mathrm{Int}(\mathcal{K})$ are open subsets of $\mathbb{R}^n$, we have

$$\mathrm{Int}(\mathcal{K}_{m,\times}) \cap \pi_{\vec{u}} = \mathrm{Int}(\mathcal{K}_{m,\times} \cap \pi_{\vec{u}}) \quad \text{and} \quad \mathrm{Int}(\mathcal{K}) \cap \pi_{\vec{u}} = \mathrm{Int}(\mathcal{K} \cap \pi_{\vec{u}}).$$

As the origin of $\mathbb{R}^n$ belongs to the interior of $\mathcal{F}_m$, the intersection $\mathcal{E}_{m,\vec{u}} := \mathcal{F}_m \cap \pi_{\vec{u}}$ is an edge of $\mathcal{K} \cap \pi_{\vec{u}}$ and its interior $\mathrm{Int}(\mathcal{E}_{m,\vec{u}})$ contains the origin of $\mathbb{R}^n$. We are reduced to prove that

$$\mathtt{F}_{\mathcal{K}}(\pi_{\vec{u}} \setminus \mathrm{Int}(\mathcal{K}_{m,\times} \cap \pi_{\vec{u}})) = \pi_{\vec{u}} \setminus \mathrm{Int}(\mathcal{K} \cap \pi_{\vec{u}}) \tag{D.5}$$

for all $\vec{u} \in \mathbb{S}^{n-2}$.

4.1.5 Conclusion of Proof

Fix now an arbitrary $\vec{u} \in \mathbb{S}^{n-2}$ and write

$$\begin{aligned}
\bar{\mathtt{l}}_i(\mathtt{y},\mathtt{z}) &:= \bar{\mathtt{h}}_i(\mathtt{y}\vec{u},\mathtt{z}) \\
&= \begin{cases} \mathtt{l}_{0k}(\mathtt{y}) & \text{if } k=1,\dots,r, \\ \epsilon_k \mathtt{l}_k(\mathtt{y},\mathtt{z}) := \epsilon_k(\mathtt{l}_{0k}(\mathtt{y})-\mathtt{z}) & \text{if } k=r+1,\dots,m-1, \\ -\mathtt{z} & \text{if } k=m. \end{cases} \\
\mathtt{q}(\mathtt{y},\mathtt{z}) &:= \mathtt{Q}(\mathtt{y}\vec{u},\mathtt{z}) = \mathtt{z}-\mathtt{y}^2-1-\sum_{i=r+1}^{m-1}\frac{\mathtt{l}_{0i}^2(\mathtt{y})+1}{2}, \\
\mathtt{g}(\mathtt{y},\mathtt{z}) &:= \mathtt{G}(\mathtt{y}\vec{u},\mathtt{z}) = \Big(\prod_{j=1}^{r}\mathtt{l}_{0j}(\mathtt{y})\Big)^2\cdot\Big(\prod_{i=r+1}^{m-1}\mathtt{l}_i(\mathtt{y},\mathtt{z})\Big), \\
\mathtt{p}(\mathtt{y},\mathtt{z}) &:= \mathtt{P}(\mathtt{y}\vec{u},\mathtt{z}) = 1-\mathtt{q}(\mathtt{y},\mathtt{z})\,\mathtt{g}^2(\mathtt{y},\mathtt{z}), \\
\mathtt{f}_1(\mathtt{y},\mathtt{z}) &:= \mathtt{F}_1(\mathtt{y}\vec{u},\mathtt{z}) = \mathtt{y}((\mathtt{p}(\mathtt{y},\mathtt{z})-1)^2+\mathtt{p}^2(\mathtt{y},\mathtt{z})), \\
\mathtt{f}_2(\mathtt{y},\mathtt{z}) &:= \mathtt{F}_2(\mathtt{y}\vec{u},\mathtt{z}) = \mathtt{z}\mathtt{p}^2(\mathtt{y},\mathtt{z}).
\end{aligned} \tag{D.6}$$

The linear polynomials $\bar{\mathtt{l}}_i$ can be interpreted as the restrictions to the plane $\pi_{\vec{u}}$ of the linear polynomials $\bar{\mathtt{h}}_i$. Recall that for some index $k \in \{r+1,\dots,m-1\}$ we have $\epsilon_k = +1$. We have settled in $\pi_{\vec{u}} \equiv \mathbb{R}^2$ coordinates $(\mathtt{y},\mathtt{z})$ with respect to the vectors $\{\vec{u},\vec{e}_n\}$. Analogously, the functions $\mathtt{q}$, $\mathtt{g}$, $\mathtt{p}$, $\mathtt{f}_1$ and $\mathtt{f}_2$ can be understood respectively as the restrictions to the plane $\pi_{\vec{u}}$ of the polynomials $\mathtt{Q}$, $\mathtt{G}$, $\mathtt{P}$, $\mathtt{F}_1$ and $\mathtt{F}_2$, appearing in (D.1) through (D.4). But restating (D.5) in terms of the plane $\mathbb{R}^2$ leads precisely to Theorem 3.3. This settles the claim §4.1.3 and completes the proof of Theorem 1.4. □

Example 4.1 To show a concrete example on how the previous algorithm works, we sketch here how to construct a polynomial map $\mathtt{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose image is the complement of the interior of the unbounded convex polyhedron $\mathcal{K} := \{\mathtt{x}_1 \geq 0, \mathtt{x}_2 \geq 0, \mathtt{x}_3 \geq 0\} \subset \mathbb{R}^3$. We start with a shortcut provided by the polynomial map

$$\mathtt{F}_{\mathcal{K}_1}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) := \left(\mathtt{x}_1^6-3\mathtt{x}_1^2\mathtt{x}_2^4, -3\mathtt{x}_1^4\mathtt{x}_2^2+\mathtt{x}_2^6, \mathtt{x}_3\right),$$

which satisfies $\mathtt{F}_{\mathcal{K}_1}(\mathbb{R}^3) = \mathbb{R}^3 \setminus \mathrm{Int}(\mathcal{K}_1)$ where $\mathcal{K}_1 := \{\mathtt{x}_1 \geq 0, \mathtt{x}_2 \geq 0\} \subset \mathbb{R}^3$ (see Example 3.4 (ii)). We need now to apply an affine transformation ϕ to place the polyhedron $\mathcal{K}_1$ in such a way that its facets are non-vertical and $-\vec{e}_3$ is contained in (the interior of) its recession cone. This is achieved, for example, with

$$\phi(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) := (\mathtt{x}_1+\mathtt{x}_3, \mathtt{x}_2+\mathtt{x}_3, \mathtt{x}_3+1).$$

We have $\phi(\mathcal{K}_1) = \mathcal{K}_1'$, where $\mathcal{K}_1' := \{\mathtt{x}_3 \leq \mathtt{x}_1+1, \mathtt{x}_3 \leq \mathtt{x}_2+1\} \subset \mathbb{R}^3$. The two facets of $\mathcal{K}_1'$ are the planes of equations $\mathtt{x}_2+1-\mathtt{x}_3 = 0$ and $\mathtt{x}_1+1-\mathtt{x}_3 = 0$. We consider the polynomials

$$\begin{aligned}\mathtt{Q}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) &:= \mathtt{x}_3 - \mathtt{x}_1^2 - \mathtt{x}_2^2 - 1 \\ &\qquad - \frac{(\mathtt{x}_1+1)^2 + (\mathtt{x}_2+1)^2 + 2}{2}, \\ \mathtt{G}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) &:= (\mathtt{x}_1 - \mathtt{x}_3 + 1)(\mathtt{x}_2 - \mathtt{x}_3 + 1), \\ \mathtt{P}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) &:= 1 - \mathtt{Q}\mathtt{G}^2.\end{aligned}$$

Then we follow our recipe and construct the polynomial map

$$\mathtt{F}_{\mathcal{K}'}(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) := \Big(\mathtt{x}_1((\mathtt{P}-1)^2+\mathtt{P}^2), \mathtt{x}_2((\mathtt{P}-1)^2+\mathtt{P}^2), \mathtt{x}_3\mathtt{P}^2\Big),$$

which satisfies $\mathtt{F}_{\mathcal{K}'}(\mathbb{R}^3\setminus \mathrm{Int}(\mathcal{K}'_1)) = \mathbb{R}^3\setminus \mathrm{Int}(\mathcal{K}')$, where $\mathcal{K}' := \{\mathtt{x}_3 \le \mathtt{x}_1 + 1, \mathtt{x}_3 \le \mathtt{x}_2 + 1, \mathtt{x}_3 \le 0\}$. Undoing the isomorphism ϕ by means of the affine bijection

$$\psi(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3) := (\mathtt{x}_1 - \mathtt{x}_3 + 1, \mathtt{x}_2 - \mathtt{x}_3 + 1, \mathtt{x}_3)$$

we finally obtain that the composition $\mathtt{F} = \psi \circ \mathtt{F}_{\mathcal{K}'} \circ \phi \circ \mathtt{F}_{\mathcal{K}_1}$ satisfies $\mathtt{F}(\mathbb{R}^3) = \mathbb{R}^3\setminus \mathrm{Int}(\mathcal{K})$, as required. The curious reader could compare this construction with our previous one in [18, Lem. 7.2]. Expanding this composition map and expressing it in terms of the variables $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3$ produces rather large polynomials. This shows that even for polyhedra with a few number of facets our current constructive procedures lead to expressions with very high degrees. □

Remark 4.2 By inspecting the polynomials introduced in Sect. 4.1.2 we can obtain information regarding the degree of the polynomial map $\mathtt{F}_{\mathcal{K}} := (\mathtt{F}_1, \mathtt{F}_2)\colon \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}$ introduced in (D.4) and hence have and idea of the complexity of our construction (see Question 1.6).

We will assume the more general case, which takes place when, along the inductive process, we never get vertical facets. Then, for an unbounded convex polyhedron $\mathcal{K}$ with m facets, we have $\deg(\mathtt{Q}) = 2$, $\deg(\mathtt{G}) = m-1$ and $\deg(\mathtt{P}) = 2+2(m-1) = 2m$. We conclude that both $\deg(\mathtt{F}_1)$ and $\deg(\mathtt{F}_2)$ have degree $2(2m)+1 = 4m+1$ and therefore $\deg(\mathtt{F}_{\mathcal{K}}) = 4m+1$. Since we are applying induction on the number of facets of the polyhedron and the last step in the process corresponds to the half-space $\{x_n \le 0\}$, which can be obtained as the image of a polynomial map of degree 2, we finally get that the polynomial map f sending $\mathbb{R}^n$ onto $\mathbb{R}^n\setminus\mathcal{K}$ has degree

$$\deg(f) \le 2 \cdot \prod_{i=2}^{m} (4i+1).$$

As an example, for the complement of the interior of an unbounded convex polyhedron with three facets we obtain a polynomial map of degree 234, whereas if the number of facets is four the degree of the polynomial map is 3978.

Example 4.3 If we do not care about the number of variables that we introduce to represent the complement $\mathbb{R}^n\setminus \mathrm{Int}(\mathcal{P})$ of an unbounded convex polygon $\mathcal{P} \subset \mathbb{R}^n$ that does not disconnect $\mathbb{R}^2$ as a polynomial image of an Euclidean space, then we can lower substantially the degrees of the involved polynomial maps (see Question 1.7). More precisely:

Let $\mathcal{P} \subset \mathbb{R}^2$ *be an unbounded convex polygon with* $m > 1$ *edges that does not disconnect* $\mathbb{R}^2$ *and such that* $\dim(\vec{\mathfrak{C}}(\mathcal{P}))$ *is two-dimensional. Then there exists a polynomial map* $\mathtt{F}_m\colon \mathbb{R}^m \to \mathbb{R}^2$ *such that* $\mathtt{F}_m(\mathbb{R}^{m+1}) = \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$ *and* $\deg(\mathtt{F}_m) \leq 8 \cdot 3^{m-2} - 2$.

For instance, for the complement of the interior of an unbounded convex polygon with three edges we obtain a polynomial map of degree 22 involving 3 variables, whereas if the number of edges is four the degree of the polynomial map is 70 involving 4 variables.

Let us explain how the procedure of increasing the number of variables helps us to reduce the degree of the involved maps. The arguments here partly resemble those developed in Sect. 3.5. We proceed by induction on the number of edges m of the convex polygon $\mathcal{P}$.

Initial step. We refer the reader to Example 3.4. For $m = 2$, we obtain a polynomial map of degree $6 = 8 \cdot 3^0 - 2$ involving two variables.

Induction step. Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon that does not disconnect $\mathbb{R}^2$ and has $m \geq 3$ edges. Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be the edges of $\mathcal{P}$ and let $\{\mathtt{h}_i = 0\}$ be the line spanned by the edge $\mathcal{E}_i$. We may assume (reindexing the facets if necessary) that $\mathcal{P} := \{\mathtt{h}_1 \geq 0, \ldots, \mathtt{h}_m \geq 0\}$ satisfies the conditions (i)–(iv) of Lemma 2.2, that is,

(i) The edge $\mathcal{E}_m$ lies in $\{\mathtt{x}_2 = 0\}$ and $\mathcal{P} \subset \{\mathtt{x}_2 \leq 0\}$.
(ii) $(0, 0) \in \operatorname{Int}(\mathcal{E}_m)$.
(iii) Whenever $(p_1, p_2) \in \mathcal{P}$, then $(0, p_2) \in \mathcal{P}$.
(iv) The vertical vector $\vec{e}_2 \notin \vec{\mathfrak{C}}(\mathcal{P}_{m,\times})$.

We choose for each $i = 1, \ldots, m-1$ an equation of the form $\mathtt{h}_i(\mathtt{x}_1, \mathtt{x}_2) = a_i\mathtt{x}_1 + b_i\mathtt{x}_{+}c_i$, where $c_i > 0$ (because $(0, 0) \in \operatorname{Int}(\mathcal{E}) \subset \partial\mathcal{P}$). Write $\mathcal{P}_{m,\times} := \{\mathtt{h}_1 \geq 0, \ldots, \mathtt{h}_{m-1} \geq 0\}$. We claim: *The polynomial map*

$$\mathtt{f}\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (\mathtt{x}_1, \mathtt{x}_2, \mathtt{w}) \mapsto (\mathtt{x}_1((\mathtt{x}_2\mathtt{w}^2)^2 + (1 - \mathtt{x}_2\mathtt{w}^2)^2), \mathtt{x}_2(1 - \mathtt{x}_2\mathtt{w}^2)^2)$$

satisfies the equality $\mathtt{f}((\mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P}_{m,\times})) \times \mathbb{R}) = \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$.

Set $\mathcal{T} := (\mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P}_{m,\times})) \times \mathbb{R}$ and $\mathcal{S} := \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$. If we verify that $\mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 \geq 0\}) = \{\mathtt{x}_2 \geq 0\}(= \mathcal{S} \cap \{\mathtt{x}_2 \geq 0\})$ and $\mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 < 0\}) = \mathcal{T} \cap \{\mathtt{x}_2 < 0\}(= \mathcal{S} \cap \{\mathtt{x}_2 < 0\})$, then the claim follows.

For the first equality, the inclusion $\mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 \geq 0\}) \subset \{\mathtt{x}_2 \geq 0\}$ is straightforward. For the reversed inclusion, take $(q_1, q_2) \in \{\mathtt{x}_2 \geq 0\}$ and choose $p_2 > 0$ so that $0 \leq q_2/p_2 < 1$ and the segment $\{|\mathtt{x}_1| \leq 2|q_1|, \mathtt{x}_2 = p_2\}$ does not meet $\mathcal{P}_{m,\times} \subset \mathbb{R}^2$ (this is posible because $\vec{e}_2 \notin \vec{\mathfrak{C}}(\mathcal{P}_{m,\times})$). If we set

$$w_0 := +\sqrt{\frac{1}{p_2}\left(1 - \sqrt{\frac{q_2}{p_2}}\right)} \quad \text{and}$$

$$p_1 := \frac{q_1}{(p_2w_0^2)^2 + (1 - p_2w_0^2)^2} = \frac{q_1}{2\left(\sqrt{\frac{q_2}{p_2}} - \frac{1}{2}\right)^2 + \frac{1}{2}},$$

we have $\mathtt{f}(p_1, p_2, w_0) = (q_1, q_2)$ and therefore $\mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 \geq 0\}) \supset \{\mathtt{x}_2 \geq 0\}$ (notice that $|p_1| \leq 2|q_1|$, so $(p_1, p_2) \notin \mathcal{P}_{m,\times}$ and $(p_1, p_2, w_0) \in \mathcal{T} \cap \{\mathtt{x}_2 \geq 0\}$).

We prove next the second equality. The inclusion $\mathcal{S} \cap \{\mathtt{x}_2 < 0\} \subset \mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 < 0\})$ is straightforward, because for $\mathtt{w} = 0$ we have $\mathtt{f}(\mathtt{x}_1, \mathtt{x}_2, 0) \equiv \mathrm{id}_{\mathbb{R}^2}(\mathtt{x}_1, \mathtt{x}_2) = (\mathtt{x}_1, \mathtt{x}_2)$, so

$$\begin{aligned}\mathcal{S} \cap \{\mathtt{x}_2 < 0\} &= \mathrm{id}_{\mathbb{R}^2}(\mathcal{S} \cap \{\mathtt{x}_2 < 0\}) \\ &= \mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 < 0\} \cap \{\mathtt{w} = 0\}) \subset \mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 < 0\}).\end{aligned}$$

For the reversed inclusion pick $(q_1, q_2) \in \mathtt{f}(\mathcal{T} \cap \{\mathtt{x}_2 < 0\})$. This means that there exists $(p_1, p_2, w_0) \in \mathcal{T} \cap \{\mathtt{x}_2 < 0\}$ with

$$\mathtt{f}(p_1, p_2, w_0) = \big(p_1((p_2 w_0^2)^2 + (1 - p_2 w_0^2)^2), p_2(1 - p_2 w_0^2)^2\big) = (q_1, q_2).$$

As $(p_1, p_2, w_0) \in \mathcal{T}$, we have $(p_1, p_2) \notin \mathrm{Int}(\mathcal{P}_{m,\times})$, so there exists an index $i = 1, \dots, m-1$ for which $\mathtt{h}_i(p_1, p_2) = a_i p_1 + b_i p_2 + c_i \leq 0$. As in the proof of Theorem 3.3, we distinguish two cases:

Case 1: $b_i \leq 0$. As also $c_i > 0$, we deduce $a_i p_1 < 0$. As $p_2 < 0$, we deduce $1 - p_2 w_0^2 \geq 1$, so $(1 - (1 - p_2 w_0^2)^2) \leq 0$. Consequently,

$$\begin{aligned}&a_i p_1((p_2 w_0^2)^2 + (1 - p_2 w_0^2)^2) + b_i p_2 (1 - p_2 w_0^2)^2 + c_i \\ &\quad = (1 - p_2 w_0^2)^2 (a_i p_1 + b_i p_2 + c_i) \\ &\qquad\qquad + c_i (1 - (1 - p_2 w_0^2)^2) + a_i p_1 (p_2 w_0^2)^2 \leq 0,\end{aligned}$$

because all the addends are non-positive. We conclude $\mathtt{f}(p_1, p_2, w_0) \in \mathcal{T} \cap \{\mathtt{x}_2 < 0\}$.

Case 2: $b_i > 0$. Set $-M := \inf\{z : (y, z) \in \mathcal{P}\}$. If $M = \infty$, then $(0, z) \in \mathcal{P}$ for $z < 0$ and $-\vec{e}_2 \in \vec{\mathfrak{C}}(\mathcal{P})$. But this implies that $b_i \leq 0$, which is a contradiction. Hence, $M \in \mathbb{R}$ and $(0, -M) \in \mathcal{P}$ implies that $-b_i M + c_i \geq 0$. Now we distinguish two subcases: If $a_i p_1 \geq 0$, then

$$a_i p_1 + b_i p_2 + c_i \leq 0 \quad \rightsquigarrow \quad p_2 \leq -\frac{a_i p_1}{b_i} - \frac{c_i}{b_i} \leq -\frac{a_i p_1}{b_i} - M \leq -M,$$

so $p_2(1 - p_2 w_0^2)^2 \leq p_2 \leq -M$. This implies that $\mathtt{f}(p_1, p_2, w_0) \notin \mathcal{P} \subset \{\mathtt{x}_2 \geq -M\}$. On the other hand, if $a_i p_1 < 0$ a similar argument as in Case 1 leads to

$$a_i p_1((p_2 w_0^2)^2 + (1 - p_2 w_0^2)^2) + b_i p_2 (1 - p_2 w_0^2)^2 + c_i \leq 0,$$

and again $\mathtt{f}(p_1, p_2, w_0) \notin \mathcal{P}$.

Therefore, in both cases $\mathtt{f}(p_1, p_2, w_0) \in \mathcal{S} \cap \{\mathtt{x}_2 < 0\}$.

Now we use our inductive hypothesis. As $\mathcal{P}_{m,\times}$ has $m-1$ edges, there exists a polynomial map $\mathtt{g}_m : \mathbb{R}^{m-1} \to \mathbb{R}^2$ such that $\mathtt{g}_m(\mathbb{R}^{m-1}) = \mathbb{R}^2 \setminus \mathcal{P}_{m,\times}$ and $\deg(\mathtt{g}_m) \leq 8 \cdot 3^{m-3} - 2$. Next we consider the map $\mathtt{F}_m : \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}^2$ defined by $\mathtt{F}_m(\mathtt{z}, \mathtt{w}) =$

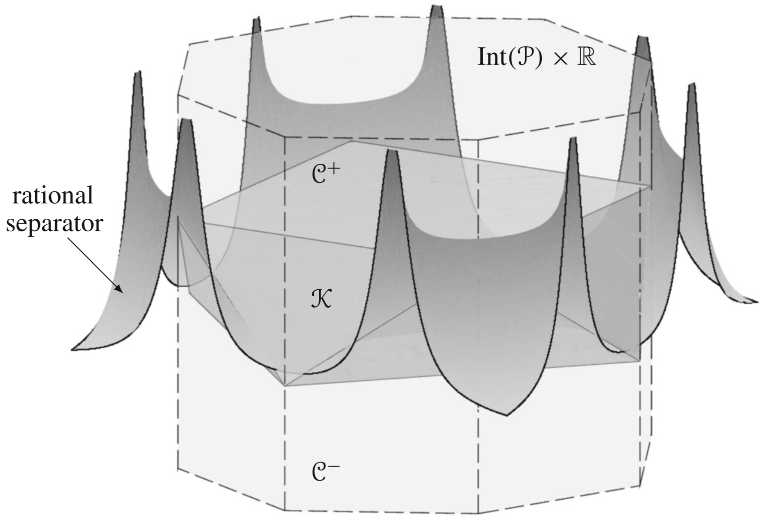


Fig. 6 Rational separator for the attic and the basement of a skyscraper

$\mathtt{f}(\mathtt{g}_m(\mathtt{z}), \mathtt{w})$. It is clear that $\mathtt{F}_m(\mathbb{R}^{m+1}) = \mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{P})$. As $\mathtt{g}_m$ has degree $\leq 8 \cdot 3^{m-3} - 2$, we have

$$\deg(\mathtt{F}_m) = \deg(\mathtt{f}(\mathtt{g}_m, \mathtt{w})) \leq 3(8 \cdot 3^{m-3} - 2) + 4 = 8 \cdot 3^{m-2} - 2$$

for $m \geq 3$, as required. □

5 Rational Separation of Distinguished Semialgebraic Sets

A crucial step to prove Theorem 1.5 is, roughly speaking, the following separation result. Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron of dimension n and let $\mathcal{P} \subset \mathbb{R}^{n-1}$ be the projection of $\mathcal{K}$ onto the first $n-1$ coordinates. Consider the 'infinite prysm' $\operatorname{Int}(\mathcal{P}) \times \mathbb{R}$, which henceforth will be called *skyscraper*. Under mild conditions on the placement of $\mathcal{K}$ in $\mathbb{R}^n$ the difference $(\operatorname{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathcal{K}$ has two connected components: the 'attic' $\mathcal{C}^+$ of the skyscraper $\operatorname{Int}(\mathcal{P}) \times \mathbb{R}$ and its 'basement' $\mathcal{C}^-$. It would be desirable (and much simpler for the exposition) to find a polynomial map $\mathtt{f} \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_{n-1}]$ that separates the attic $\mathcal{C}^+$ and the basement $\mathcal{C}^-$ of the skyscraper $\operatorname{Int}(\mathcal{P}) \times \mathbb{R}$, but the strong restrictions concerning separation of non-compact semialgebraic sets by polynomial functions suggests to use more general functions. In Proposition 5.9 we find a rational map depending on $(\mathtt{x}_1, \ldots, \mathtt{x}_{n-1})$ that separates $\mathcal{C}^+$ and $\mathcal{C}^-$, see Fig. 6. In addition, the previous rational map is regular on $\operatorname{Int}(\mathcal{P})$ and we choose a representation such that the zero set of the denominator is contained in the zero set of the numerator.

The result announced above is based on a preliminary one concerning rational separation of tuples of variables, which has interest by its own.

5.1 Rational Separation of Tuples of Variables

Fix positive integers r, s and set $\mathtt{y} := (\mathtt{y}_1, \dots, \mathtt{y}_r)$, $\mathtt{z} := (\mathtt{z}_1, \dots, \mathtt{z}_s)$. Consider the convex polyhedron

$$\mathcal{Q}_{r,s} := \{(\mathtt{y}; \mathtt{z}) \in \mathbb{R}^r \times \mathbb{R}^s : \max\{\mathtt{y}_1, \dots, \mathtt{y}_r\} \leq \min\{\mathtt{z}_1, \dots, \mathtt{z}_s\}\}.$$

It holds

$$\operatorname{Int}(\mathcal{Q}_{r,s}) = \{(\mathtt{y}; \mathtt{z}) \in \mathbb{R}^r \times \mathbb{R}^s : \max\{\mathtt{y}_1, \dots, \mathtt{y}_r\} < \min\{\mathtt{z}_1, \dots, \mathtt{z}_s\}\}$$

and $\operatorname{Cl}(\operatorname{Int}(\mathcal{Q}_{r,s})) = \mathcal{Q}_{r,s}$. If $r, s \geq 1$ and $k, \ell \geq 0$, the map

$$\begin{aligned} \rho_{r,s}^{k,\ell} : \mathbb{R}^r \times \mathbb{R}^s &\to \mathbb{R}^{r+k} \times \mathbb{R}^{s+\ell}, \\ (\mathtt{y}; \mathtt{z}) &\mapsto (\mathtt{y}_1, \dots, \mathtt{y}_{r-1}, \mathtt{y}_r, \overset{(k+1)}{\dots}, \mathtt{y}_r; \mathtt{z}_1, \dots, \mathtt{z}_{s-1}, \mathtt{z}_s, \overset{(\ell+1)}{\dots}, \mathtt{z}_s) \end{aligned}$$

is a linear embedding such that $\rho(\operatorname{Int}(\mathcal{Q}_{r,s})) \subset \operatorname{Int}(\mathcal{Q}_{r+k,s+\ell})$ and $\rho(\mathcal{Q}_{r,s}) \subset \mathcal{Q}_{r+k,s+\ell}$. Using the identities $\min(S) = -\max(-S)$ and $\max(S) = -\min(-S)$ for any finite set S, one proves that *if* $r, s \geq 1$ *the linear isomorphism*

$$\sigma : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^s \times \mathbb{R}^r, \ (\mathtt{y}; \mathtt{z}) \mapsto (-\mathtt{z}; -\mathtt{y})$$

satisfies $\sigma(\mathcal{Q}_{r,s}) = \mathcal{Q}_{s,r}$ *and* $\sigma(\operatorname{Int}(\mathcal{Q}_{r,s})) = \operatorname{Int}(\mathcal{Q}_{s,r})$.

Definition 5.1 A rational separator for the pair (r, s) *is a rational function* $\phi_{r,s} : \mathbb{R}^r \times \mathbb{R}^s \dashrightarrow \mathbb{R}$ *that is regular on* $\operatorname{Int}(\mathcal{Q}_{r,s})$, *extends to a continuous (semialgebraic) function* $\mathcal{Q}_{r,s}$ *and satisfies*

$$\max\{\mathtt{y}_1, \dots, \mathtt{y}_r\} < \phi_{r,s}(\mathtt{y}; \mathtt{z}) < \min\{\mathtt{z}_1, \dots, \mathtt{z}_s\}$$

for each $(\mathtt{y}; \mathtt{z}) \in \operatorname{Int}(\mathcal{Q}_{r,s})$.

As $\operatorname{Cl}(\operatorname{Int}(\mathcal{Q}_{r,s})) = \mathcal{Q}_{r,s}$ and $\phi_{r,s}$ extends to a continuous (semialgebraic) function $\Phi_{r,s}$ on $\mathcal{Q}_{r,s}$, we deduce

$$\max\{\mathtt{y}_1, \dots, \mathtt{y}_r\} \leq \Phi_{r,s}(\mathtt{y}; \mathtt{z}) \leq \min\{\mathtt{z}_1, \dots, \mathtt{z}_s\}$$

for each $(\mathtt{y}; \mathtt{z}) \in \mathcal{Q}_{r,s}$.

5.2 Recursive Properties of Rational Separators

We present next some recursive properties of rational separators that will ease the proof of the existence of rational separators in Proposition 5.4.

Remark 5.2 (i) Let $\phi_{r,s}\colon \mathbb{R}^r \times \mathbb{R}^s \dashrightarrow \mathbb{R}$ be a a rational separator for (r, s). Then the rational function $\phi_{s,r}(\mathtt{z}, \mathtt{y}) := -\phi_{r,s}(-\mathtt{y}; -\mathtt{z})$ is a rational separator for (s, r).

(ii) Let $r, s \geq 1$ and $k, \ell \geq 0$ and let $\phi_{r+k,s+\ell}\colon \mathbb{R}^{r+k} \times \mathbb{R}^{s+\ell} \dashrightarrow \mathbb{R}$ be a rational separator for $(r + k, s + \ell)$. Then $\phi_{r,s}(\mathtt{y}, \mathtt{z}) = \phi_{r+k,s+\ell}(\rho^{k,\ell}_{r,s}(\mathtt{y}, \mathtt{z}))$ is a rational separator for (r, s).

Lemma 5.3 *Let $r, s, k \geq 1$ be such that $r \geq k$. Let $\phi_{r-k+1,s}$ be a rational separator for $(r - k + 1, s)$ and $\phi_{k,s}$ a rational separator for (k, s). Then*

$$\phi_{r,s}(\mathtt{y}; \mathtt{z}) = \phi_{r-k+1,s}(\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \phi_{k,s}(\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z}); \mathtt{z})$$

is a rational separator for (r, s).

Proof Define

$$\mathcal{M} := \{(\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z}) \in \mathbb{R}^r \times \mathbb{R}^s : \max\{\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r\} \\ \leq \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}\}.$$

The rational map

$$\Theta\colon \mathbb{R}^r \times \mathbb{R}^s \dashrightarrow \mathbb{R}^{r-k+1} \times \mathbb{R}^s, \\ (\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z}) \mapsto (\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \phi_{k,s}(\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z}); \mathtt{z})$$

is regular on $\operatorname{Int}(\mathcal{M})$ and extends continuously to $\mathcal{M} = \operatorname{Cl}(\operatorname{Int}(\mathcal{M}))$. Clearly, $\mathcal{Q}_{r,s} \subset \mathcal{M}$. We claim: $\Theta(\operatorname{Int}(\mathcal{Q}_{r,s})) \subset \operatorname{Int}(\mathcal{Q}_{r-k+1,s})$.

If $(\mathtt{y}; \mathtt{z}) \in \operatorname{Int}(\mathcal{Q}_{r,s})$, then $\max\{\mathtt{y}_1, \ldots, \mathtt{y}_r\} < \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}$, so

$$\max\{\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r\} < \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}$$

and consequently $\phi_{k,s}(\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z}) < \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}$. Thus,

$$\max\{\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \phi_{k,s}(\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z})\} < \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\} \tag{E.1}$$

and $\Theta(\mathtt{y}; \mathtt{z}) \in \operatorname{Int}(\mathcal{Q}_{r-k+1,s})$. Therefore $\phi_{r,s}$ is regular on $\operatorname{Int}(\mathcal{Q}_{r,s})$ and extends continuously to $\mathcal{Q}_{r,s}$. Let us check: $\max\{\mathtt{y}_1, \ldots, \mathtt{y}_r\} < \phi_{r,s}(\mathtt{y}; \mathtt{z}) < \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}$ *for each* $(\mathtt{y}; \mathtt{z}) \in \operatorname{Int}(\mathcal{Q}_{r,s})$.

Indeed, by (E.1) we know that $\Theta(\mathtt{y}; \mathtt{z}) \in \operatorname{Int}(\mathcal{Q}_{r-k+1,s})$, so

$$\begin{aligned}\max\{\mathtt{y}_1, \ldots, \mathtt{y}_r\} &\leq \max\{\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \phi_{k,s}(\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z})\} \\ &< \phi_{r,s}(\mathtt{y}; \mathtt{z}) := \phi_{r-k+1,s}(\mathtt{y}_1, \ldots, \mathtt{y}_{r-k}, \phi_{k,s}(\mathtt{y}_{r-k+1}, \ldots, \mathtt{y}_r; \mathtt{z}); \mathtt{z}) \\ &< \min\{\mathtt{z}_1, \ldots, \mathtt{z}_s\}.\end{aligned}$$

We conclude that $\phi_{r,s}$ is a rational separator for (r, s). □

5.3 Existence of Rational Separators

We prove next that, for each pair (r, s) of positive integers, there actually are rational separators. Denote $\ell(r, s) := \max\{r - 1, 1\} \cdot \max\{s - 1, 1\}$.

Proposition 5.4 *For each pair* (r, s) *of positive integers there exists a rational separator* $\phi_{r,s}$. *In addition, we may assume that* $\phi_{r,s}$ *is the quotient* $\frac{P_{r,s}}{Q_{r,s}}$ *of two homogeneous polynomials* $P_{r,s}, Q_{r,s} \in \mathbb{R}[\mathtt{y}; \mathtt{z}]$ *such that* $\deg(P_{r,s}) = 3^{\ell(r,s)}$, $\deg(Q_{r,s}) = \deg(P_{r,s}) - 1$ *and* $\{Q_{r,s} = 0\} \subset \{P_{r,s} = 0\}$.

Proof The proof is conducted by induction on $t := \min\{r, s\}$. We begin with some initial cases:
CASE $t = r = s = 2$. Define

$$\phi_{2,2}(\mathtt{y}_1, \mathtt{y}_2; \mathtt{z}_1, \mathtt{z}_2) := \frac{\mathtt{z}_1\mathtt{z}_2 - \mathtt{y}_1\mathtt{y}_2}{(\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2)} = \frac{(\mathtt{z}_1\mathtt{z}_2 - \mathtt{y}_1\mathtt{y}_2)((\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2))}{((\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2))^2},$$

which is regular in $\mathrm{Int}(\mathcal{Q}_{2,2})$ and it is the quotient of two homogeneous polynomials $P_{2,2} := (\mathtt{z}_1\mathtt{z}_2 - \mathtt{y}_1\mathtt{y}_2)((\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2))$ and $Q_{2,2} := ((\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2))^2$ such that $\deg(P_{2,2}) = 3$, $\deg(Q_{2,2}) = 2$ and $\{Q_{2,2} = 0\} \subset \{P_{2,2} = 0\}$.

Let us prove that if $(\mathtt{y}_1, \mathtt{y}_2; \mathtt{z}_1, \mathtt{z}_2) \in \mathrm{Int}(\mathcal{Q}_{2,2})$, then

$$\mathtt{y}_i < \phi_{2,2}(\mathtt{y}_1, \mathtt{y}_2; \mathtt{z}_1, \mathtt{z}_2) < \mathtt{z}_j \quad \text{for } i = 1, 2 \text{ and } j = 1, 2. \tag{E.2}$$

As $\phi_{2,2}$ is symmetric with respect to the variables $(\mathtt{y}_1, \mathtt{y}_2)$ and $(\mathtt{z}_1, \mathtt{z}_2)$, we only consider $i = 1$, $j = 1$. We have to prove

$$\mathtt{y}_1((\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2)) < \mathtt{z}_1\mathtt{z}_2 - \mathtt{y}_1\mathtt{y}_2 < \mathtt{z}_1((\mathtt{z}_1 + \mathtt{z}_2) - (\mathtt{y}_1 + \mathtt{y}_2)).$$

The first inequality is equivalent to

$$\mathtt{z}_1\mathtt{z}_2 > \mathtt{y}_1\mathtt{z}_1 + \mathtt{y}_1\mathtt{z}_2 - \mathtt{y}_1^2 \iff (\mathtt{z}_1 - \mathtt{y}_1)(\mathtt{z}_2 - \mathtt{y}_1) > 0,$$

hence it holds. The second inequality is equivalent to

$$\begin{aligned} -\mathtt{y}_1\mathtt{y}_2 &< \mathtt{z}_1^2 - \mathtt{y}_1\mathtt{z}_1 - \mathtt{y}_2\mathtt{z}_1 \\ &\iff \mathtt{z}_1^2 - \mathtt{y}_1\mathtt{z}_1 - \mathtt{y}_2\mathtt{z}_1 + \mathtt{y}_1\mathtt{y}_2 = (\mathtt{z}_1 - \mathtt{y}_1)(\mathtt{z}_1 - \mathtt{y}_2) > 0 \end{aligned}$$

and it also holds. Note that $\mathcal{Q}_{2,2} \cap \{\mathtt{z}_1 + \mathtt{z}_2 - \mathtt{y}_1 - \mathtt{y}_2 = 0\} = \{\mathtt{z}_1 = \mathtt{z}_2 = \mathtt{y}_1 = \mathtt{y}_2\}$. By (E.2) the rational map $\phi_{2,2}$ extends continuously to $\mathcal{Q}_{2,2}$ as

$$\Phi_{2,2}(\mathtt{y}_1, \mathtt{y}_2; \mathtt{z}_1, \mathtt{z}_2) = \begin{cases} \phi_{2,2}(\mathtt{y}_1, \mathtt{y}_2; \mathtt{z}_1, \mathtt{z}_2) & \text{if } \mathtt{z}_1 + \mathtt{z}_2 - \mathtt{y}_1 - \mathtt{y}_2 > 0, \\ \mathtt{z}_1 & \text{if } \mathtt{z}_1 = \mathtt{z}_2 = \mathtt{y}_1 = \mathtt{y}_2. \end{cases}$$

Case $t=2$. By Remark 5.2(i) we may assume that $r \geq s = 2$. We proceed by induction on r. We have constructed above a rational separator for (2, 2) (even satisfying the additional conditions in the statement), so the initial case $r=2$ has been already approached. By induction hypothesis there exists a rational separator $\phi_{r-1,2}$ for $(r-1,2)$ if $r \geq 3$. We may assume in addition that $\phi_{r-1,2} = \frac{P_{r-1,2}}{Q_{r-1,2}}$ is a quotient of homogeneous polynomials $P_{r-1,2}, Q_{r-1,2}$ such that $\deg(P_{r-1,2}) = 3^{r-2}$, $\deg(Q_{r-1,2}) = \deg(P_{r-1,2}) - 1$ and $\{Q_{r-1,2}=0\} \subset \{P_{r-1,2}=0\}$. By Lemma 5.3 the function

$$\begin{aligned}
&\phi_{r,2}(\mathtt{y};\mathtt{z})\\
&\quad= \phi_{r-1,2}(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z})\\
&\quad= \frac{P_{r-1,2}(Q_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z}))}{Q_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot Q_{r-1,2}(Q_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z}))}
\end{aligned}$$

is a rational separator for $(r,2)$. Define

$$\begin{aligned}
P_{r,2} &:= P_{r-1,2}(Q_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z})),\\
Q_{r,2} &:= Q_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\\
&\qquad\cdot Q_{r-1,2}(Q_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,2}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z})).
\end{aligned}$$

As the polynomials $P_{r-1,2}, Q_{r-1,2}$ are homogeneous, $\{Q_{r-1,2}=0\} \subset \{P_{r-1,2}=0\}$ and $\{Q_{2,2}=0\} \subset \{P_{2,2}=0\}$, we deduce $\{Q_{r,2}=0\} \subset \{P_{r,2}=0\}$. In addition, as $\deg(P_{2,2}) = 3$ and $\deg(Q_{2,2}) = 2$, we have $\deg(P_{r,2}) = 3\deg(P_{r-1,2}) = 3^{r-1}$ and

$$\begin{aligned}
\deg(Q_{r,2}) &= 3\deg(Q_{r-1,2}) + 2 = 3(\deg(P_{r-1,2}) - 1) + 2\\
&= 3\deg(P_{r-1,2}) - 1 = \deg(P_{r,2}) - 1.
\end{aligned}$$

Case $t \geq 3$. By Remark 5.2(i) we may assume that $s \geq r \geq 3$. Using Remark 5.2(i) and the construction for $t=2$ we have a rational separator for $(2,s)$ if $s \geq 2$. By induction hypothesis there exists a rational separator $\phi_{r-1,s}$ for $(r-1,s)$ if $r \geq 3$. We may assume in addition that $\phi_{r-1,s} = \frac{P_{r-1,s}}{Q_{r-1,s}}$ is a quotient of homogeneous polynomials $P_{r-1,s}, Q_{r-1,s}$ such that $\deg(P_{r-1,s}) = 3^{(s-1)(r-2)}$, $\deg(Q_{r-1,s}) = \deg(P_{r-1,s}) - 1$ and $\{Q_{r-1,s}=0\} \subset \{P_{r-1,s}=0\}$. By Lemma 5.3 the function

$$\begin{aligned}
&\phi_{r,s}(\mathtt{y};\mathtt{z})\\
&\quad= \phi_{r-1,s}(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z})\\
&\quad= \frac{P_{r-1,s}(Q_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z}))}{Q_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot Q_{r-1,s}(Q_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z}))}
\end{aligned}$$

is a rational separator for (r,s). Define

$$P_{r,s} := P_{r-1,s}(Q_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z})\cdot(\mathtt{y}_1,\ldots,\mathtt{y}_{r-2},\phi_{2,s}(\mathtt{y}_{r-1},\mathtt{y}_r;\mathtt{z});\mathtt{z})),$$

$$Q_{r,s} := Q_{2,s}(\mathtt{y}_{r-1}, \mathtt{y}_r; \mathtt{z}) \\ \cdot Q_{r-1,s}(Q_{2,s}(\mathtt{y}_{r-1}, \mathtt{y}_r; \mathtt{z}) \cdot (\mathtt{y}_1, \dots, \mathtt{y}_{r-2}, \phi_{2,s}(\mathtt{y}_{r-1}, \mathtt{y}_r; \mathtt{z}); \mathtt{z})).$$

As the polynomials $P_{r-1,s}$, $Q_{r-1,s}$ are homogeneous, $\{Q_{r-1,s} = 0\} \subset \{P_{r-1,s} = 0\}$ and $\{Q_{2,s} = 0\} \subset \{P_{2,s} = 0\}$, we deduce $\{Q_{r,s} = 0\} \subset \{P_{r,s} = 0\}$. In addition, as $\deg(P_{2,s}) = 3^{s-1}$ and $\deg(Q_{2,s}) = 3^{s-1} - 1$, we have $\deg(P_{r,s}) = 3^{s-1}\deg(P_{r-1,s}) = 3^{(s-1)(r-1)}$ and

$$\begin{aligned} \deg(Q_{r,s}) &= 3^{s-1}\deg(Q_{r-1,s}) + (3^{s-1} - 1) \\ &= 3^{s-1}(\deg(P_{r-1,2}) - 1) + (3^{s-1} - 1) \\ &= 3^{s-1}\deg(P_{r-1,2}) - 1 = \deg(P_{r,2}) - 1. \end{aligned}$$

Case $t = 1$. By Remark 5.2 (ii)

$$\begin{aligned} \phi_{1,s}(\mathtt{y}_1; \mathtt{z}) &= \phi_{2,s}(\rho^{1,0}_{1,s}(\mathtt{y}_1; \mathtt{z})), \\ \phi_{r,1}(\mathtt{y}; \mathtt{z}_1) &= \phi_{r,2}(\rho^{0,1}_{r,1}(\mathtt{y}; \mathtt{z}_1)) \end{aligned}$$

are respective rational separators for $(1, s)$ and $(r, 1)$ if $r, s \geq 1$. The additional conditions in the statement are satisfied because they hold for $\phi_{2,s}$ and $\phi_{r,2}$, as required. □

5.4 Rational Separation of the Attic and the Basement of a Skyscraper

Let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\mathtt{x} := (\mathtt{x}_1, \dots, \mathtt{x}_n) \mapsto \mathtt{x}' := (\mathtt{x}_1, \dots, \mathtt{x}_{n-1})$ be the projection onto the first $n - 1$ coordinates. The following position for an n-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ guarantees that both differences $(\mathrm{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathcal{K}$ and $(\mathrm{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathrm{Int}(\mathcal{K})$, where $\mathcal{P} := \pi_n(\mathcal{K})$, have two connected components.

Definition 5.5 Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron. We say that $\mathcal{K}$ is in $\vec{\ell}_n$ *-bounded position* if the intersection of $\mathcal{K}$ with any vertical line ℓ is either empty or a bounded interval.

Remark 5.6 One proves straightforwardly: *If $\mathcal{K}$ is in $\vec{\ell}_n$-bounded position, then* $(\mathrm{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathcal{K}$ *has two connected components.*

The following result provides an easy test to determine if a convex polyhedron is in $\vec{\ell}_n$-bounded position.

Lemma 5.7 *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and let ℓ be a vertical line. Suppose that $\ell \cap \mathcal{K}$ is a non-empty bounded segment, which may reduce to a point. Then $\mathcal{K}$ is in $\vec{\ell}_n$-bounded position.*

Proof Under the hypotheses $\vec{e}_n, -\vec{e}_n \notin \vec{\mathfrak{C}}(\mathcal{K})$, so the intersection of $\mathcal{K}$ with any vertical line ℓ is either empty or a bounded interval. □

As a straightforward consequence one shows that each n-dimensional convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$ with at least two facets can be placed in $\vec{\ell}_n$-bounded position. Moreover, we have

Corollary 5.8 *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron and let $\mathcal{F}$ be a facet of $\mathcal{K}$ that has itself at least two facets. Then $\mathcal{K}$ can be placed in $\vec{\ell}_n$-bounded position in such a way that $\mathcal{F} \subset \{x_{n-1} = 0\}$ and $\mathcal{K} \subset \{x_{n-1} \leq 0\}$.*

Proof After an affine change of coordinates we may assume that $\mathcal{F} \subset \{x_{n-1} = 0\}$ and $\mathcal{K} \subset \{x_{n-1} \leq 0\}$. As $\mathcal{F}$ has itself at least two facets, there exist an affine change of coordinates that keeps invariant the half space $x_{n-1} \leq 0$ and such that $\mathcal{F}$ is in $\vec{\ell}_n$-bounded position (inside $\{x_{n-1} = 0\}$). By Lemma 5.7 also $\mathcal{K}$ is in $\vec{\ell}_n$-bounded position. □

5.4.1 *Convex polyhedron placed in $\vec{\ell}_n$-bounded position.* Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional convex polyhedron in $\vec{\ell}_n$-bounded position and let $\mathcal{P} := \pi_n(\mathcal{K}) \subset \mathbb{R}^{n-1}$ be its projection onto the hyperplane $\{x_n = 0\}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be the facets of $\mathcal{K}$ and let $H_i := \{h_i = 0\}$ be the hyperplane generated by $\mathcal{F}_i$. Assume $\mathcal{K} = \bigcap_{i=1}^m H_i^+$ and set $x := (x_1, \ldots, x_{n-1}, x_n) := (x', x_n)$. We can write

$$h_i(x) = -a_i(x') + x_n \quad \forall\, i = 1, \ldots, r, \tag{E.3}$$

$$h_{r+j}(x) = b_j(x') - x_n \quad \forall\, j = 1, \ldots, s, \tag{E.4}$$

$$h_{r+s+k}(x) = c_k(x') \quad \forall\, k = 1, \ldots, m - r - s, \tag{E.5}$$

where $a_i, b_j(x'), c_k(x') \in \mathbb{R}[x']$ are linear polynomials in $n-1$ variables. Equations (E.3) correspond to the non-vertical, lower facets of $\mathcal{K}$ making up its 'floor', while (E.4) and (E.5) correspond respectively to the non-vertical, upper facets making up its 'ceiling' and those that constitute its vertical 'walls'. As $\mathcal{K} \subset \mathbb{R}^n$ is in $\vec{\ell}_n$-bounded position, we have $r, s \geq 1$. Observe that $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ where

$$\mathcal{K}_1 := \big\{(x', x_n) \in \mathbb{R}^n : \max\{a_1(x'), \ldots, a_r(x')\} \leq x_n \leq \min\{b_1(x'), \ldots, b_s(x')\}\big\},$$
$$\mathcal{K}_2 := \big\{(x', x_n) \in \mathbb{R}^n : c_1(x') \geq 0, \ldots, c_{m-r-s}(x') \geq 0\big\}.$$

Notice that

$$\pi_n(\mathcal{K}_1) = \big\{x' \in \mathbb{R}^{n-1} : \max\{a_1(x'), \ldots, a_r(x')\} \leq \min\{b_1(x'), \ldots, b_s(x')\}\big\},$$
$$\pi_n(\mathcal{K}_2) = \big\{x' \in \mathbb{R}^{n-1} : c_1(x') \geq 0, \ldots, c_{m-r-s}(x') \geq 0\big\}.$$

By [29, II, Thm. 6.5] $\mathrm{Int}(\mathcal{K}) = \mathrm{Int}(\mathcal{K}_1) \cap \mathrm{Int}(\mathcal{K}_2)$. As $\pi_n^{-1}(\pi_n(\mathcal{K}_2)) = \mathcal{K}_2$, the equality $\pi_n(\mathcal{K}) = \pi_n(\mathcal{K}_1) \cap \pi_n(\mathcal{K}_2)$ holds. Analogously, $\pi_n(\mathrm{Int}(\mathcal{K})) = \pi_n(\mathrm{Int}(\mathcal{K}_1)) \cap \pi_n(\mathrm{Int}(\mathcal{K}_2))$. In addition

$$\mathrm{Int}(\mathcal{K}_1) = \big\{(x', x_n) \in \mathbb{R}^n : \max\{a_1(x'), \ldots, a_r(x')\} < x_n < \min\{b_1(x'), \ldots, b_s(x')\}\big\},$$
$$\mathrm{Int}(\mathcal{K}_2) = \big\{(x', x_n) \in \mathbb{R}^n : c_1(x') > 0, \ldots, c_{m-r-s}(x') > 0\big\}.$$

Notice that

$$\pi_n(\mathrm{Int}(\mathcal{K}_1)) = \big\{x' \in \mathbb{R}^{n-1} : \max\{a_1(x'), \ldots, a_r(x')\} < \min\{b_1(x'), \ldots, b_s(x')\}\big\},$$

$\pi_n(\mathrm{Int}(\mathcal{K}_2)) = \{\mathtt{x}' \in \mathbb{R}^{n-1} : \mathtt{c}_1(\mathtt{x}') > 0, \dots, \mathtt{c}_{m-r-s}(\mathtt{x}') > 0\}.$

By [29, II, Thm. 6.6] we have $\mathrm{Int}(\mathcal{P}) = \pi_n(\mathrm{Int}(\mathcal{K}))$. Thus

$$\mathrm{Int}(\mathcal{P}) = \pi_n(\mathrm{Int}(\mathcal{K})) = \pi_n(\mathrm{Int}(\mathcal{K}_1)) \cap \pi_n(\mathrm{Int}(\mathcal{K}_2)). \tag{E.6}$$

By Remark 5.6 the difference $(\mathrm{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathcal{K}$ has two connected components. Namely

$$\mathcal{C}^- := \{(\mathtt{x}', \mathtt{x}_n) \in \mathrm{Int}(\mathcal{P}) \times \mathbb{R} : \mathtt{x}_n < \max\{\mathtt{a}_1(\mathtt{x}'), \dots, \mathtt{a}_r(\mathtt{x}')\}\} \quad \text{(basement)},$$
$$\mathcal{C}^+ := \{(\mathtt{x}', \mathtt{x}_n) \in \mathrm{Int}(\mathcal{P}) \times \mathbb{R} : \min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\} < \mathtt{x}_n\} \quad \text{(attic)}.$$

The next result provides a rational function $\frac{\mathtt{f}_2}{\mathtt{f}_1} \in \mathbb{R}(\mathtt{x}_1, \dots, \mathtt{x}_{n-1})$ that separates $\mathcal{C}^-$, $\mathcal{C}^+$ and is regular on $\mathrm{Int}(\mathcal{P})$.

Proposition 5.9 (Rational separation of the attic and the basement) *Write* $\ell := \ell(r, s) = \max\{r-1, 1\} \cdot \max\{s-1, 1\} \le \frac{(m-1)^2}{4}$. *Then there exists a polynomial* $\mathtt{P}(x) := \mathtt{f}_1(x')\,x_n - \mathtt{f}_2(x') \in \mathbb{R}[x', x_n]$ *of degree* $\le 3^\ell$ *such that*

- $\{\mathtt{f}_1 = 0\} \subset \{\mathtt{f}_2 = 0\}$,
- $\mathtt{P}|_{\mathcal{C}^-} < 0$ *and* $\mathtt{P}|_{\mathcal{C}^+} > 0$,
- $\mathtt{f}_1|_{\mathrm{Int}(\mathcal{P})} > 0$ *and* $\mathtt{f}_1|_{\mathcal{P}} \ge 0$.

Proof Let $\phi_{r,s}(\mathtt{y}; \mathtt{z}) = \frac{\mathtt{g}_2(\mathtt{y};\mathtt{z})}{\mathtt{g}_1(\mathtt{y};\mathtt{z})}$ be a rational separator for (r, s), where $\mathtt{g}_1, \mathtt{g}_2 \in \mathbb{R}[\mathtt{y}; \mathtt{z}]$ are (non-zero) homogeneous polynomials of respective degrees $3^\ell - 1$ and 3^ℓ such that $\{\mathtt{g}_1 = 0\} \subset \{\mathtt{g}_2 = 0\}$. Recall that $\phi_{r,s}$ is regular on $\mathrm{Int}(\mathcal{Q}_{r,s})$ and extends to a continuous semialgebraic function $\Phi_{r,s}$ on $\mathcal{Q}_{r,s}$. As $\mathcal{Q}_{r,s}$ is by definition a convex polyhedron and $\phi_{r,s}$ is regular on $\mathrm{Int}(\mathcal{Q}_{r,s})$, we may assume that $\mathtt{g}_1$ has constant sign on $\mathrm{Int}(\mathcal{Q}_{r,s})$ and in fact that $\mathtt{g}_1$ is strictly positive on $\mathrm{Int}(\mathcal{Q}_{r,s})$. Define

$$\begin{aligned} \mathtt{P}(\mathtt{x}) &:= \mathtt{f}_1(\mathtt{x}')\,\mathtt{x}_n - \mathtt{f}_2(\mathtt{x}') \quad \text{where} \\ \mathtt{f}_k(\mathtt{x}') &= \mathtt{g}_k(\mathtt{a}_1(\mathtt{x}'), \dots, \mathtt{a}_r(\mathtt{x}'); \mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')) \end{aligned}$$

for $k = 1, 2$. Note that $\{\mathtt{f}_1 = 0\} \subset \{\mathtt{f}_2 = 0\}$, $\deg(\mathtt{f}_1) \le 3^\ell - 1$ and $\deg(\mathtt{f}_2) \le 3^\ell$. As $\phi_{r,s}$ is a rational separator for (r, s) and $\mathrm{Int}(\mathcal{P}) \subset \pi_n(\mathrm{Int}(\mathcal{K}_1))$, we deduce: *if* $\mathtt{x}' \in \mathrm{Int}(\mathcal{P})$, *then*

$$\max\{\mathtt{a}_1(\mathtt{x}'), \dots, \mathtt{a}_r(\mathtt{x}')\} < \frac{\mathtt{f}_2(\mathtt{x}')}{\mathtt{f}_1(\mathtt{x}')} < \min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\}.$$

Besides, $\mathtt{f}_1(\mathtt{x}') := \mathtt{g}_1(\mathtt{a}_1(\mathtt{x}'), \dots, \mathtt{a}_r(\mathtt{x}'); \mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')) > 0$ *for each* $\mathtt{x}' \in \mathrm{Int}(\mathcal{P})$.

Now, if $\mathtt{x} := (\mathtt{x}', \mathtt{x}_n) \in \mathcal{C}^-$, we have $\mathtt{x}' \in \mathrm{Int}(\mathcal{P})$ and there exists some $i = 1, \dots, r$ such that $\mathtt{x}_n < \mathtt{a}_i(\mathtt{x}')$, so $\mathtt{P}(\mathtt{x}) < 0$. Similarly, if $\mathtt{x} := (\mathtt{x}', \mathtt{x}_n) \in \mathcal{C}^+$, we have $\mathtt{x}' \in \mathrm{Int}(\mathcal{P})$ and there exists some $j = 1, \dots, s$ such that $\mathtt{b}_j(\mathtt{x}') < \mathtt{x}_n$, so $\mathtt{P}(\mathtt{x}) > 0$, as required. □

Some technicalities arising from the proof of Theorem 1.5 force us to find a kind of analogous result to Proposition 5.9 when we are dealing with non-degenerate convex polyhedra not placed in $\vec{\ell}_n$-bounded position.

5.4.2 *Non-degenerate convex polyhedra not placed in $\vec{\ell}_n$-bounded position*. Let $\mathcal{K} \subset \mathbb{R}^n$ be a non-degenerate n-dimensional convex polyhedron not placed in $\vec{\ell}_n$-bounded position. Let $\mathcal{F}_1, \dots, \mathcal{F}_m$ be the facets of $\mathcal{K}$ and let $H_i := \{\mathtt{h}_i = 0\}$ be the hyperplane generated by $\mathcal{F}_i$. Assume that $\mathcal{K} = \bigcap_{i=1}^m H_i^+$. We may write

$$\mathtt{h}_i(\mathtt{x}) = \mathtt{b}_i(\mathtt{x}') + \varepsilon_i \mathtt{x}_n \quad \forall\, i = 1, \dots, s,$$
$$\mathtt{h}_{s+k}(\mathtt{x}) = \mathtt{c}'_k(\mathtt{x}') \quad \forall\, k = 1, \dots, m-s,$$

where $\varepsilon_i = \pm 1$. As $\mathcal{K}$ is non-degenerate, $s \geq 1$ and as $\mathcal{K} \subset \mathbb{R}^n$ is not placed in $\vec{\ell}_n$-bounded position, all ε_i are either equal to 1 or -1 (that is, all non-vertical facets constitute either the floor or the ceiling of $\mathcal{K}$). Assume that all $\varepsilon_i = -1$. Observe that $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ where

$$\mathcal{K}_1 := \{(\mathtt{x}', \mathtt{x}_n) \in \mathbb{R}^n : \mathtt{x}_n \leq \min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\}\},$$
$$\mathcal{K}_2 := \{(\mathtt{x}', \mathtt{x}_n) \in \mathbb{R}^n : \mathtt{c}_1(\mathtt{x}') \geq 0, \dots, \mathtt{c}_{m-s}(\mathtt{x}') \geq 0\}.$$

In addition

$$\mathrm{Int}(\mathcal{K}_1) = \{(\mathtt{x}', \mathtt{x}_n) \in \mathbb{R}^n : \mathtt{x}_n < \min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\}\},$$
$$\mathrm{Int}(\mathcal{K}_2) = \{(\mathtt{x}', \mathtt{x}_n) \in \mathbb{R}^n : \mathtt{c}_1(\mathtt{x}') > 0, \dots, \mathtt{c}_{m-s}(\mathtt{x}') > 0\}.$$

Notice that $\pi_n(\mathcal{K}_1) = \mathbb{R}^{n-1}$ and

$$\pi_n(\mathcal{K}_2) = \{\mathtt{x}' \in \mathbb{R}^{n-1} : \mathtt{c}_1(\mathtt{x}') \geq 0, \dots, \mathtt{c}_{m-s}(\mathtt{x}') \geq 0\}.$$

As $\pi_n^{-1}(\pi_n(\mathcal{K}_2)) = \mathcal{K}_2$, we have $\mathcal{P} := \pi_n(\mathcal{K}) = \pi_n(\mathcal{K}_1) \cap \pi_n(\mathcal{K}_2) = \pi_n(\mathcal{K}_2)$. Thus, as $\pi_n(\mathrm{Int}(\mathcal{K})) = \pi_n(\mathrm{Int}(\mathcal{K}_1)) \cap \pi_n(\mathrm{Int}(\mathcal{K}_2)) = \pi_n(\mathrm{Int}(\mathcal{K}_2))$,

$$\begin{aligned}\mathrm{Int}(\mathcal{P}) = \mathrm{Int}(\pi_n(\mathcal{K})) &= \{\mathtt{x}' \in \mathbb{R}^{n-1} : \mathtt{c}_1(\mathtt{x}') > 0, \dots, \mathtt{c}_{m-s}(\mathtt{x}') > 0\} \\ &= \pi_n(\mathrm{Int}(\mathcal{K}_2)) = \pi_n(\mathrm{Int}(\mathcal{K})).\end{aligned}$$

In particular, $\mathcal{K}_2 = \mathcal{P} \times \mathbb{R}$ and $\mathrm{Int}(\mathcal{K}_2) = \mathrm{Int}(\mathcal{P}) \times \mathbb{R}$. Notice that: *the semialgebraic set*

$$\mathcal{C}^+ := (\mathrm{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathcal{K} = \{(\mathtt{x}', \mathtt{x}_n) \in \mathcal{K}_2 : \min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\} < \mathtt{x}_n\}$$

is connected.

Proposition 5.10 *There exists a polynomial* $\mathtt{P}(x) := \mathtt{f}_1(x')\, x_n - \mathtt{f}_2(x') \in \mathbb{R}[x', x_n]$ *of degree* 2 *such that* $\mathtt{f}_1(x') = 1$, $-\mathtt{f}_2 > 0$ *on* $\mathbb{R}^{n-1}$ *and* $\mathtt{P}|_{\mathcal{C}^+} > 0$.

Proof Let $\mathtt{f}_2(\mathtt{x}') := -\sum_{j=1}^{s} \frac{\mathtt{b}_j^2(\mathtt{x}')+1}{2}$ and $\mathtt{P}(\mathtt{x}) := \mathtt{x}_n - \mathtt{f}_2(\mathtt{x}')$. Observe that if $\mathtt{x} = (\mathtt{x}', \mathtt{x}_n) \in \mathcal{C}^+ := (\mathrm{Int}(\mathcal{P}) \times \mathbb{R}) \setminus \mathcal{K}$, then $\mathtt{x}_n - \min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\} > 0$. Thus

$$\begin{aligned} -\mathtt{x}_n &< -\min\{\mathtt{b}_1(\mathtt{x}'), \dots, \mathtt{b}_s(\mathtt{x}')\} = \max\{-\mathtt{b}_1(\mathtt{x}'), \dots, -\mathtt{b}_s(\mathtt{x}')\} \\ &< \sum_{j=1}^{s} \frac{\mathtt{b}_j^2(\mathtt{x}') + 1}{2} \end{aligned}$$

for each $\mathtt{x}' \in \mathbb{R}^{n-1}$, so $\mathtt{P}(\mathtt{x}', \mathtt{x}_n) > 0$ for each $(\mathtt{x}', \mathtt{x}_n) \in \mathcal{C}^+$. □

6 Complements of Convex Polyhedra

In this section we prove constructively Theorem 1.5. In fact, we prove a more general statement, since we do not assume in the sequel the unboundedness condition. We develop first in Sect. 6.1 some basic tools. In this regard, Lemma 6.2 describes the properties of polynomial maps $\mathtt{T}_{\mathcal{K}}$ (see (F.3)), which involve the rational separators we have dealt with in the previous section. The ideal situation would be that, given a convex polyhedron $\mathcal{K}$ and one of its facets $\mathcal{F}_i$, a polynomial map $\mathtt{T}_{\mathcal{K}}$ would map $\mathbb{R}^n \setminus \mathcal{K}_{i,\times}$ onto $\mathbb{R}^n \setminus \mathcal{K}$ (see Sect. 2.1 for the definition of $\mathcal{K}_{i,\times}$). This would allow a neat inductive proof of Theorem 1.5. However, the zero set of the denominator of the chosen representation of the involved rational separator produces some difficulties. To take care of them we introduce Corollaries 6.3, 6.5 and 6.6. We next consider the complement $\mathbb{R}^n \setminus \mathrm{Int}(\mathcal{K}_0)$, where $\mathcal{K}_0$ is a polyhedron that contains $\mathcal{K}$ tightly (in a sense to be described in Sect. 6.2). This complement is by Theorem 1.4 a polynomial image of $\mathbb{R}^n$. Then we are in a position to apply a sequence of polynomial maps of type (F.3) whose images progressively fill the remaining gap $\mathrm{Int}(\mathcal{K}_0) \setminus \mathcal{K}$, until we finally accomplish the representation of $\mathbb{R}^n \setminus \mathcal{K}$ as the image of a finite composition of polynomial maps. We hope this brief explanation will soften the technicalities of the process.

6.1 Basic Tools for the Inductive Process

Let $\mathcal{K} \subset \mathbb{R}^n$ be a non-degenerate n-dimensional convex polyhedron and let $\mathcal{P} := \pi_n(\mathcal{K})$. Assume that $\vec{e}_n \notin \vec{\mathfrak{C}}(\mathcal{K})$. Let

$$\mathtt{P}(\mathtt{x}', \mathtt{x}_n) := \mathtt{f}_1(\mathtt{x}')\,\mathtt{x}_n - \mathtt{f}_2(\mathtt{x}') \in \mathbb{R}[\mathtt{x}', \mathtt{x}_n] \tag{F.1}$$

be a polynomial satisfying the conditions of Proposition 5.9 if $\mathcal{K}$ is placed in $\vec{\ell}_n$-bounded position and the conditions of Proposition 5.10 otherwise. Write $\mathcal{K} := \bigcap_{i=1}^{m} \{\mathtt{h}_i \geq 0\}$ (minimal presentation) where each $\mathtt{h}_i$ is a linear equation. We may assume that the coefficient of $\mathtt{x}_n$ is non-zero for $\mathtt{h}_i$ if and only if $i = 1, \dots, d \leq m$. As $\mathcal{K}$ is non-degenerate, $d \geq 1$. Consider the polynomial

$$\mathtt{h} := \prod_{i=1}^{m} \mathtt{h}_i \in \mathbb{R}[\mathtt{x}] \tag{F.2}$$

and the polynomial map

$$\begin{aligned} &\mathtt{T}_{\mathcal{K}}\colon \mathbb{R}^n \to \mathbb{R}^n,\\ &\quad \mathtt{x} := (\mathtt{x}', \mathtt{x}_n) = (\mathtt{x}_1, \dots, \mathtt{x}_{n-1}, \mathtt{x}_n) \mapsto (\mathtt{x}', \mathtt{x}_n - \mathtt{x}_{n-1}\mathtt{h}^2(\mathtt{x})\,\mathtt{P}(\mathtt{x})). \end{aligned} \tag{F.3}$$

Note that if $H_i := \{\mathtt{h}_i = 0\}$ then $\mathtt{T}_{\mathcal{K}}|_{H_i} = \mathrm{id}_{H_i}$ because $\mathtt{h}|_{H_i} \equiv 0$. Fix $a' := (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ and denote the vertical line through the point $(a', 0)$ by $\ell_{a'} := (a', 0) + \vec{\ell}_n$.

Lemma 6.1 *For each $a' \in \mathbb{R}^{n-1}$ we have $\mathtt{T}_{\mathcal{K}}(\ell_{a'}) = \ell_{a'}$.*

Proof In order to see this we prove: *The polynomial $\mathtt{Q}_{a'}(\mathtt{t}) := \mathtt{t} - a_{n-1}\mathtt{h}^2(a', \mathtt{t})\,\mathtt{P}(a', \mathtt{t})$ has odd degree for each $a' \in \mathbb{R}^{n-1}$*. In addition, $\mathtt{Q}_{a'}(\mathtt{t}) = \mathtt{t}$ *if and only if either* $a_{n-1}\mathtt{h}^2(a', \mathtt{t}) \equiv 0$ *or* $\mathtt{f}_1(a') = 0$.

Indeed, if $\mathcal{K}$ is placed in $\vec{\ell}_n$-bounded position and $\mathtt{f}_1(a') = 0$, then by Proposition 5.9 $\mathtt{f}_2(a') = 0$, so $\mathtt{P}(a', \mathtt{t}) \equiv 0$ and $\mathtt{Q}_{a'}(\mathtt{t}) = \mathtt{t}$ has odd degree. If $\mathcal{K}$ is not placed in $\vec{\ell}_n$-bounded position, $\mathtt{P}(a', \mathtt{t})$ is a monic polynomial of degree 1. Therefore we may assume that $\mathtt{f}_1(a') \neq 0$ and $\mathtt{P}(a', \mathtt{t})$ is a polynomial of degree 1. As $\mathcal{K}$ is non-degenerate, we have that $\mathtt{h}^2(a', \mathtt{t})$ is either identically zero or a polynomial of positive degree $2d > 0$. Therefore, $\mathtt{Q}_{a'}(\mathtt{t})$ is either a polynomial of degree 1 (if $a_{n-1}\mathtt{h}^2(a', \mathtt{t}) \equiv 0$) or of degree $2d + 1 > 1$ (otherwise). □

Let us analyze the behavior of $\mathtt{T}_{\mathcal{K}}$ over certain subsets of the line $\ell_{a'}$ attending to the position of the latter with respect to $\mathcal{K}$ (Fig. 7).

Proposition 6.2 *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional non-degenerate convex polyhedron and let $\mathcal{P} := \pi_n(\mathcal{K})$. Let $\mathcal{G}, \mathcal{R}$ be sets such that* $\mathrm{Int}(\mathcal{K}) \subset \mathcal{G} \subset \mathcal{K}$ *and $\mathcal{R} \subset \partial\mathcal{K}$. Given $a' := (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, we have:*

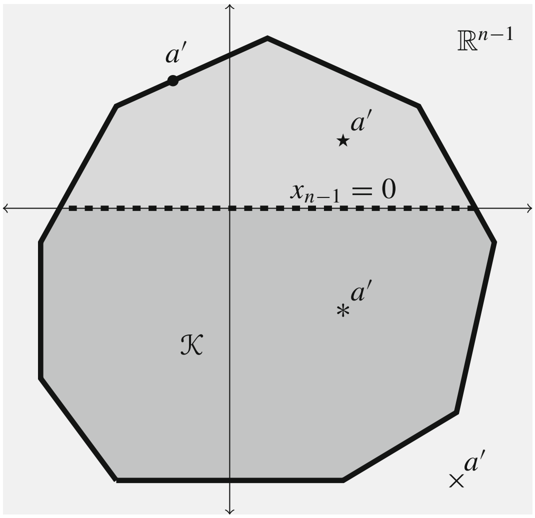


Fig. 7 Analyzed positions (i)–(iv) for $a' \in \mathbb{R}^{n-1}$

(i) *If* $a' \notin \mathcal{P}$, *then* $\ell_{a'} \backslash \mathcal{K} = \ell_{a'}$ *and* $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{K}) = \mathtt{T}_{\mathcal{K}}(\ell_{a'}) = \ell_{a'}$.
(ii) *If* $a' \in \mathcal{P}$ *and* $a_{n-1} \leq 0$, *then* $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}) = \ell_{a'} \backslash \mathcal{G}$.
(iii) *If* $a' \in \mathrm{Int}(\mathcal{P})$ *and* $a_{n-1} > 0$, *then* $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'}$ *and* $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}) = \ell_{a'}$ *if* $\ell_{a'} \cap \mathcal{G}$ *is a bounded set.*
(iv) *If* $a' \in \partial\mathcal{P}$ *and* $a_{n-1} > 0$, *then*

$$\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \begin{cases} \ell_{a'} \backslash \mathcal{R} & \textit{if } \mathtt{f}_1(a') = 0 \textit{ or } \mathtt{h}(a', \mathtt{t}) \equiv 0, \\ \ell_{a'} & \textit{if } \mathtt{f}_1(a') \neq 0 \textit{ and } \mathtt{h}(a', \mathtt{t}) \not\equiv 0. \end{cases}$$

In particular, $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'} \backslash \mathcal{R}$ *if* $\ell_{a'} \cap \mathcal{R}$ *contains at least two points.*

Proof (i) As $\ell_{a'} \backslash \mathcal{K} = \ell_{a'}$, if follows from Lemma 6.1 that $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{K}) = \mathtt{T}_{\mathcal{K}}(\ell_{a'}) = \ell_{a'}$.

We prove next statements (ii), (iii) and (iv). For all of them $a' \in \mathcal{P}$. Then the set $\mathcal{K} \cap \ell_{a'}$ is either a segment (which can reduce to a point) or a ray. The end point(s) of $\mathcal{K} \cap \ell_{a'}$ belong(s) to $\bigcup_{i=1}^{d} \{\mathtt{h}_i = 0\}$. If $\mathcal{K} \cap \ell_{a'}$ is a segment, we denote the end points of $\mathcal{K} \cap \ell_{a'}$ by $p^- := (a', t^-)$ and $p^+ := (a', t^+)$, where $t^- \leq t^+$. If $\mathcal{K} \cap \ell_{a'}$ is a ray, we denote its end point by $p^+ := (a', t^+)$. To avoid unnecessary repetitions, we abuse notation and set $\mathcal{C}^- = \varnothing$ and $p^- := (a', t^-)$ where $t^- := -\infty$ if $\mathcal{K} \cap \ell_{a'}$ is a ray. Consequently,

$$\ell_{a'} = (\ell_{a'} \cap \mathcal{C}^-) \cup \{p^-\} \cup (\ell_{a'} \cap \mathrm{Int}(\mathcal{K})) \cup \{p^+\} \cup (\ell_{a'} \cap \mathcal{C}^+).$$

We can write the differences $\ell_{a'} \backslash \mathcal{G}$ and $\ell_{a'} \backslash \mathcal{R}$ as follows:

$$\begin{aligned} \ell_{a'} \backslash \mathcal{G} &= (\ell_{a'} \cap \mathcal{C}^-) \cup (\ell_{a'} \cap \mathcal{C}^+) \cup \mathfrak{F}_{\mathcal{G}}, \\ \ell_{a'} \backslash \mathcal{R} &= (\ell_{a'} \cap \mathcal{C}^-) \cup (\ell_{a'} \cap \mathrm{Int}(\mathcal{K})) \cup (\ell_{a'} \cap \mathcal{C}^+) \cup \mathfrak{F}_{\mathcal{R}} \end{aligned}$$

for some $\mathfrak{F}_{\mathcal{G}}, \mathfrak{F}_{\mathcal{R}} \subset \{p^-, p^+\}$. Observe that

$$\ell_{a'} \cap \mathcal{C}^- = \{(a, t) : t < t^-\} \quad \text{and} \quad \ell_{a'} \cap \mathcal{C}^+ := \{(a, t) : t > t^+\}.$$

By Proposition 5.9 we have $\mathtt{P}|_{\mathcal{C}^-} < 0$ and $\mathtt{P}|_{\mathcal{C}^+} > 0$. Also the equality $\mathtt{h}(p^+) = 0$ holds, so $\mathtt{T}_{\mathcal{K}}(p^+) = p^+$. Analogously, if $t^- > -\infty$ then $\mathtt{T}_{\mathcal{K}}(p^-) = p^-$.

(ii) If $a_{n-1} = 0$, then $\mathtt{Q}_{a'}(\mathtt{t}) = \mathtt{t}$, so $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}) = \ell_{a'} \backslash \mathcal{G}$. Assume next that $a_{n-1} < 0$. If $t > t^+$, then $\mathtt{P}(a', t) > 0$, so $\mathtt{Q}_{a'}(t) \geq t > t^+$ and $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \cap \mathcal{C}^+) = \ell_{a'} \cap \mathcal{C}^+$ (because $\mathtt{T}_{\mathcal{K}}(p^+) = p^+$). Similarly, if $\ell_{a'} \cap \mathcal{C}^- \neq \varnothing$ and $t < t^-$ then $\mathtt{P}(a', t) < 0$, so $\mathtt{Q}_{a'}(t) \leq t < t^-$ and $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \cap \mathcal{C}^-) = \ell_{a'} \cap \mathcal{C}^-$ (because $\mathtt{T}_{\mathcal{K}}(p^-) = p^-$). Consequently, $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}) = \ell_{a'} \backslash \mathcal{G}$.

(iii) By Proposition 5.9 we have $\mathtt{f}_1(a') > 0$, so the univariate polynomial $\mathtt{P}(a', \mathtt{t}) = \mathtt{f}_1(a')\mathtt{t} - \mathtt{f}_2(a')$ has degree one and its leading coefficient is positive. As $a' \in \mathrm{Int}(\mathcal{P})$, the polynomial $\mathtt{h}^2(a, \mathtt{t})$ has even positive degree in $\mathtt{t}$ and positive leading coefficient. Consequently, $\mathtt{Q}_{a'}(\mathtt{t})$ is a univariate polynomial of odd degree whose leading coefficient is negative, so $\lim_{t \to \pm\infty} \mathtt{Q}_{a'}(t) = \mp\infty$.

If $\ell_{a'} \cap \mathcal{G}$ is a bounded set, then $p^- \in \partial\mathcal{K}$. As $\mathtt{Q}_{a'}(t^-) = t^-$, $\mathtt{Q}_{a'}(t^+) = t^+$ and $t^- < t^+$ (because $a' \in \operatorname{Int}(\mathcal{P})$), we deduce

$$\mathbb{R} =]-\infty, t^+[\cup]t^-, +\infty[\subset \mathtt{Q}_{a'}(]t^+, +\infty[) \cup \mathtt{Q}_{a'}(]-\infty, t^-[) \subset \mathbb{R}.$$

Consequently, $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}) = \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{K}) = \ell_{a'}$.

As $a' \in \operatorname{Int}(\mathcal{P})$, we have that $\ell_{a'} \cap \mathcal{R} \subset \ell_{a'} \cap \partial\mathcal{K}$ consists of at most two points.

Case 1: If $\ell_{a'} \cap \mathcal{R}$ contains two different points, then $\ell_{a'} \cap \mathcal{G}$ is a bounded set and $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{K}) = \ell_{a'}$, as we have seen above.
Case 2: If $\ell_{a'} \cap \mathcal{R}$ is a singleton $\{(a', t_0)\}$, we observe that $\mathtt{h}(a', t_0) = 0$ and

$$\frac{\partial}{\partial \mathtt{t}} \mathtt{Q}_{a'}(\mathtt{t}) = 1 - a_{n-1}\mathtt{h}(a', \mathtt{t}) \left(2\, \frac{\partial \mathtt{h}}{\partial \mathtt{x}_n}(a', \mathtt{t})\, \mathtt{P}(a', \mathtt{t}) + \mathtt{h}(a', \mathtt{t})\, \frac{\partial \mathtt{P}}{\partial \mathtt{x}_n}(a', \mathtt{t}) \right)$$
$$\rightsquigarrow \frac{\partial}{\partial \mathtt{t}}\, \mathtt{Q}_{a'}(t_0) = 1.$$

Thus, there exists $\varepsilon > 0$ such that $\mathtt{Q}_{a'}(t_0 - \varepsilon) =: s_1 < t_0 < s_2 := \mathtt{Q}_{a'}(t_0 + \varepsilon)$. As $\lim_{t \to \pm\infty} \mathtt{Q}_{a'}(t) = \mp\infty$, we have

$$\mathbb{R} =]-\infty, s_2[\cup]s_1, +\infty[\subset \mathtt{Q}_{a'}(]t_0 - \varepsilon, +\infty[) \cup \mathtt{Q}_{a'}(]-\infty, t_0 + \varepsilon[) \subset \mathbb{R}.$$

Consequently, $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'}$.

(iv) By Sect. 2.2 we have $\ell_{a'} \cap \mathcal{K} \subset \partial\mathcal{K}$. If $\mathtt{f}_1(a') = 0$, then by Proposition 5.9 also $\mathtt{f}_2(a') = 0$ and $\mathtt{Q}_{a'}(\mathtt{t}) = \mathtt{t}$, so $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'} \backslash \mathcal{R}$. Analogously, if $\mathtt{h}(a', \mathtt{t}) \equiv 0$, then $\mathtt{Q}_{a'}(\mathtt{t}) = \mathtt{t}$, so $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'} \backslash \mathcal{R}$.

If $\mathtt{f}_1(a') \neq 0$ and $\mathtt{h}(a', \mathtt{t}) \not\equiv 0$, then $\ell_{a'} \cap \mathcal{K}$ reduces to a point and $\mathcal{R} \cap \ell_{a'}$ can be either empty or a singleton. If $\mathcal{R} \cap \ell_{a'} = \varnothing$ it is clear by Lemma 6.1 that $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'}$. If $\mathcal{R}$ is a singleton, it follows by Case 2 that $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'}$.

Next, if $\mathcal{R} \cap \ell_{a'}$ contains two different points, then by Sect. 2.2 the segment connecting these two points is contained in $\ell_{a'} \cap \mathcal{K} \subset \partial\mathcal{K}$. Thus, $\mathcal{R}$ is contained in a vertical facet of $\mathcal{K}$. Consequently, $\mathtt{h}(a', \mathtt{t}) \equiv 0$, so $\mathtt{Q}_{a'}(\mathtt{t}) = \mathtt{t}$ and $\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}) = \ell_{a'} \backslash \mathcal{R}$. □

We present next some technical consequences of the previous result, which are useful for the proof of Theorem 1.5. We need to understand how are the images under $\mathtt{T}_{\mathcal{K}}$ of certain sets $\mathcal{G}$ satisfying $\operatorname{Int}(\mathcal{K}) \subset \mathcal{G} \subset \mathcal{K}$, which consist of the interior of the polyhedron $\mathcal{K}$ together with portions of some of its faces. From now on we denote the hyperplane $\{\mathtt{x}_{n-1} = 0\}$ by Π_{n-1} and the half-spaces $\{\mathtt{x}_{n-1} \leq 0\}$ and $\{\mathtt{x}_{n-1} \geq 0\}$ by Π_{n-1}^- and Π_{n-1}^+ respectively.

Corollary 6.3 *Let $\mathcal{K} \subset \mathbb{R}^n$ be an n-dimensional non-degenerate convex polyhedron placed in $\vec{\ell}_n$-bounded position and let $\mathcal{P} := \pi_n(\mathcal{K})$. Let $\mathcal{G}$ be a set such that* $\operatorname{Int}(\mathcal{K}) \subset \mathcal{G} \subset \mathcal{K}$. *Let us set $\mathcal{P} := \pi_n(\mathcal{K})$ and $\mathcal{G}^- := \mathcal{G} \cap \Pi_{n-1}^-$. Then*

$$(\mathbb{R}^n\backslash \mathcal{G}^-) \setminus (\mathcal{G} \cap \pi_n^{-1}(\partial \mathcal{P}) \cap \mathrm{Int}(\Pi_{n-1}^+)) \subset \mathtt{T}_{\mathcal{K}}(\mathbb{R}^n\backslash \mathcal{G}) \subset \mathbb{R}^n\backslash \mathcal{G}^- \tag{F.4}$$

and $\mathtt{T}_{\mathcal{K}}(\mathbb{R}^n\backslash \mathcal{G}^-) = \mathbb{R}^n\backslash \mathcal{G}^-$.

Proof Let $a' \in \mathbb{R}^{n-1}$ and let $\ell_{a'}$ be the vertical line through $(a', 0)$. If $a_{n-1} \le 0$, we have by Proposition 6.2 (i) and (ii)

$$\mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G}) = \ell_{a'}\backslash \mathcal{G} = \ell_{a'}\backslash \mathcal{G}^-.$$

Consequently

$$\begin{aligned} \mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash \mathcal{G}) \cap \Pi_{n-1}^-) &= \bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}\le 0}} \mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G}) \\ &= \bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}\le 0}} \ell_{a'}\backslash \mathcal{G}^- = (\mathbb{R}^n\backslash \mathcal{G}^-) \cap \Pi_{n-1}^-. \end{aligned} \tag{F.5}$$

If $a_{n-1} > 0$, we have by Lemma 6.1 and Proposition 6.2,

$$\mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G}) = \begin{cases} \ell_{a'} & \text{if } \ell \cap \mathcal{G} = \varnothing, \\ \ell_{a'} & \text{if } a' \in \mathrm{Int}\,\mathcal{P} \text{ (because } \ell_{a'} \cap \mathcal{K} \text{ is bounded)}, \\ \ell_{a'}\backslash \mathcal{G} & \text{if } a' \in \partial\mathcal{P} \text{ and } \ell \cap \mathcal{G} \text{ contains at least two points.} \end{cases}$$

In case $a' \in \partial\mathcal{P}$ and $\ell \cap \mathcal{G}$ is a singleton we have $\ell_{a'}\backslash \mathcal{G} \subset \mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G}) \subset \ell_{a'}$. Thus,

$$\begin{cases} \mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G}) = \ell_{a'} & \text{if } a' \notin \partial\mathcal{P},\ a_{n-1} > 0, \\ \ell_{a'}\backslash \mathcal{G} \subset \mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G}) \subset \ell_{a'} & \text{if } a' \in \partial\mathcal{P},\ a_{n-1} > 0. \end{cases} \tag{F.6}$$

Using the following equality:

$$\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash \mathcal{G}) \cap \mathrm{Int}(\Pi_{n-1}^+)) = \bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}> 0}} \mathtt{T}_{\mathcal{K}}(\ell_{a'}\backslash \mathcal{G})$$

and (F.6) we conclude

$$\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash \mathcal{G}) \cap \mathrm{Int}(\Pi_{n-1}^+)) \subset \bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}> 0}} \ell_{a'} = \mathrm{Int}(\Pi_{n-1}^+) = (\mathbb{R}^n\backslash \mathcal{G}^-) \cap \mathrm{Int}(\Pi_{n-1}^+),$$

$$\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash \mathcal{G}) \cap \mathrm{Int}(\Pi_{n-1}^+)) \supset \bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}> 0, a'\notin\partial\mathcal{P}}} \ell_{a'} \cup \bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}> 0, a'\in\partial\mathcal{P}}} \ell_{a'}\backslash \mathcal{G}$$

$$= \mathrm{Int}(\Pi^+_{n-1})\backslash(\mathcal{G}\cap\pi_n^{-1}(\partial\mathcal{P}))$$
$$= ((\mathbb{R}^n\backslash\mathcal{G}^-)\cap\mathrm{Int}(\Pi^+_{n-1}))\backslash(\mathcal{G}\cap\pi_n^{-1}(\partial\mathcal{P})\cap\mathrm{Int}(\Pi^+_{n-1})). \tag{F.7}$$

The last equality in the statement follows from (F.5) and Lemma 6.1. Therefore, by (F.5) and (F.7) the inclusions in (F.4) follow, as required. □

Lemma 6.4 *Let $\mathcal{E}$ be a face of an n-dimensional convex polyhedron $\mathcal{K}\subset\mathbb{R}^n$ and let $p, q\in\mathrm{Int}(\mathcal{E})$. Let ℓ_p be a line through p that meets $\mathrm{Int}(\mathcal{K})$ and let ℓ_q be a line through q and parallel to ℓ_p. Then ℓ_q also meets $\mathrm{Int}(\mathcal{K})$.*

Proof Write $\mathcal{K}=\bigcap_{i=1}^r\{\mathtt{h}_i\geq 0\}$ where each $\mathtt{h}_i$ is a linear equation and $\ell_p:=\{p+\mathtt{t}\vec{v}:\mathtt{t}\in\mathbb{R}\}$. We may assume $\mathrm{Int}(\mathcal{E})=\bigcap_{i=1}^k\{\mathtt{h}_i=0\}\cap\bigcap_{i=k+1}^r\{\mathtt{h}_i>0\}$, so $\mathtt{h}_i(p)=0$ for $i=1,\dots,k$ and $\mathtt{h}_i(p)>0$ for $i=k+1,\dots,r$. As $\ell_p\cap\mathrm{Int}(\mathcal{K})\neq\varnothing$, we may assume (changing $\vec{v}$ by $-\vec{v}$ if necessary) that there exists $t>0$ such that $\mathtt{h}_i(p+t\vec{v})>0$ for $i=1,\dots,r$. Consequently, $\vec{\mathtt{h}}_i(\vec{v})>0$ for $i=1,\dots,k$. As $\mathtt{h}_i(q)=0$ for $i=1,\dots,k$ and $\mathtt{h}_i(q)>0$ for $i=k+1,\dots,r$, we have $\mathtt{h}_i(q+t'\vec{v})>0$ for $i=1,\dots,r$ and $t'>0$ small enough. Thus, $\ell_q\cap\mathrm{Int}(\mathcal{K})\neq\varnothing$, as required. □

Corollary 6.5 *Let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional non-degenerate convex polyhedron and let $\mathcal{E}$ be a face of $\mathcal{K}$ that is non-parallel to Π_{n-1} and meets the open half-space $\mathrm{Int}(\Pi^+_{n-1})$. Let $\mathcal{R}_0\subset\partial\mathcal{K}$ be such that $\mathrm{Int}(\mathcal{E})\cap\mathcal{R}_0=\varnothing$ and let $\mathcal{G}$ be a set such that $\mathrm{Int}(\mathcal{K})\cup\mathrm{Int}(\mathcal{E})\cup\mathcal{R}_0\subset\mathcal{G}\subset\mathcal{K}$. Denote $\mathcal{G}^-:=\mathcal{G}\cap\Pi^-_{n-1}$ and $\mathcal{P}:=\pi_n(\mathcal{K})$. Then, after an affine change of coordinates that keeps the hyperplane Π_{n-1} invariant (and only depends on $\mathcal{E}$), we have*

$$(\mathbb{R}^n\backslash\mathcal{G}^-)\setminus(\mathcal{R}_0\cap\pi_n^{-1}(\partial\mathcal{P}))\subset\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus(\mathcal{R}_0\cup\mathrm{Int}(\mathcal{E})))\subset\mathbb{R}^n\backslash\mathcal{G}^- \tag{F.8}$$

and $\mathtt{T}_{\mathcal{K}}(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathbb{R}^n\backslash\mathcal{G}^-$.

Proof Take a point $p\in\mathrm{Int}(\mathcal{E})$ and let $\Pi_{n-1,p}$ be the hyperplane through p parallel to Π_{n-1}. As $\mathcal{E}$ is not parallel to Π_{n-1}, the intersection $\Pi_{n-1,p}\cap\mathrm{Int}(\mathcal{K})\neq\varnothing$. Otherwise, as $p\in\mathcal{K}\cap\Pi_{n-1,p}\subset\partial\mathcal{K}$, we have that $\Pi_{n-1,p}$ is a supporting hyperplane for $\mathcal{K}$. As $p\in\mathrm{Int}(\mathcal{E})$, we have $\mathrm{Int}(\mathcal{E})\subset\Pi_{n-1,p}$, so $\mathcal{E}$ is parallel to Π_{n-1}, which is a contradiction.

Let $q\in\mathrm{Int}(\mathcal{K})\cap\Pi_{n-1,p}$. After an affine change of coordinates that keeps the hyperplane Π_{n-1} invariant, we may assume that the line through p and q is vertical and $\vec{e}_n\notin\vec{\mathfrak{C}}(\mathcal{K})$. This latter condition is possible because as $\mathcal{K}$ is non-degenerate, either $\vec{e}_n$ or $-\vec{e}_n$ does not belong to $\vec{\mathfrak{C}}(\mathcal{K})$. Therefore, $\pi_n(p)\in\mathrm{Int}(\mathcal{P})=\pi_n(\mathrm{Int}(\mathcal{K}))$ and by Lemma 6.4 we have $\pi_n(\mathrm{Int}(\mathcal{E}))\subset\mathrm{Int}(\mathcal{P})$.

The rest of the proof is similar to the one of Corollary 6.3. We include all the technicalities for the sake of completeness. Observe that

$$\mathcal{G}^-=(\mathcal{G}^-\cup\mathcal{R}_0\cup\mathrm{Int}(\mathcal{E}))\cap\Pi^-_{n-1}. \tag{F.9}$$

Let $a' \in \mathbb{R}^{n-1}$ and let $\ell_{a'}$ be the vertical line through $(a', 0)$. If $a_{n-1} \leq 0$ we have by Proposition 6.2 (i) and (ii)

$$\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}^-) = \ell_{a'} \backslash \mathcal{G}^-.$$

Consequently

$$\begin{aligned} \mathtt{T}_{\mathcal{K}}((\mathbb{R}^n \backslash \mathcal{G}^-) \cap \Pi^-_{n-1}) &= \bigcup_{\substack{a' \in \mathbb{R}^{n-1} \\ a_{n-1} \leq 0}} \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{G}) \\ &= \bigcup_{\substack{a' \in \mathbb{R}^{n-1} \\ a_{n-1} \leq 0}} \ell_{a'} \backslash \mathcal{G}^- = (\mathbb{R}^n \backslash \mathcal{G}^-) \cap \Pi^-_{n-1}. \end{aligned} \tag{F.10}$$

By (F.9) it holds

$$\begin{aligned} &\mathtt{T}_{\mathcal{K}}(((\mathbb{R}^n \backslash \mathcal{G}^-) \setminus (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) \cap \Pi^-_{n-1}) \\ &\quad = \mathtt{T}_{\mathcal{K}}((\mathbb{R}^n \backslash \mathcal{G}^-) \cap \Pi^-_{n-1}) = (\mathbb{R}^n \backslash \mathcal{G}^-) \cap \Pi^-_{n-1}. \end{aligned} \tag{F.11}$$

Observe also that

$$(\mathcal{G}^- \cup \mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E})) \cap \mathrm{Int}(\Pi^+_{n-1}) = (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E})) \cap \mathrm{Int}(\Pi^+_{n-1}) \subset \partial\mathcal{K} \cap \mathrm{Int}(\Pi^+_{n-1}).$$

We have shown above that $\pi_n(\mathrm{Int}(\mathcal{E})) \subset \mathrm{Int}(\mathcal{P})$. If $a_{n-1} > 0$ we have by Proposition 6.2

$$\mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) = \begin{cases} \ell_{a'} & \text{if } \ell \cap (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E})) = \varnothing, \\ \ell_{a'} & \text{if } a' \in \mathrm{Int}\,\mathcal{P} \text{ (because } \mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}) \subset \partial\mathcal{K}), \\ \ell_{a'} \backslash \mathcal{R}_0 & \text{if } a' \in \partial\mathcal{P} \text{ and } \ell \cap \mathcal{R}_0 \text{ contains at least two points.} \end{cases}$$

In case $a' \in \partial\mathcal{P}$ and $\ell \cap \mathcal{R}_0$ is a singleton, we have $\ell_{a'} \backslash \mathcal{R}_0 \subset \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash \mathcal{R}_0) \subset \ell_{a'}$. Thus,

$$\begin{cases} \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) = \ell_{a'} & \text{if } a' \notin \partial\mathcal{P},\ a_{n-1} > 0, \\ \ell_{a'} \backslash \mathcal{R}_0 \subset \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) \subset \ell_{a'} & \text{if } a' \in \partial\mathcal{P},\ a_{n-1} > 0. \end{cases} \tag{F.12}$$

Using the following equalities:

$$\begin{aligned} &\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n \backslash \mathcal{G}^-) \setminus (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) \cap \mathrm{Int}(\Pi^+_{n-1})) \\ &\quad = \mathtt{T}_{\mathcal{K}}((\mathbb{R}^n \backslash (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) \cap \mathrm{Int}(\Pi^+_{n-1})) \\ &\quad = \bigcup_{\substack{a' \in \mathbb{R}^{n-1} \\ a_{n-1} > 0}} \mathtt{T}_{\mathcal{K}}(\ell_{a'} \backslash (\mathcal{R}_0 \cup \mathrm{Int}(\mathcal{E}))) \end{aligned} \tag{F.13}$$

and (F.12) we conclude

$$
\begin{aligned}
\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash(\mathcal{R}_0\cup\mathrm{Int}(\mathcal{E})))\cap\mathrm{Int}(\Pi^+_{n-1}))&\subset\bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}>0}}\ell_{a'}\\
&=\mathrm{Int}(\Pi^+_{n-1})=(\mathbb{R}^n\backslash\mathcal{G}^-)\cap\mathrm{Int}(\Pi^+_{n-1})\\
\mathtt{T}_{\mathcal{K}}((\mathbb{R}^n\backslash(\mathcal{R}_0\cup\mathrm{Int}(\mathcal{E})))\cap\mathrm{Int}(\Pi^+_{n-1}))&\supset\bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}>0,a'\notin\partial\mathcal{P}}}\ell_{a'}\cup\bigcup_{\substack{a'\in\mathbb{R}^{n-1}\\ a_{n-1}>0,a'\in\partial\mathcal{P}}}\ell_{a'}\backslash\mathcal{R}_0\\
&=(\mathrm{Int}(\Pi^+_{n-1})\backslash(\mathcal{R}_0\cap\pi_n^{-1}(\partial\mathcal{P}))\\
&=(\mathrm{Int}(\Pi^+_{n-1})\cap(\mathbb{R}^n\backslash\mathcal{G}^-))\setminus(\mathcal{R}_0\cap\pi_n^{-1}(\partial\mathcal{P})).
\end{aligned}
\tag{F.14}
$$

From (F.11), (F.13) and (F.14) we obtain (F.8). The last equality in the statement follows from Eq. (F.10) and the fact that $\mathtt{T}_{\mathcal{K}}(\ell_{a'})=\ell_{a'}$ for each $a'\in\mathbb{R}^{n-1}$, as required. □

Corollary 6.6 *Let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional non-degenerate convex polyhedron and let $\mathcal{E}_1,\dots,\mathcal{E}_s$ be finitely many faces of $\mathcal{K}$ such that each one is non-parallel to Π_{n-1} and meets the open half-space $\mathrm{Int}(\Pi^+_{n-1})$. Let $\mathcal{R}:=\bigsqcup_{k=1}^s\mathrm{Int}(\mathcal{E}_k)$ and let $\mathcal{G}$ be a set such that $\mathrm{Int}(\mathcal{K})\cup\mathcal{R}\subset\mathcal{G}\subset\mathcal{K}$. Denote $\mathcal{G}^-:=\mathcal{G}\cap\Pi^-_{n-1}$. Then, there exists a polynomial map $\mathtt{T}\colon\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathtt{T}((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus\mathcal{R})=\mathbb{R}^n\backslash\mathcal{G}^-$ and $\mathtt{T}(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathbb{R}^n\backslash\mathcal{G}^-$.*

Proof We proceed by induction on s. If $s=1$, the statement follows from Corollary 6.5. So let us assume that $s>1$ and let $\mathcal{R}_0:=\bigcup_{k=1}^{s-1}\mathrm{Int}(\mathcal{E}_k)$. By induction hypothesis there exists a polynomial map $\mathtt{T}_0\colon\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathtt{T}_0((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus\mathcal{R}_0))=\mathbb{R}^n\backslash\mathcal{G}^-$ and $\mathtt{T}_0(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathbb{R}^n\backslash\mathcal{G}^-$. On the other hand by Corollary 6.5 there exists a polynomial map $\mathtt{T}_1\colon\mathbb{R}^n\to\mathbb{R}^n$ such that

$$(\mathbb{R}^n\backslash\mathcal{G}^-)\setminus\mathcal{R}_0\subset\mathtt{T}_1((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus(\mathcal{R}_0\cup\mathrm{Int}(\mathcal{E})))\subset\mathbb{R}^n\backslash\mathcal{G}^-$$

and $\mathtt{T}_1(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathbb{R}^n\backslash\mathcal{G}^-$. Consider the polynomial map $\mathtt{T}=\mathtt{T}_0\circ\mathtt{T}_1\colon\mathbb{R}^n\to\mathbb{R}^n$. We have

$$
\begin{aligned}
\mathbb{R}^n\backslash\mathcal{G}^-&=\mathtt{T}_0((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus\mathcal{R}_0)\subset\mathtt{T}_0(\mathtt{T}_1((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus(\mathcal{R}_0\cup\mathrm{Int}(\mathcal{E}))))\\
&=\mathtt{T}((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus\mathcal{R})\subset\mathtt{T}_0(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathbb{R}^n\backslash\mathcal{G}^-,
\end{aligned}
$$

so $\mathtt{T}((\mathbb{R}^n\backslash\mathcal{G}^-)\setminus\mathcal{R}))=\mathbb{R}^n\backslash\mathcal{G}^-$. Finally,

$$\mathtt{T}(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathtt{T}_0(\mathtt{T}_1(\mathbb{R}^n\backslash\mathcal{G}^-))=\mathtt{T}_0(\mathbb{R}^n\backslash\mathcal{G}^-)=\mathbb{R}^n\backslash\mathcal{G}^-,$$

as required. □

6.2 Enveloping Polyhedron for $\mathcal{K}$

Keeping the same notation as before, let $\mathcal{K}\subset\mathbb{R}^n$ be an n-dimensional non-degenerate convex polyhedron and let $\mathcal{F}_1,\dots,\mathcal{F}_m$ be its facets. Let H_i be the hyperplane generated by $\mathcal{F}_i$ and let $\mathtt{h}_i$ be a linear equation of H_i such that $\mathcal{K} = \bigcap_{i=1}^m \{\mathtt{h}_i \geq 0\}$. Now, for each $\varepsilon > 0$ denote by $H_i(\varepsilon)$ the hyperplane of linear equation $\mathtt{h}_i + \varepsilon = 0$. The hyperplanes H_i and $H_i(\varepsilon)$ are parallel and $H_i^+ \subset H_i^+(\epsilon)$.

Let $I := \{i_1, \dots, i_k, i_{k+1}\} \subset \{1, \dots, m\}$ be such that

(1) The vectorial hyperplanes $\vec{H}_{i_1}, \dots, \vec{H}_{i_k}$ are linearly independent.
(2) $W_I := \bigcap_{j=1}^k H_{i_j}$ is parallel to $H_{i_{k+1}}$
(3) $W_I \subset \operatorname{Int}(H_{i_{k+1}}^-)$.

Define $\delta(I) := \operatorname{dist}(W_I, H_{i_{k+1}}^+)$. Let $\mathfrak{I}$ be the collection of all the subsets $I \subset \{1, \dots, m\}$ satisfying conditions (1), (2) and (3). Define

$$\delta := \min\{\delta(I) : I \in \mathfrak{I}\} > 0.$$

Fix $\varepsilon > 0$ such that $\operatorname{dist}(H_i^+, H_i(\varepsilon)) < \frac{\delta}{2}$ for $i = 1, \dots, m$ and define an enveloping polyhedron for $\mathcal{K}$ as

$$\mathcal{K}_0 := \bigcap_{i=1}^m H_i^+(\epsilon).$$

Observe that $\mathcal{K} \subset \operatorname{Int}(\mathcal{K}_0)$ and notice that $\mathcal{K}_0$ is an n-dimensional non-degenerate convex polyhedron. Define

$$\mathcal{G}_0 := \operatorname{Int}(\mathcal{K}_0), \quad \mathcal{G}_i := \mathcal{G}_{i-1} \cap H_i^+ \quad \text{and} \quad \mathcal{G}_m = \mathcal{K}.$$

Set $\mathcal{K}_i := \operatorname{Cl}(\mathcal{G}_i)$ and observe that

$$\operatorname{Int}(\mathcal{K}_i) \subset \mathcal{G}_i = H_1^+ \cap \dots \cap H_i^+ \cap \operatorname{Int}(H_{i+1}^+(\epsilon)) \cap \dots \cap \operatorname{Int}(H_m^+(\epsilon)) \subset \mathcal{K}_i.$$

By Theorems 1.2 and 1.4 the semialgebraic set $\mathbb{R}^n \setminus \mathcal{G}_0$ is a polynomial image of $\mathbb{R}^n$. Our goal is to represent $\mathbb{R}^n \setminus \mathcal{K}$ as a polynomial image of $\mathbb{R}^n$ by applying a sequence of polynomial maps to the initial polynomial image $\mathbb{R}^n \setminus \mathcal{G}_0$, making use of Corollaries 6.3 and 6.6 along the process. We need first the following property of the sets $\mathcal{G}_i$.

Lemma 6.7 *Fix $i = 1, \dots, m$ and let $\mathcal{E}$ be a face of $\mathcal{K}_i := \operatorname{Cl}(\mathcal{G}_i)$ that lies in $\operatorname{Int}(H_{i+1}^-)$ and is parallel to H_{i+1}. Then $\mathcal{E} \cap \mathcal{G}_i = \varnothing$.*

Proof Let W be the affine subspace generated by $\mathcal{E}$. We claim: $W \subset H_\ell(\varepsilon)$ *for some* $\ell = i+1, \dots, m$. Note that if this is the case then $W \cap \mathcal{G}_i = \varnothing$ and so $\mathcal{E} \cap \mathcal{G}_i = \varnothing$.

To prove our claim, assume that none of the aforementioned inclusions hold, so that $W = \bigcap_{j=1}^k H_{i_j}$ for some $1 \leq i_1, \dots, i_k \leq i$ such that $\vec{H}_{i_1}, \dots, \vec{H}_{i_k}$ are linearly independent. Thus, $W = W_I$ for $I = \{i_1, \dots, i_k, i+1\}$ following the notation in Sect. 6.2. We show now that this cannot happen. As $\mathcal{E} \subset \operatorname{Int}(H_{i+1}^-)$ and $\mathcal{E}$ is parallel to H_{i+1}, we have $\operatorname{dist}(W_I, H_{i+1}^+) \geq \delta$. Since $\operatorname{dist}(H_{i+1}^+, H_{i+1}(\varepsilon)) < \frac{\delta}{2}$, we deduce $\mathcal{E} \subset W_I \subset \operatorname{Int}(H_{i+1}(\varepsilon)^-)$. On the other hand, we must have $\mathcal{E} \subset \mathcal{K}_i \subset H_{i+1}(\varepsilon)^+$, which is a contradiction. □

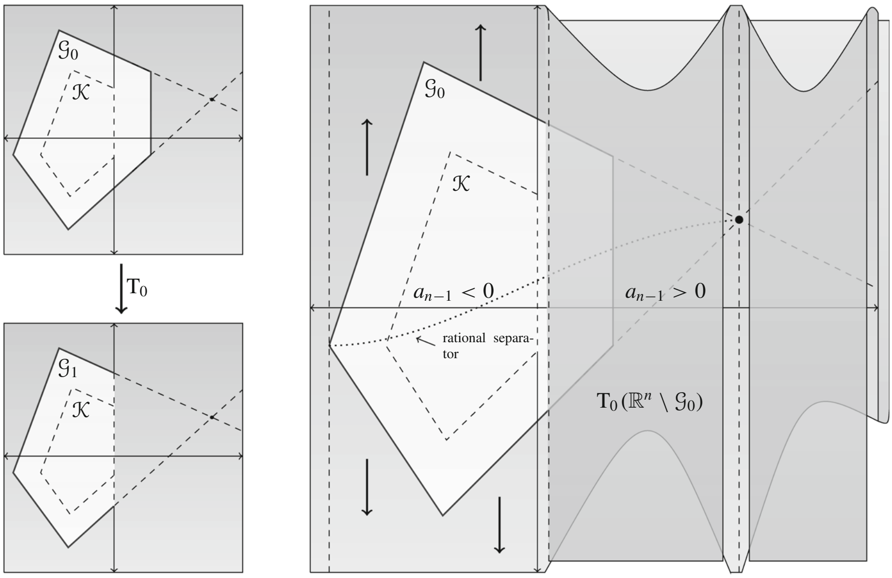


Fig. 8 A two-dimensional sketch of the behavior of the polynomial map $\mathtt{T}_0$

6.3 Proof of Theorem 1.5

Now that we have developed the necessary machinery, we are ready to confront the proof of Theorem 1.5. According to the degeneracy of $\mathcal{K}$, we distinguish two cases.

6.3.1 Case of a Non-degenerate *n*-Dimensional Convex Polyhedron

Consider the non-degenerate n-dimensional convex polyhedron $\mathcal{K}_0$ and the semialgebraic sets $\mathcal{G}_0, \ldots, \mathcal{G}_m$ described in Sect. 6.2. Place $\mathcal{K}$ in $\vec{\ell}_n$-bounded position in such a way that the facet $\mathcal{F}_1$ is contained in Π_{n-1} and $\mathcal{K} \subset \Pi_{n-1}^-$ (see Corollary 5.8). Notice that $\mathcal{K}_0 = \mathrm{Cl}(\mathcal{G}_0)$ is also in $\vec{\ell}_n$-bounded position. Observe that $H_1 = \Pi_{n-1}$, $H_1^+ = \Pi_{n-1}^-$ and $\mathcal{G}_0 \cap H_1^+ = \mathcal{G}_1$. By Corollary 6.3 applied to $\mathcal{G}_0$ there exists a polynomial map $\mathtt{T}_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\mathbb{R}^n \backslash \mathcal{G}_1) \setminus (\mathrm{Int}(H_1^-) \cap \mathcal{G}_0 \cap \pi_n^{-1}(\partial \mathcal{P}_0)) \subset \mathtt{T}_0(\mathbb{R}^n \backslash \mathcal{G}_0) \subset \mathbb{R}^n \backslash \mathcal{G}_1$$

where $\mathcal{P}_0 := \pi_n(\mathcal{K}_0)$. As $\mathcal{G}_0 = \mathrm{Int}(\mathcal{K}_0)$, we have $\mathcal{G}_0 \cap \pi_n^{-1}(\partial \mathcal{P}_0) = \varnothing$, so $\mathtt{T}_0(\mathbb{R}^n \backslash \mathcal{G}_0) = \mathcal{G}_1$. Figure 8 illustrates the behaviour of the polynomial map $\mathtt{T}_0$ in a two-dimensional setting.

Assume by induction that $\mathbb{R}^n \backslash \mathcal{G}_i$ is a polynomial image of $\mathbb{R}^n$. Place $\mathcal{K}_{i+1}$ in $\vec{\ell}_n$-bounded position in such a way that the facet $\mathcal{F}_{i+1}$ is contained in $\Pi_{n-1} = H_{i+1}$ and $\mathcal{K} \subset \Pi_{n-1}^- = H_{i+1}^+$ (see Corollary 5.8). Note that $\mathcal{K}_i = \mathrm{Cl}(\mathcal{G}_i)$ is also in $\vec{\ell}_n$-bounded position. By Corollary 6.3 applied to $\mathcal{G}_i$ we obtain a polynomial map $\mathtt{T}_{i,0} \colon \mathbb{R}^n \to \mathbb{R}^n$ with

$$(\mathbb{R}^n\setminus\mathcal{G}_{i+1})\setminus(\operatorname{Int}(H_{i+1}^-)\cap\mathcal{G}_i\cap\pi_n^{-1}(\partial\mathcal{P}_i))\subset\mathtt{T}_{i,0}(\mathbb{R}^n\setminus\mathcal{G}_i)\subset\mathbb{R}^n\setminus\mathcal{G}_{i+1}$$

and $\mathtt{T}_{i,0}(\mathbb{R}^n\setminus\mathcal{G}_{i+1})=\mathbb{R}^n\setminus\mathcal{G}_{i+1}$ where $\mathcal{P}_i:=\pi_n(\operatorname{Cl}(\mathcal{G}_i))$. By Sect. 2.2 the set $\mathcal{G}_i\cap\pi_n^{-1}(\partial\mathcal{P}_i)$ can be expressed as a finite, (disjoint) union of some interiors of faces of $\mathcal{K}_i=\operatorname{Cl}(\mathcal{G}_i)$. Thus, there exist finitely many facets $\mathcal{E}_1,\dots,\mathcal{E}_r$ of $\mathcal{K}_i$ such that $\mathcal{G}_i\cap\pi_n^{-1}(\partial\mathcal{P}_i)=\operatorname{Int}(\mathcal{E}_1)\sqcup\dots\sqcup\operatorname{Int}(\mathcal{E}_r)$. We may assume that only the faces $\mathcal{E}_1,\dots,\mathcal{E}_s$ where $0\le s\le r$ intersect $\operatorname{Int}(H_{i+1}^-)$. We claim: *$\mathcal{E}_j$ is non-parallel to H_{i+1} for $j=1,\dots,s$.*

Suppose that $\mathcal{E}_j$ is parallel to H_{i+1} for some $j=1,\dots,s$. As $\mathcal{E}_j\cap\operatorname{Int}(H_{i+1}^-)\neq\varnothing$, we deduce $\mathcal{E}_j\subset\operatorname{Int}(H_{i+1}^-)$. By Lemma 6.7 $\operatorname{Int}(\mathcal{E}_j)\subset\mathcal{E}_j\cap\mathcal{G}_i=\varnothing$, which is a contradiction.

Define $\mathcal{R}_i:=\operatorname{Int}(\mathcal{E}_1)\sqcup\dots\sqcup\operatorname{Int}(\mathcal{E}_s)$. By Corollary 6.6 there exists a polynomial map $\mathtt{T}_{i,1}\colon\mathbb{R}^n\to\mathbb{R}^n$ such that

$$\mathtt{T}_{i,1}((\mathbb{R}^n\setminus\mathcal{G}_{i+1})\setminus\mathcal{R}_i)=\mathbb{R}^n\setminus\mathcal{G}_{i+1}$$

and $\mathtt{T}_{i,1}(\mathbb{R}^n\setminus\mathcal{G}_{i+1})=\mathbb{R}^n\setminus\mathcal{G}_{i+1}$. Thus, the polynomial map $\mathtt{T}_i:=\mathtt{T}_{i,1}\circ\mathtt{T}_{i,0}\colon\mathbb{R}^n\to\mathbb{R}^n$ satisfies

$$\begin{aligned}\mathbb{R}^n\setminus\mathcal{G}_{i+1}&=\mathtt{T}_{i,1}((\mathbb{R}^n\setminus\mathcal{G}_{i+1})\setminus\mathcal{R}_i)\subset\mathtt{T}_{i,1}(\mathtt{T}_{i,0}(\mathbb{R}^n\setminus\mathcal{G}_i))\\&=\mathtt{T}_i(\mathbb{R}^n\setminus\mathcal{G}_i)\subset\mathtt{T}_{i,1}(\mathbb{R}^n\setminus\mathcal{G}_{i+1})=\mathbb{R}^n\setminus\mathcal{G}_{i+1},\end{aligned}$$

that is, $\mathtt{T}_i(\mathbb{R}^n\setminus\mathcal{G}_i)=\mathbb{R}^n\setminus\mathcal{G}_{i+1}$.

The composition $\mathtt{T}:=\mathtt{T}_{m-1}\circ\dots\circ\mathtt{T}_0$ satisfies $\mathtt{T}(\mathbb{R}^n\setminus\operatorname{Int}(\mathcal{K}_0))=\mathbb{R}^n\setminus\mathcal{K}$ and since $\mathbb{R}^n\setminus\operatorname{Int}(\mathcal{K}_0)$ is by Theorems 1.2 and 1.4 a polynomial image of $\mathbb{R}^n$, we are done.

6.3.2 Case of a Degenerate Polyhedron

We assume now that $\mathcal{K}$ is degenerate. We may write $\mathcal{K}$, after an affine change of coordinates, as $\mathcal{K}=\mathcal{P}\times\mathbb{R}^{n-k}$ where $\mathcal{P}\subset\mathbb{R}^k$ is a k-dimensional non-degenerate convex polyhedron. If $k\ge 2$, there exists by 6.3.1 a polynomial map $\mathtt{T}_0\colon\mathbb{R}^k\to\mathbb{R}^k$ such that $\mathtt{T}_0(\mathbb{R}^k)=\mathbb{R}^k\setminus\mathcal{P}$, so the polynomial map

$$\mathtt{T}\colon\mathbb{R}^k\times\mathbb{R}^{n-k}\to\mathbb{R}^k\times\mathbb{R}^{n-k},\ x:=(x',x'')\mapsto(\mathtt{T}_0(x'),x'')$$

satisfies $\mathtt{T}(\mathbb{R}^n)=\mathbb{R}^n\setminus\mathcal{K}$.

If $k=0$, then $\mathcal{K}=\mathbb{R}^n$ and there is nothing to say. If $k=1$, we may assume that $\mathcal{P}=(-\infty,0]$ (because $\mathcal{K}$ is not a layer). This means that $\mathcal{K}=\{\mathtt{x}_1\le 0\}$, so $\mathbb{R}^n\setminus\mathcal{K}=\{\mathtt{x}_1>0\}$. As it is well-known this set is a polynomial image of $\mathbb{R}^n$. Take for instance the polynomial map

$$\begin{aligned}&\mathtt{T}:=(\mathtt{T}_1,\dots,\mathtt{T}_n)\colon\mathbb{R}^n\to\mathbb{R}^n,\\&\mathtt{x}=(\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}'')\mapsto((\mathtt{x}_1\mathtt{x}_2-1)^2+\mathtt{x}_1^2,\mathtt{x}_2(\mathtt{x}_1\mathtt{x}_2-1),\mathtt{x}''),\end{aligned}$$

where $\mathtt{x}'':=(\mathtt{x}_3,\dots,\mathtt{x}_n)$ and whose image is $\mathbb{R}^n\setminus\mathcal{K}=\{\mathtt{x}_1>0\}$ (see [11]), as required. □

Remark 6.8 With respect to the degrees of the polynomial maps appearing in this section, we make a few considerations (see Question 1.6). In Proposition 5.9 we have seen that, for an unbounded convex polyhedron $\mathcal{K}$ with m facets, of which r are lower facets and s upper facets, the degree of a separating polynomial $\mathtt{P}$ is bounded above by 3^ℓ, where $\ell = \max\{r-1,1\}\cdot\max\{s-1,1\} \le \frac{(m-1)^2}{4}$. Since the polynomial in (F.2) has degree m, we conclude that the degree of the polynomial map $\mathtt{T}_{\mathcal{K}}$ appearing in (F.3) is bounded by $3^\ell + 2m + 1 \le 3^{(m-1)^2/4} + 2m + 1$.

The final map $\mathtt{T}$ that we have constructed in the proof of Theorem 1.5 results from a composition of polynomial maps $\mathtt{T}_{\mathcal{K}_i}$ for various intermediate polyhedra $\mathcal{K}_i$, as well as affine transformations. Besides, the number of these maps rely upon the geometry of the targeted polyhedron, because we not only use them for reducing the number of facets, but also for handling the problems arising with its lower dimensional faces. The multiplicative nature of polynomial degrees under composition leads us to really high exponents for the final map $\mathtt{T}$ and it is not easy to obtain a sharp bound for its degree. Nevertheless, let us provide at least a very coarse bound: Assume for simplicity that $\mathcal{K}$ is a generic non-degenerate unbounded convex polyhedron such that, when we slightly modify $\mathcal{K}$ (as we do in Sect. 6.2) all the faces of the polyhedra $\mathcal{K}_i$ have the same number of faces as those of $\mathcal{K}$. In particular, $\mathcal{K}$ and $\mathcal{K}_i$ have the same total number q of faces. If we follow the proofs of Corollary 6.6 and Sect. 6.3.1 and we have in mind the bound in Remark 4.2, we obtain the coarse bound

$$\deg(\mathtt{T}) \le (3^{(m-1)^2/4} + 2m + 1)^{mq} \cdot 2 \cdot \prod_{i=2}^{m} (4i+1).$$

Again, finding more optimal approaches (as proposed in Question 1.6) seems a challenging task for further research on the topic.

Example 6.9 If we do not care about the number of variables that we introduce to represent as a polynomial image of an Euclidean space the complement $\mathbb{R}^2\setminus\mathcal{P}$ of a convex polygon $\mathcal{P}\subset\mathbb{R}^2$ that does not disconnect $\mathbb{R}^2$, we can simplify substantially the complexity of the involved polynomial map and in particular its degree (see Question 1.7). More precisely:

Let $\mathcal{P}\subset\mathbb{R}^2$ be a convex polygon with m edges that does not disconnected $\mathbb{R}^2$. Then there exists a polynomial map $\mathtt{F}_m:\mathbb{R}^{m+2}\to\mathbb{R}^2$ such that $\mathtt{F}_m(\mathbb{R}^{m+2}) = \mathbb{R}^2 \to \mathcal{P} =: \mathcal{S}$ and $\deg(\mathtt{F}_m)\le 2m+2$.

We proceed by induction on the number m of edges of $\mathcal{P}$.

Initial case. If $m=1$, then after an affine change of coordinates we may assume $\mathcal{P} = \{\mathtt{x}_2 \le 0\}$, so $\mathcal{S} = \{\mathtt{x}_2 > 0\} = \mathbb{R}\times\,]0,+\infty[$. We take the polynomial map

$$\mathtt{F}_1:\mathbb{R}^3\to\mathbb{R}^2,\ (\mathtt{x}_1,\mathtt{x}_2,\mathtt{x}_3)\mapsto(\mathtt{x}_1,(1-\mathtt{x}_2\mathtt{x}_3)^2+\mathtt{x}_2^2),$$

whose image is $\mathcal{S}$ and has degree 4.

Induction step. Let $\mathcal{P}\subset\mathbb{R}^2$ be a convex polygon that does not disconnect $\mathbb{R}^2$ and has $m\ge 2$ edges. Let $\mathcal{E}_1,\dots,\mathcal{E}_m$ be the edges of $\mathcal{P}$ and let $H_i := \{\mathtt{h}_i = 0\}$ be the line spanned by the edge $\mathcal{E}_i$. We may assume $\mathcal{P} := \{\mathtt{h}_1\ge 0,\dots,\mathtt{h}_m\ge 0\}$. As $\mathcal{P}$ has at least two edges, it is non-degenerate. After an affine change of coordinates, we may assume that:

- The origin of $\mathbb{R}^2$ is a vertex of $\mathcal{P}$.
- $\mathtt{h}_1 = \mathtt{x}_1, \mathtt{h}_2 = \mathtt{x}_1 - \alpha \mathtt{x}_2$ for some $\alpha > 0$.
- If $\mathcal{P}$ is unbounded, then $\mathcal{E}_1$ is an unbounded edge of $\mathcal{P}$ and $\mathcal{T} \cap \{\mathtt{x}_1 > 0\} \subset \{\mathtt{x}_2 > 0\}$.
- If $\mathcal{P}$ is bounded, the furthest vertex of $\mathcal{P}$ to the edge $\mathcal{E}_1$ (or one of them if there exist two) belongs to the line $\mathtt{x}_2 = 0$. This means that a vertical line $\ell_a := \{(a, \mathtt{t}) : \mathtt{t} \in \mathbb{R}\}$ that meets $\mathcal{P}$ satisfies $\mathcal{P} \cap \ell_a = \{(a, \mathtt{t}) : b_a \leq \mathtt{t} \leq c_a\}$ where $b_a \leq 0 \leq c_a$.

Define $\mathcal{P}_0 := \{\mathtt{h}_2 \geq 0, \dots, \mathtt{h}_m \geq 0\}$, which is a convex polygon with $m - 1 \geq 1$ edges that does not disconnect $\mathbb{R}^2$. By induction hypothesis there exists a polynomial map $\mathtt{g}'_m : \mathbb{R}^m \to \mathbb{R}^n$ of degree $\leq 2(m-1)+2$ such that $\mathcal{T} := \mathtt{g}'_m(\mathbb{R}^m) = \mathbb{R}^2 \setminus \mathcal{P}_0$. Consider the polynomial maps

$$\mathtt{g}_m : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^3,\ (\mathtt{z}, \mathtt{w}) \mapsto (\mathtt{g}'_m(\mathtt{z}), \mathtt{w}),$$
$$\mathtt{f} : \mathbb{R}^3 \to \mathbb{R}^2,\ (\mathtt{x}_1, \mathtt{x}_2, \mathtt{w}) \mapsto (\mathtt{x}_1, \mathtt{x}_2(1+\mathtt{w}^2)).$$

Let us prove: $\mathtt{F}_m(\mathtt{f} \circ \mathtt{g}_m)(\mathbb{R}^m \times \mathbb{R}) = \mathcal{S}$. To that end we prove: $\mathtt{f}(\mathcal{T} \times \mathbb{R}) = \mathcal{S}$.

Let $\ell_a := \{(a, \mathtt{t}) : \mathtt{t} \in \mathbb{R}\}$ be a vertical line of $\mathbb{R}^2$ and consider the plane $H_a := \ell_a \times \mathbb{R}$. Then $\mathtt{f}|_{H_a} : H_a \to \mathbb{R}^2,\ (a, \mathtt{t}, \mathtt{w}) \mapsto (a, \mathtt{t}(1+\mathtt{w}^2))$. If we fix a point $(a, t_0) \in \ell_a$, then the image under $\mathtt{f}$ of the line $\{(a, t_0)\} \times \mathbb{R}$ is

$$\begin{cases} \{(a, t_0 + \lambda) : \lambda \geq 0\} & \text{if } t_0 > 0, \\ \{(a, 0)\} & \text{if } t_0 = 0, \\ \{(a, t_0 - \lambda) : \lambda \geq 0\} & \text{if } t_0 < 0. \end{cases}$$

As $\mathtt{h}_2 := \mathtt{x}_1 - \alpha \mathtt{x}_2$, for each $a < 0$ the open ray $r_a := \{(a, \mathtt{t}) : \mathtt{t} > \frac{a}{\alpha}\} \subset \mathcal{T}$, so $\mathtt{f}(r_a \times \mathbb{R}) = \{a\} \times \mathbb{R}$. Thus, $\mathtt{f}((\mathcal{T} \cap \{\mathtt{x}_1 < 0\}) \times \mathbb{R}) = \{\mathtt{x}_1 < 0\} = \mathcal{S} \cap \{\mathtt{x}_1 < 0\}$. Observe that

$$\mathcal{T} \cap \{\mathtt{x}_1 = 0\} = \begin{cases} \{(0, \mathtt{t}) : \mathtt{t} > 0\} & \text{if } \mathcal{P} \text{ is unbounded,} \\ \{(0, \mathtt{t}) : \mathtt{t} \in]-\infty, b_0[\cup \{]0, +\infty[\}\} & \text{for some } b_0 < 0 \text{ if } \mathcal{P} \text{ is bounded.} \end{cases}$$

So $\mathtt{f}((\mathcal{T} \cap \{\mathtt{x}_1 = 0\}) \times \mathbb{R}) = \mathcal{T} \cap \{\mathtt{x}_1 = 0\} = \mathcal{S} \cap \{\mathtt{x}_1 = 0\}$. In addition, if $a > 0$, then $\ell_a \cap \mathcal{T} = \{(a, \mathtt{t}) : \mathtt{t} \in S_a\}$, where

$$S_a = \begin{cases}]c_a, +\infty[& \text{for some } c_a > 0 \text{ if } \mathcal{P} \text{ is unbounded,} \\ ]-\infty, b_a[\cup]c_a, +\infty[& \text{for some } b_a \leq 0 \leq c_a \text{ if } \mathcal{P} \text{ is bounded.} \end{cases}$$

Thus, $\mathtt{f}((\ell_a \cap \mathcal{T}) \times \mathbb{R}) = \ell_a \cap \mathcal{T}$ for each $a > 0$. Consequently, $\mathtt{f}((\mathcal{T} \cap \{\mathtt{x}_1 > 0\}) \times \mathbb{R}) = \mathcal{T} \cap \{\mathtt{x}_1 > 0\} = \mathcal{S} \cap \{\mathtt{x}_1 > 0\}$ and

$$\begin{aligned} \mathtt{f}(\mathcal{T} \times \mathbb{R}) &= \mathtt{f}((\mathcal{T} \cap \{\mathtt{x}_1 < 0\}) \times \mathbb{R}) \sqcup \mathtt{f}((\mathcal{T} \cap \{\mathtt{x}_1 = 0\}) \times \mathbb{R}) \sqcup \mathtt{f}((\mathcal{T} \cap \{\mathtt{x}_1 > 0\}) \times \mathbb{R}) \\ &= (\mathcal{S} \cap \{\mathtt{x}_1 < 0\}) \sqcup (\mathcal{S} \cap \{\mathtt{x}_1 = 0\}) \sqcup (\mathcal{S} \cap \{\mathtt{x}_1 > 0\}) = \mathcal{S}. \end{aligned}$$

We have also $\deg(\mathtt{f}\circ\mathtt{g}_m)\leq\deg(\mathtt{g}'_m)+2\leq 2(m-1)+2+2=2m+2$. Therefore, the polynomial map $\mathtt{F}_m=\mathtt{f}\circ\mathtt{g}_m$ satisfies all our claims. □

Acknowledgements The authors are strongly indebted with the anonymous referees for their suggestions that have helped them to improve significantly the final version of this article.

References

1. Acquistapace, F., Andradas, C., Broglia, F.: Separation of semialgebraic sets. J. Am. Math. Soc. **12**(3), 703–728 (1999)
2. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic Geometry. Algorithms and Computation in Mathematics, vol. 10, 2nd edn. Springer, Berlin (2006)
3. Berger, M.: Geometry I & II. Universitext. Springer, Berlin (1987)
4. Bing, R.H.: Generalizations of two theorems of Janiszewski. Bull. Am. Math. Soc. **51**(12), 954–960 (1945)
5. Bochnak, J., Coste, M., Roy, M.F.: Real Algebraic Geometry. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36. Springer, Berlin (1998)
6. Eisenbud, D.: Open problems in computational algebraic geometry. In: Eisenbud, D., Robbiano, L. (eds.) Computational Algebraic Geometry and Commutative Algebra. Symposia Mathematica, vol. 34, pp. 49–70. Cambridge University Press, Cambridge (1993)
7. Fawzi, H.: On polyhedral approximations of the positive semidefinite cone. arXiv:1811.09649 (2018)
8. Fernando, J.F.: On the one dimensional polynomial and regular images of $\mathbb{R}^n$. J. Pure Appl. Algebra **218**(9), 1745–1753 (2014)
9. Fernando, J.F.: On Nash images of Euclidean spaces. Adv. Math. **331**, 627–719 (2018)
10. Fernando, J.F., Fichou, G., Quarez, R., Ueno, C.: On regulous and regular images of Euclidean spaces. Q. J. Math. **69**(4), 1327–1351 (2018)
11. Fernando, J.F., Gamboa, J.M.: Polynomial images of $\mathbb{R}^n$. J. Pure Appl. Algebra **179**(3), 241–254 (2003)
12. Fernando, J.F., Gamboa, J.M.: Polynomial and regular images of $\mathbb{R}^n$. Israel J. Math. **153**, 61–92 (2006)
13. Fernando, J.F., Gamboa, J.M., Ueno, C.: On convex polyhedra as regular images of $\mathbb{R}^n$. Proc. Lond. Math. Soc. **103**(5), 847–878 (2011)
14. Fernando, J.F., Gamboa, J.M., Ueno, C.: The open quadrant problem: a topological proof. In: A Mathematical Tribute to Professor José María Montesinos Amilibia. Dep. Geom. Topol. Fac. Cien. Mat., pp. 337–350. UCM, Madrid (2016)
15. Fernando, J.F., Gamboa, J.M., Ueno, C.: Polynomial, regular and Nash images of Euclidean spaces. In: Broglia, F., et al. (eds.) Ordered Algebraic Structures and Related Topics. Contemporary Mathematics, vol. 697, pp. 145–167. American Mathematical Society, Providence (2017)
16. Fernando, J.F., Gamboa, J.M., Ueno, C.: Unbounded convex polyhedra as polynomial images of Euclidean spaces. Ann. Sc. Norm. Super. Pisa Cl. Sci. **19**(2), 509–565 (2019)
17. Fernando, J.F., Ueno, C.: On the set of points at infinity of a polynomial image of $\mathbb{R}^n$. Discrete Comput. Geom. **52**(4), 583–611 (2014)
18. Fernando, J.F., Ueno, C.: On complements of convex polyhedra as polynomial and regular images of $\mathbb{R}^n$. Int. Math. Res. Not. IMRN **2014**(18), 5084–5123 (2014)
19. Fernando, J.F., Ueno, C.: On the complements of 3-dimensional convex polyhedra as polynomial images of $\mathbb{R}^3$. Int. J. Math. **25**(7), 1450071 (2014)
20. Fernando, J.F., Ueno, C.: A short proof for the open quadrant problem. J. Symb. Comput. **79**(1), 57–64 (2017)
21. Föllmer, H., Schied, A.: Stochastic Finance: An Introduction in Discrete Time. de Gruyter Studies in Mathematics, vol. 27, 2nd edn. de Gruyter, Berlin (2004)
22. Gamboa, J.M.: Reelle Algebraische Geometrie. June, 10th–16th. Oberwolfach (1990)
23. González-López, M.J., Recio, T., Santos, F.: Parametrization of semialgebraic sets. Math. Comput. Simul. **42**(4–6), 353–362 (1996)
24. Handelman, D.: Representing polynomials by positive linear functions on compact convex polyhedra. Pac. J. Math. **132**(1), 35–62 (1988)
25. Janiszewski, Z.: Sur les coupures du plan faites par les continus. Prace Mat.-Fiz. **26**, 11–63 (1913)

26. Kubjas, K., Parrilo, P.A., Sturmfels, B.: How to flatten a soccer ball. In: Conca, A., Gubeladze, J., Römer, T. (eds.) Homological and Computational Methods in Commutative Algebra. Springer INdAM Series, vol. 20, pp. 141–162. Springer, Cham (2017)
27. Nie, J., Demmel, J., Sturmfels, B.: Minimizing polynomials via sum of squares over the gradient ideal. Math. Program. **106**(3), 587–606 (2006)
28. Parrilo, P.A., Sturmfels, B.: Minimizing polynomial functions. In: Basu, S., Gonzalez-Vega, L. (eds.) Algorithmic and Quantitative Real Algebraic Geometry. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 60, pp. 83–99. American Mathematical Society, Providence (2003)
29. Rockafellar, T.R.: Convex Analysis. Princeton Mathematical Series, vol. 28. Princeton University Press, Princeton (1970)
30. Schweighofer, M.: Global optimization of polynomials using gradient tentacles and sums of squares. SIAM J. Optim. **17**(3), 920–942 (2006)
31. Stengle, G.: A Nullstellensatz and a Positivstellensatz in semialgebraic geometry. Math. Ann. **207**, 87–97 (1974)
32. Ueno, C.: A note on boundaries of open polynomial images of $\mathbb{R}^2$. Rev. Mat. Iberoam. **24**(3), 981–988 (2008)
33. Ueno, C.: On convex polygons and their complements as images of regular and polynomial maps of $\mathbb{R}^2$. J. Pure Appl. Algebra **216**(11), 2436–2448 (2012)
34. Vui, Hà Huy, So'n, Phạm Tiên: Global optimization of polynomials using the truncated tangency variety and sums of squares. SIAM J. Optim. **19**(2), 941–951 (2008)
35. Yannakakis, M.: Expressing combinatorial optimization problems by linear programs. J. Comput. Syst. Sci. **43**(3), 441–466 (1991)
36. Ziegler, G.M.: Lectures on Polytopes. Graduate Texts in Mathematics, vol. 152. Springer, New York (1995)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

José F. Fernando[1] · **Carlos Ueno**[2]

José F. Fernando
josefer@mat.ucm.es

Carlos Ueno
cuenjac@gmail.com

[1] Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

[2] Matematika Munkaközösség, Pécsi Kodály Zoltán Gimnázium, Dobó István utca 35-37, Pécs 7629, Hungary