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A short proof for the open quadrant problem

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ABSTRACT

In 2003 it was proved that the open quadrant $\mathcal{Q} := \{x > 0, y > 0\}$ of $\mathbb{R}^2$ is a polynomial image of $\mathbb{R}^2$. This result was the origin of an ulterior more systematic study of polynomial images of Euclidean spaces. In this article we provide a short proof of the previous fact that does not involve computer calculations, in contrast with the original one. The strategy here is to represent the open quadrant as the image of a polynomial map that can be expressed as the composition of three simple polynomial maps whose images can be easily understood.

1. Introduction

In the 1990 Reelle Algebraische Geometrie Seminar held in Oberwolfach Gamboa (1990) proposed the following problem:

Characterize geometrically the images of polynomial maps between Euclidean spaces.

The effective representation of a subset $\mathcal{S} \subset \mathbb{R}^m$ as a polynomial or regular image of $\mathbb{R}^n$ reduces the study of certain classical problems in Real Geometry to its study in $\mathbb{R}^n$, with the advantage of

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avoiding contour conditions. Examples of these problems arise in Optimization or in the search for Positivstellensätze certificates (Fernando and Gamboa, 2006; Fernando and Ueno, 2014a).

When facing the problem above, the fact of working over the field of real numbers introduces extra difficulties that are not present when working over the field of complex numbers. As a simple example, it is a basic result in the theory of one complex variable that the image of a non-constant polynomial map $f : \mathbb{C} \to \mathbb{C}$ is always equal to $\mathbb{C}$. However, the equivalent statement in the real setting no longer holds. The reader can easily verify that: *The image of a real, non-constant polynomial function is an unbounded closed interval.*

If we broaden our interest to polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^m$ between Euclidean spaces the characterization of their images becomes a tougher task. By Tarski–Seidenberg's principle (Bochnak et al., 1998, 1.4) the image of an either polynomial or regular map is a semialgebraic set. A subset $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* when it has a description by a finite boolean combination of polynomial equations and inequalities. During the last decade we have approached the problem of characterizing which (semialgebraic) subsets $\mathcal{S} \subset \mathbb{R}^m$ are polynomial or regular images of $\mathbb{R}^n$. On the one hand, we have obtained some necessary conditions that a semialgebraic set must satisfy in order to be a polynomial or regular image of $\mathbb{R}^n$ (Fernando, 2014; Fernando and Gamboa, 2003; Fernando and Gamboa, 2006; Fernando and Ueno, 2014c). On the other hand, we have described how to obtain constructively notable families of semialgebraic sets as images of polynomial or regular maps. In particular, we have focused our attention in convex polyhedra, their interiors and their complementaries (Fernando et al., 2011; Fernando and Ueno, 2014a; Fernando and Ueno, 2014b; Ueno, 2012).

Even in low dimensions we have to deal with situations that at first sight look harmless, but when considered more carefully become unexpectedly hard to handle because of the lack of precise tools to determine the image of a polynomial map. A particular case is the positive answer to the famous 'quadrant problem':

Theorem 1. *The open quadrant $\mathcal{Q} := \{x > 0, y > 0\}$ of $\mathbb{R}^2$ is a polynomial image of $\mathbb{R}^2$.*

This problem was stated in Gamboa (1990) and solved in Fernando and Gamboa (2003). The proof proposed in Fernando and Gamboa (2003) makes use of Sturm's algorithm applied to a high degree polynomial and the complexity of the involved calculations required computer assistance. This fact makes the reading of the proof rather disappointing, for it becomes a tedious task to verify that all the performed computations are indeed correct.

We have always wondered whether a less technical and demanding approach was possible. In this work we present a very short and elementary proof for the quadrant problem, which completely avoids the use of computers. Our approach is different to the one chosen in Fernando and Gamboa (2003). Our strategy here is to provide a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ that can be expressed as the composition of three simple polynomial maps whose images are easily estimated and has the open quadrant as image. To be more precise, we will show that $\mathcal{Q}$ is the image of the polynomial map $f := H \circ G \circ F : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\begin{aligned} &F : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto ((xy-1)^2 + x^2, (xy-1)^2 + y^2), \\ &G : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x, y(xy-2)^2 + x(xy-1)^2), \\ &H : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x(xy-2)^2 + \tfrac{1}{2}xy^2, y). \end{aligned} \tag{1}$$

It is well known that expressing a polynomial as a composition of simpler polynomials has proved useful in attacking diverse computational problems such as evaluating them or finding their roots, providing faster algorithms and allowing proofs that otherwise become more tedious or even infeasible (Alonso et al., 1995; von zur Gathen et al., 2003; Kozen and Landau, 1989). Similar benefits can be expected from expressing polynomial maps as compositions of simpler polynomial maps. Notice here that the each of the polynomial coordinates appearing in (1) have at most total degree 5, and all of them have at most degree 2 with respect to one of the variables x, y. This is an improvement with respect to the previous example known (Fernando and Gamboa, 2003), where the composition factors used in order to obtain the open quadrant as a polynomial image contain polynomials of total degree up to 10 and having degre of at least 4 in both variables. Even though in Section 3 we will see that, when completely expanded, our new polynomial map looks unhandier, from a computational perspective this ability of having a nicer expression in terms of composition factors helps to provide a shorter and more comprehensible proof of the open quadrant problem, and also improves some of the computational aspects involved. Moreover, we have other reasons to revisit the issue. One is related to its importance: *The representation of the open quadrant as a polynomial image is a key step in order to construct polynomial or regular images of higher complexity*, as is the case for the family of convex polyhedra that we mentioned before. This is because in order to construct either polynomial or regular maps $f : \mathbb{R}^n \to \mathbb{R}^n$ producing polyhedral images a first step passes through the construction of a polynomial map whose image is the open orthant $\{x_1 > 0, \dots, x_n > 0\}$, and this heavily relies on using polynomial maps on the plane with the half-plane and the open quadrant as images, as shows for example Fernando et al. (2011, Lemma 4.1) or Fernando and Ueno (2014a, Lemma 5.2). We briefly describe the strategy for obtaining polynomial maps with (convex) polyhedral images (Fernando et al., 2011): Observe that the orthant is the simplest convex polyhedron containing at least one vertex. We show then that an open n-simplex is a regular image of the orthant, and we proceed to “sculpt” the desired convex polyhedron by means of an inductive process which adds a new vertex of the convex polyhedron by composing with a suitable regular map, until obtaining the targeted polyhedron. Thus, any improvement in the complexity of the involved polynomial maps would lead to a better output, and this also applies to the first step of the process, directly connected to the open quadrant problem. A similar argument applies to prove that the complement in $\mathbb{R}^n$ of an n-dimensional convex polyhedron is a polynomial image of $\mathbb{R}^n$ (Fernando and Ueno, 2014a), which uses an inductive process starting with the complement of the open orthant. In order to represent this complement of the orthant as a polynomial image we need first to obtain the complement of the (closed) quadrant in $\mathbb{R}^2$ as a polynomial image, and this is achieved by applying to the open quadrant (considered in $\mathbb{C}$) the polynomial map $z \mapsto z^3$. Thus, the representation of the open quadrant as a polynomial image of $\mathbb{R}^2$ is the seed in many inductive processes that allows us to determine the semialgebraic sets with piecewise linear boundary that are polynomial or regular images of $\mathbb{R}^n$.

This brings us to another reason to return to this problem, the still pending question of finding an optimal polynomial map that achieves the goal. In other words:

Which is the simplest polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose image is the open quadrant?

Here the term ‘simplest’ is rather vague and ambiguous. In Section 3 we try to be more specific and provide questions that at present we are unable to answer.

2. The new proof

In order to prove Theorem 1 we need some preliminary work. As we have already announced, $\mathcal{Q}$ is the image of a composition of three simple polynomial maps. We present next three auxiliary lemmas that show some properties of the images of the polynomial maps F, G, H introduced in (1). These lemmas involve the semialgebraic sets shown in Fig. 1.

Lemma 2. *Let $\mathcal{A} := \{xy - 1 \geq 0\} \cap \mathcal{Q}$. Then the image of*

$$F := (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto ((xy-1)^2 + x^2, (xy-1)^2 + y^2)$$

satisfies $\mathcal{A} \subset F(\mathbb{R}^2) \subset \mathcal{Q}$.

Proof. It is clear that F_1, F_2 are strictly positive on $\mathbb{R}^2$. Consequently, $F(\mathbb{R}^2) \subset \mathcal{Q}$. To prove the first inclusion we show that if $a > 0$, $b > 0$ satisfy $ab - 1 \geq 0$, then the system of equations

$$\begin{cases} (xy-1)^2 + x^2 = a, \\ (xy-1)^2 + y^2 = b \end{cases} \tag{2}$$

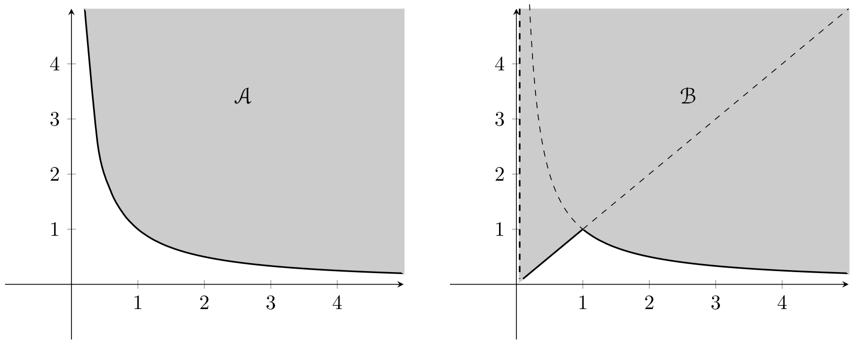

Fig. 1. The sets $\mathcal{A} := \{xy - 1 \geq 0\} \cap \mathcal{Q}$ and $\mathcal{B} := \mathcal{A} \cup \{y \geq x > 0\}$.

has a solution $(x, y) \in \mathbb{R}^2$. Set $z := xy - 1$ and rewrite the system (2) in terms of the variables $\{x, z\}$. We have $y = \frac{z+1}{x}$ and (2) becomes

$$\begin{cases} z^2 + x^2 = a, \\ z^2 + \dfrac{(z+1)^2}{x^2} = b. \end{cases}$$

We eliminate x and deduce that z must satisfy the polynomial equation

$$P(z) := z^4 - (a + b + 1)z^2 - 2z + (ab - 1) = 0.$$

Observe that P is a monic polynomial of even degree such that

$$P(0) = ab - 1 \geq 0 \quad \text{and} \quad P(\sqrt{a}) = -2\sqrt{a} - a - 1 < 0.$$

Thus, P *has a real root* z_0 *such that* $0 \leq z_0 < \sqrt{a}$. Set $x_0 := \sqrt{a - z_0^2}$ and $y_0 := \frac{z_0+1}{x_0}$. We have $F(x_0, y_0) = (a, b)$, so $\mathcal{A} \subset F(\mathbb{R}^2)$, as required. □

Lemma 3. *Let* $\mathcal{B} := \mathcal{A} \cup \{y \geq x > 0\}$. *Then the image of*

$$G := (G_1, G_2) : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (x, y(xy - 2)^2 + x(xy - 1)^2)$$

satisfies $\mathcal{B} \subset G(\mathcal{A}) \subset G(\mathcal{Q}) \subset \mathcal{Q}$.

Proof. The inclusion $G(\mathcal{A}) \subset G(\mathcal{Q})$ is obvious. Observe that G_1 and G_2 are strictly positive on $\mathcal{Q}$. Consequently, $G(\mathcal{Q}) \subset \mathcal{Q}$.

Next, we prove the inclusion $\mathcal{B} \subset G(\mathcal{A})$. Notice first that we can express $\mathcal{B}$ as follows:

$$\mathcal{B} = \bigsqcup_{x>0} (\{x\} \times [y_x, +\infty[) := \bigsqcup_{x>0} (\{x\} \times \mathcal{B}_x) \quad \text{where } y_x := \min\{x, \tfrac{1}{x}\}.$$

For each $x > 0$ consider the polynomial function in the variable $\mathtt{y}$

$$\phi_x(\mathtt{y}) := \mathtt{y}(x\mathtt{y} - 2)^2 + x(x\mathtt{y} - 1)^2 = x^2\mathtt{y}^3 + (x^3 - 4x)\mathtt{y}^2 + (4 - 2x^2)\mathtt{y} + x.$$

These polynomials have odd degree and positive leading coefficient because $x > 0$. Observe also that $\phi_x(\frac{1}{x}) = \frac{1}{x}$ and $\phi_x(\frac{2}{x}) = x$. Consequently

$$\mathcal{B}_x = [y_x, +\infty[\subset \phi_x([\tfrac{1}{x}, +\infty[).$$

Therefore

$$\begin{aligned}\mathcal{B}&=\bigsqcup_{x>0}(\{x\}\times\mathcal{B}_x)\subset\bigsqcup_{x>0}\{x\}\times\phi_x([\tfrac{1}{x},+\infty[)\\&=\bigsqcup_{x>0}G(\{x\}\times[\tfrac{1}{x},+\infty[)=G(\mathcal{A}),\end{aligned}$$

as required. □

Lemma 4. *The polynomial map*

$$H:=(H_1,H_2):\mathbb{R}^2\to\mathbb{R}^2,\ (x,y)\mapsto(x(xy-2)^2+\tfrac{1}{2}xy^2,y)$$

satisfies $H(\mathcal{B})=H(\mathcal{Q})=\mathcal{Q}$.

Proof. The inclusion $H(\mathcal{B})\subset H(\mathcal{Q})$ is obvious. Observe that H_1 and H_2 are strictly positive on $\mathcal{Q}$. Consequently, $H(\mathcal{Q})\subset\mathcal{Q}$.

Next, we prove $\mathcal{Q}\subset H(\mathcal{B})$ and consequently we will have $\mathcal{Q}\subset H(\mathcal{B})\subset H(\mathcal{Q})\subset\mathcal{Q}$, so $H(\mathcal{B})=H(\mathcal{Q})=\mathcal{Q}$.

For each $y>0$ consider the polynomial in the variable $\mathtt{x}$

$$\psi_y(\mathtt{x}):=\mathtt{x}(\mathtt{x}y-2)^2+\tfrac{1}{2}\mathtt{x}y^2=y^2\mathtt{x}^3-4y\mathtt{x}^2+(4+\tfrac{1}{2}y^2)\mathtt{x}.$$

Notice that the set $\mathcal{B}$ can be expressed as

$$\mathcal{B}=\bigsqcup_{y>0}(\mathcal{B}_y\times\{y\})\quad\text{where }\mathcal{B}_y:=\begin{cases}]0,+\infty[& \text{if } y\geq 1,\\ ]0,y]\cup[\tfrac{1}{y},+\infty[& \text{if } 0<y<1.\end{cases}$$

As $\psi_y(\mathtt{x})$ has odd degree and positive leading coefficient, we have $\lim_{x\to\infty}\psi_y(x)=+\infty$. Moreover, it holds

$$\psi_y(0)=0,\quad \psi_y(y)=y(y^2-2)^2+\tfrac{1}{2}y^3\quad\text{and}\quad\psi_y(\tfrac{2}{y})=y.$$

For $0<y<1$ we have

$$\psi_y(y)=y(y^2-2)^2+\tfrac{1}{2}y^3=y((y^2-2)^2+\tfrac{1}{2}y^2)>y$$

because $(y^2-2)^2+\frac{1}{2}y^2>1$ if $0<y<1$. As ψ_y is strictly positive on $]0,+\infty[$, we deduce

$$\psi_y(\mathcal{B}_y)=\begin{cases}\psi(]0,+\infty[)=]0,+\infty[& \text{if } y\geq 1\\ \psi_y(]0,y]\cup[\tfrac{1}{y},+\infty[)\supset]0,\psi_y(y)]\cup[\psi_y(\tfrac{2}{y}),+\infty[=]0,+\infty[& \text{if } 0<y<1.\end{cases}$$

Consequently,

$$\mathcal{Q}=\bigsqcup_{y>0}(]0,+\infty[\times\{y\})\subset\bigsqcup_{y>0}(\psi_y(\mathcal{B}_y)\times\{y\})=\bigsqcup_{y>0}H(\mathcal{B}_y\times\{y\})=H(\mathcal{B}),$$

as required. □

Finally, Theorem 1 follows straightforwardly from the previous three Lemmas.

Proof of Theorem 1. Applying Lemmas 2, 3 and 4 we deduce that

$$\mathcal{Q}=H(\mathcal{B})\subset(H\circ G)(\mathcal{A})\subset(H\circ G\circ F)(\mathbb{R}^2)\subset(H\circ G)(\mathcal{Q})\subset H(\mathcal{Q})=\mathcal{Q},$$

that is, $(H\circ G\circ F)(\mathbb{R}^2)=\mathcal{Q}$, as required. □

Table 1
The old polynomial map.

$g_1(x,y) = (x^{18} + 2x^{16} + x^{14})y^{10} + (-14x^{17} - 30x^{15} + 4x^{14} - 18x^{13} + 6x^{12} - 2x^{11} + 2x^{10})y^9 + (87x^{16} + 202x^{14} - 44x^{13} + 143x^{12} - 72x^{11} + 34x^{10} - 30x^9 + 7x^8 - 2x^7 + x^6)y^8 + (-316x^{15} - 804x^{13} + 208x^{12} - 662x^{11} + 378x^{10} - 226x^9 + 192x^8 - 66x^7 + 26x^6 - 12x^5 + 2x^4)y^7 + (743x^{14} + 2094x^{12} - 552x^{11} + 1985x^{10} - 1134x^9 + 828x^8 - 688x^7 + 269x^6 - 128x^5 + 58x^4 - 12x^3 + x^2)y^6 + (-1182x^{13} - 3726x^{11} + 900x^{10} - 4046x^9 + 2124x^8 - 1922x^7 + 1522x^6 - 622x^5 + 340x^4 - 146x^3 + 28x^2 - 2x)y^5 + (1289x^{12} + 4582x^{10} - 924x^9 + 5702x^8 - 2538x^7 + 3022x^6 - 2150x^5 + 906x^4 - 558x^3 + 207x^2 - 30x + 1)y^4 + (-952x^{11} - 3840x^9 + 584x^8 - 5504x^7 + 1884x^6 - 3286x^5 + 1910x^4 - 888x^3 + 586x^2 - 162x + 12)y^3 + (456x^{10} + 2096x^8 - 208x^7 + 3487x^6 - 792x^5 + 2408x^4 - 978x^3 + 621x^2 - 372x + 55)y^2 + (-128x^9 - 672x^7 + 32x^6 - 1308x^5 + 144x^4 - 1080x^3 + 220x^2 - 308x + 112)y + (16x^8 + 96x^6 + 220x^4 + 224x^2 + 85)$,

$g_2(x,y) = x^{16}y^{12} + (-14x^{15} - 2x^{13} + 2x^{12})y^{11} + (89x^{14} + 26x^{12} - 22x^{11} + x^{10} - 2x^9 + x^8)y^{10} + (-338x^{13} - 152x^{11} + 108x^{10} - 12x^9 + 20x^8 - 8x^7)y^9 + (849x^{12} + 524x^{10} - 308x^9 + 64x^8 - 88x^7 + 28x^6)y^8 + (-1476x^{11} - 1176x^9 + 558x^8 - 198x^7 + 220x^6 - 54x^5)y^7 + (1808x^{10} + 1792x^8 - 662x^7 + 391x^6 - 340x^5 + 61x^4)y^6 + (-1562x^9 - 1878x^7 + 514x^6 - 512x^5 + 332x^4 - 40x^3)y^5 + (944x^8 + 1344x^6 - 258x^5 + 447x^4 - 202x^3 + 15x^2)y^4 + (-398x^7 - 644x^5 + 86x^4 - 254x^3 + 74x^2 - 4x)y^3 + (121x^6 + 206x^4 - 22x^3 + 90x^2 - 18x + 1)y^2 + (-28x^5 - 48x^3 + 4x^2 - 20x + 4)y + (4x^4 + 8x^2 + 4)$.

3. Effectiveness of the new map

The problem of the open quadrant, together with its already known positive constructive answers, invites to search for alternative polynomial maps that also solve the problem and are optimal with respect to their algebraic complexity. This algebraic complexity can be understood in several ways. We briefly describe two possible approaches to this question.

(A) *Optimal algebraic structure of the polynomial map*. On a first look it is natural to wonder how our new obtained polynomial map looks like when completely expanded and how it compares to the previous known example in Fernando and Gamboa (2003). We care about the total degree of the involved polynomial map (the sum of the degrees of its components) and its total number of (non-zero) monomials. We would like to find a polynomial map with the least possible total degree and the least possible number of monomials.

Table 1 shows the components of the map $g(x,y) := (g_1(x,y), g_2(x,y))$ proposed in Fernando and Gamboa (2003), while Table 2 shows those of our new map $f(x,y) := (f_1(x,y), f_2(x,y))$. Observe that the total degree of g is 56 while the total degree of f is 72. In addition, the total number of monomials of g is 168 while the total number of monomials of f is 350. We wonder:

Question 5. (1) *Which is the minimum total degree for the set of polynomial maps $\mathbb{R}^2 \to \mathbb{R}^2$ whose image is the open quadrant?*

(2) *Which is the sparsest polynomial map $\mathbb{R}^2 \to \mathbb{R}^2$ whose image is the open quadrant?*

A possible roadmap to answer these questions would come from finding polynomial maps solving the open quadrant problem with polynomial components of much lower degree than those known at present, followed by checking the minimality of these new examples with respect to total degree or sparseness.

(B) *Optimal (multiplicative) complexity*. The so-called Straight-Line Programs (SLP's) formalize step-by-step computations that do not require branching and can be applied to the evaluation of polynomials (see Bürgisser et al., 1997, chap. 4 and Winograd, 1970). Here we are particularly interested in evaluating the polynomial coordinates of our map in an effective way. As multiplications have a higher cost to compute than additions/subtractions, non-scalar complexity seems a reasonable approach to consider in the first place. In our particular case, expressing our map f as a composition of three simpler maps helps to lower the complexity required to evaluate f at a point. More precisely, if we rewrite (1) as

$$F : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto ((xy-1)^2 + x^2, (xy-1)^2 + y^2),$$
$$G : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto (x, y((xy)^2 - 4xy + 4) + x((xy)^2 - 2xy + 1)), \tag{3}$$

Table 2
The new polynomial map.

$$\begin{aligned}
f_1(x,y) ={}& (4x^{26}+20x^{24}+41x^{22}+44x^{20}+26x^{18}+8x^{16}+x^{14})y^{26}+(-104x^{25}-480x^{23}-902x^{21}-880x^{19}-\\
&468x^{17}-128x^{15}-14x^{13})y^{25}+(32x^{26}+1458x^{24}+5839x^{22}+9807x^{20}+8554x^{18}+4036x^{16}+967x^{14}+91x^{12})y^{24}+\\
&(-768x^{25}-13876x^{23}-46860x^{21}-69188x^{19}-53264x^{17}-22028x^{15}-4564x^{13}-364x^{11})y^{23}+(113x^{26}+9382x^{24}+\\
&97367x^{22}+274164x^{20}+\tfrac{703621}{2}x^{18}+236532x^{16}+84820x^{14}+15018x^{12}+\tfrac{2003}{2}x^{10})y^{22}+(-2486x^{25}-75768x^{23}-\\
&525594x^{21}-1229924x^{19}-1360121x^{17}-791240x^{15}-243544x^{13}-36436x^{11}-2007x^{9})y^{21}+(231x^{26}+27209x^{24}+\\
&446529x^{22}+2240074x^{20}+\tfrac{8710925}{2}x^{18}+\tfrac{8244713}{2}x^{16}+2057114x^{14}+538091x^{12}+\tfrac{134535}{2}x^{10}+\tfrac{6051}{2}x^{8})y^{20}+(-4620x^{25}-\\
&193928x^{23}-2018472x^{21}-7667498x^{19}-12393340x^{17}-9976824x^{15}-4233956x^{13}-931594x^{11}-96204x^{9}-3492x^{7})y^{19}+\\
&(301x^{26}+45304x^{24}+\tfrac{1996081}{2}x^{22}+7201629x^{20}+21316526x^{18}+28643722x^{16}+\tfrac{38977381}{2}x^{14}+6970385x^{12}+1276041x^{10}+\\
&107514x^{8}+3108x^{6})y^{18}+(-5418x^{25}-285964x^{23}-3907609x^{21}-20635184x^{19}-48472730x^{17}-54092612x^{15}-\\
&30891281x^{13}-9219636x^{11}-1387160x^{9}-94004x^{7}-2128x^{5})y^{17}+(259x^{26}+47243x^{24}+\tfrac{2580609}{2}x^{22}+\tfrac{23978613}{2}x^{20}+\\
&47980885x^{18}+90479081x^{16}+\tfrac{167285973}{2}x^{14}+\tfrac{79543913}{2}x^{12}+9792158x^{10}+1193474x^{8}+63896x^{6}+1106x^{4})y^{16}+\\
&(-4144x^{25}-262276x^{23}-4387372x^{21}-29332620x^{19}-91032136x^{17}-138703760x^{15}-105761608x^{13}-41463220x^{11}-\\
&8305148x^{9}-805172x^{7}-33232x^{5}-424x^{3})y^{15}+(147x^{26}+31738x^{24}+1029734x^{22}+11578342x^{20}+\tfrac{115563913}{2}x^{18}+\\
&141146980x^{16}+174192331x^{14}+108825883x^{12}+\tfrac{69479137}{2}x^{10}+5560611x^{8}+418307x^{6}+12832x^{4}+\tfrac{227}{2}x^{2})y^{14}+\\
&(-2058x^{25}-152936x^{23}-3012828x^{21}-24111820x^{19}-92027457x^{17}-178500672x^{15}-178178038x^{13}-90313924x^{11}-\\
&23084717x^{9}-2880652x^{7}-162050x^{5}-3480x^{3}-19x)y^{13}+(53x^{26}+13607x^{24}+514692x^{22}+6758603x^{20}+\tfrac{79906193}{2}x^{18}+\\
&\tfrac{236896489}{2}x^{16}+183083134x^{14}+146995602x^{12}+\tfrac{119160879}{2}x^{10}+\tfrac{23809093}{2}x^{8}+1115393x^{6}+44130x^{4}+\tfrac{1187}{2}x^{2}+\tfrac{3}{2})y^{12}+\\
&(-636x^{25}-55808x^{23}-1273116x^{21}-11799438x^{19}-52781280x^{17}-122608140x^{15}-150791284x^{13}-96320302x^{11}-\\
&30530204x^{9}-4590824x^{7}-302620x^{5}-7466x^{3}-48x)y^{11}+(11x^{26}+3544x^{24}+\tfrac{314743}{2}x^{22}+2376631x^{20}+16126054x^{18}+\\
&55398078x^{16}+\tfrac{202163613}{2}x^{14}+98174729x^{12}+48935769x^{10}+11683851x^{8}+\tfrac{2465761}{2}x^{6}+49595x^{4}+534x^{2})y^{10}+\\
&(-110x^{25}-12028x^{23}-320343x^{21}-3389132x^{19}-17228062x^{17}-45771560x^{15}-65325089x^{13}-49297772x^{11}-\\
&18508228x^{9}-3090272x^{7}-192077x^{5}-2508x^{3}+26x)y^{9}+(x^{26}+499x^{24}+\tfrac{54971}{2}x^{22}+\tfrac{963471}{2}x^{20}+3698783x^{18}+\\
&14267155x^{16}+\tfrac{58589293}{2}x^{14}+\tfrac{64546337}{2}x^{12}+18324283x^{10}+4804470x^{8}+\tfrac{907373}{2}x^{6}+\tfrac{7397}{2}x^{4}-553x^{2}-3)y^{8}+(-8x^{25}-\\
&1344x^{23}-44228x^{21}-539160x^{19}-3067508x^{17}-9009888x^{15}-14152800x^{13}-11695504x^{11}-4652360x^{9}-672008x^{7}+\\
&7308x^{5}+4412x^{3}+72x)y^{7}+(28x^{24}+2366x^{22}+50998x^{20}+446884x^{18}+1901034x^{16}+4222671x^{14}+4948005x^{12}+\\
&\tfrac{5751153}{2}x^{10}+639605x^{8}-\tfrac{52037}{2}x^{6}-16587x^{4}-\tfrac{1249}{2}x^{2}-1)y^{6}+(-56x^{23}-2828x^{21}-42092x^{19}-269596x^{17}-854940x^{15}-\\
&1404926x^{13}-1159452x^{11}-389461x^{9}+18128x^{7}+30043x^{5}+2488x^{3}+9x)y^{5}+(70x^{22}+2310x^{20}+24418x^{18}+\\
&114536x^{16}+265718x^{14}+306840x^{12}+\tfrac{302491}{2}x^{10}-\tfrac{3703}{2}x^{8}-\tfrac{49077}{2}x^{6}-\tfrac{9289}{2}-\tfrac{39x^{2}}{2}x^{4}+\tfrac{3}{2})y^{4}+(-56x^{21}-1264x^{19}-\\
&9568x^{17}-32360x^{15}-52408x^{13}-37034x^{11}-2472x^{9}+10104x^{7}+3718x^{5}+16x^{3}-20x)y^{3}+(28x^{20}+439x^{18}+2350x^{16}+\\
&\tfrac{10971}{2}x^{14}+5505x^{12}+1064x^{10}-2161x^{8}-1411x^{6}-46x^{4}+\tfrac{127}{2}x^{2}+1)y^{2}+(-8x^{19}-86x^{17}-312x^{15}-451x^{13}-168x^{11}+\\
&222x^{9}+252x^{7}+26x^{5}-34x^{3}-5x)y+x^{18}+7x^{16}+\tfrac{31}{2}x^{14}+\tfrac{19}{2}x^{12}-8x^{10}-17x^{8}-4x^{6}+5x^{4}+\tfrac{3}{2}x^{2}+\tfrac{3}{2},
\end{aligned}$$

$$\begin{aligned}
f_2(x,y) ={}& (2x^{10}+5x^{8}+4x^{6}+x^{4})y^{10}+(-20x^{9}-40x^{7}-24x^{5}-4x^{3})y^{9}+(5x^{10}+102x^{8}+149x^{6}+62x^{4}+6x^{2})y^{8}+\\
&(-40x^{9}-312x^{7}-316x^{5}-84x^{3}-4x)y^{7}+(4x^{10}+149x^{8}+600x^{6}+395x^{4}+58x^{2}+1)y^{6}+(-24x^{9}-316x^{7}-720x^{5}-\\
&276x^{3}-16x)y^{5}+(x^{10}+62x^{8}+397x^{6}+504x^{4}+85x^{2})y^{4}+(-4x^{9}-84x^{7}-284x^{5}-168x^{3})y^{3}+(6x^{8}+60x^{6}+99x^{4}+\\
&5x^{2}-1)y^{2}+(-4x^{7}-20x^{5}-8x^{3}+6x)y+x^{6}+2x^{4}-2x^{2}+1.
\end{aligned}$$

$$H:\mathbb{R}^2\to\mathbb{R}^2,\ (x,y)\mapsto (xy(x\cdot(xy)-4x+4+\tfrac{1}{2}y),\,y),$$

an ocular inspection gives us an upper bound for the non-scalar complexity (working with real coefficients[3]) of $4+4+3=11$. It is not so clear whether this bound can be achieved with the polynomial map proposed in Fernando and Gamboa (2003). At this point, we wonder:

Question 6. *Which is the minimum non-scalar complexity for the set of polynomial maps $\mathbb{R}^2\to\mathbb{R}^2$ whose image is the open quadrant?*

Of course, we can formulate diverse variants of this question if we consider other measures of complexity. In any case, regardless of the different approaches considered, the authors are convinced that more effective examples can be found for the open quadrant problem, and perhaps even shorter proofs.

[3] If we consider coefficients in $\mathbb{C}$, we can lower the non-scalar complexity bound of the map F by one.

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References

Alonso, C., Gutierrez, J., Recio, T., 1995. A rational function decomposition algorithm by near-separated polynomials. J. Symb. Comput. 19, 527–544. http://dx.doi.org/10.1006/jsco.1995.1030.

Bochnak, J., Coste, M., Roy, M.F., 1998. Real Algebraic Geometry. Ergeb. Math. Ihrer Grenzgeb., vol. 36. Springer-Verlag, Berlin.

Bürgisser, P., Clausen, M., Shokrollahi, M.A., 1997. Algebraic Complexity Theory. Grundlehren Math. Wiss., vol. 315. Springer, Berlin.

Fernando, J.F., 2014. On the one dimensional polynomial and regular images of $\mathbb{R}^n$. J. Pure Appl. Algebra 218, 1745–1753. http://dx.doi.org/10.1016/j.jpaa.2014.01.011.

Fernando, J.F., Gamboa, J.M., 2003. Polynomial images of $\mathbb{R}^n$. J. Pure Appl. Algebra 179, 241–254. http://dx.doi.org/10.1016/S0022-4049(02)00121-4.

Fernando, J.F., Gamboa, J.M., 2006. Polynomial and regular images of $\mathbb{R}^n$. Isr. J. Math. 153, 61–92. http://dx.doi.org/10.1007/BF02771779.

Fernando, J.F., Ueno, C., 2014a. On complements of convex polyhedra as polynomial and regular images of $\mathbb{R}^n$. Int. Math. Res. Not. 2014, 5084–5123. http://dx.doi.org/10.1093/imrn/rnt112.

Fernando, J.F., Ueno, C., 2014b. On the complements of 3-dimensional convex polyhedra as polynomial images of $\mathbb{R}^3$. Int. J. Math. 25, 1450071. http://dx.doi.org/10.1142/S0129167X14500712.

Fernando, J.F., Ueno, C., 2014c. On the set of points at infinity of a polynomial image of $\mathbb{R}^n$. Discrete Comput. Geom. 52, 583–611. http://dx.doi.org/10.1007/s00454-014-9620-7.

Fernando, J.F., Gamboa, J.M., Ueno, C., 2011. On convex polyhedra as regular images of $\mathbb{R}^n$. Proc. R. Soc., Math. Phys. Eng. Sci. 103, 847–878. http://dx.doi.org/10.1112/plms/pdr015.

Gamboa, J.M., 1990. Reelle Algebraische Geometrie. June, 10th–16th, Oberwolfach.

von zur Gathen, J., Gutierrez, J., Rubio, R., 2003. Multivariate polynomial decomposition. Appl. Algebra Eng. Commun. Comput. 14, 11–31. http://dx.doi.org/10.1007/s00200-003-0122-8.

Kozen, D., Landau, S., 1989. Polynomial decomposition algorithms. J. Symb. Comput. 7, 445–456. http://dx.doi.org/10.1016/S0747-7171(89)80027-6.

Ueno, C., 2012. On convex polygons and their complements as images of regular and polynomial maps of $\mathbb{R}^2$. J. Pure Appl. Algebra 216, 2436–2448. http://dx.doi.org/10.1016/j.jpaa.2012.03.016.

Winograd, S., 1970. On the number of multiplications necessary to compute certain functions. Commun. Pure Appl. Math. 23, 165–179. http://dx.doi.org/10.1002/cpa.3160230204.