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Differentiable approximation of continuous semialgebraic maps

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Abstract

In this work we approach the problem of approximating uniformly continuous semialgebraic maps $f : S \to T$ from a compact semialgebraic set S to an arbitrary semialgebraic set T by semialgebraic maps $g : S \to T$ that are differentiable of class $\mathcal{C}^\nu$ for a fixed integer $\nu \geq 1$. As the reader can expect, the difficulty arises mainly when one tries to keep the same target space after approximation. For $\nu = 1$ we give a complete affirmative solution to the problem: such a uniform approximation is always possible. For $\nu \geq 2$ we obtain density results in the following two relevant situations: either T is compact and locally $\mathcal{C}^\nu$ semialgebraically equivalent to a polyhedron, for instance when T is a compact polyhedron; or T is an open semialgebraic subset of a Nash set, for instance when T is a Nash set. Our density results are based on a recent $\mathcal{C}^1$-triangulation theorem for semialgebraic sets due to Ohmoto and Shiota, and on new approximation techniques we develop in the present paper. Our results are sharp in a sense we specify by explicit examples.

Keywords Approximation of semialgebraic maps · Approximation of maps between polyhedra

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1 Introduction and main results

The importance of approximation of continuous functions is a natural question that arises from Stone–Weierstrass result of uniform approximation of continuous functions over compact sets by polynomial functions. This result provides naturally a polynomial approximation result for $\mathbb{R}^n$-valued continuous maps $f : K \to \mathbb{R}^n$ defined on a compact set K. Difficulties arise when trying to restrict the image of the approximating map. One way to proceed is to be more flexible with the type of approximating maps but also by considering domains of definition and target spaces in suitable tame categories.

This paper deals with the approximation problem in the semialgebraic category. Recall that a set $S \subset \mathbb{R}^m$ is *semialgebraic* if it is a Boolean combination of sets defined by polynomial equalities and inequalities. Let $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ be (non-empty) semialgebraic sets. A map $f : S \to T$ is *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{m+n}$. Continuous semialgebraic maps are called $\mathcal{S}^0$ *maps* and we denote $\mathcal{S}^0(S, T)$ the set of continuous semialgebraic maps from S to T. We endow $\mathcal{S}^0(S, T)$ with its Whitney's $\mathcal{S}^0$ topology, which has as a basis of neighborhoods of an $\mathcal{S}^0$ map $f : S \to T$ the sets $\mathcal{U}_\varepsilon(f) := \{g \in \mathcal{S}^0(S, T) : \|f(x) - g(x)\|_n < \varepsilon(x)\ \forall x \in S\}$, where $\|v\|_n$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$ and $\varepsilon : S \to \mathbb{R}$ is a strictly positive $\mathcal{S}^0$ function. In the following when we say that $f, g \in \mathcal{S}^0(S, T)$ are *close*, we mean that there exists a small strictly positive $\mathcal{S}^0$ function $\varepsilon : S \to \mathbb{R}$ such that $\|f(x) - g(x)\|_n < \varepsilon(x)$ for each $x \in S$.

Fix an integer $\nu \geq 1$ or $\nu = \infty$. A semialgebraic map $f := (f_1, \ldots, f_n) : S \to T \subset \mathbb{R}^n$ is said to be $\mathcal{S}^\nu$ (or *Nash* if $\nu = \infty$) if there exist an open semialgebraic neighborhood Ω of S in $\mathbb{R}^n$ and real-valued semialgebraic functions $F_1, \ldots, F_n$ defined on Ω that are differentiable of class $\mathcal{C}^\nu$ such that $f(x) = (F_1(x), \ldots, F_n(x))$ for each $x \in S$. We denote $\mathcal{S}^\nu(S, T)$ the set of these maps (or $\mathcal{N}(S, T)$ if $\nu = \infty$). To lighten notations we use respectively the symbols $\mathcal{S}^\nu(S)$ and $\mathcal{N}(S)$ in place of $\mathcal{S}^\nu(S, \mathbb{R})$ and $\mathcal{N}(S, \mathbb{R})$.

A semialgebraic set $M \subset \mathbb{R}^m$ is called an (*affine*) $\mathcal{S}^\nu$ *manifold* (or a *Nash manifold* if $\nu = \infty$) if it is in addition a $\mathcal{C}^\nu$ submanifold of (an open subset of) $\mathbb{R}^m$. As in this paper all manifolds are affine we often drop the adjective affine. A map $f : M \to N$ between $\mathcal{S}^\nu$ manifolds is $\mathcal{S}^\nu$ in the sense of the above paragraph if and only if it is semialgebraic and differentiable of class $\mathcal{C}^\nu$ in the usual sense for $\mathcal{C}^\nu$ manifolds involving charts. For more details concerning the spaces of $\mathcal{S}^\nu$ maps, we refer the reader to [4, 2.C & 2.D]. We recall also that if U is an open semialgebraic subset of $\mathbb{R}^n$, a set $Y \subset U$ is called *Nash subset of* U if there exists a Nash function $g \in \mathcal{N}(U)$ such that Y is the zero locus of g. A *Nash set* is a Nash subset of an open semialgebraic subset of some $\mathbb{R}^n$. Naturally, a *Nash subset* Z of a Nash set Y is a Nash set Z that is closed in Y. A Nash set is semialgebraic and a Nash manifold is a (non-singular) Nash set.

1.1 Some relevant known approximation results

In the literature there are many approximation results of algebraic/semialgebraic nature and we recall here some of them.

$\mathcal{S}^\nu$ maps by Nash maps

Efroymson's approximation theorem [16, §1] ensures that continuous semialgebraic functions can be approximated by Nash functions on a Nash manifold. This statement was improved by Shiota in many directions [42], for instance, providing a similar approximation result for $\mathcal{S}^\nu$ functions using Whitney's $\mathcal{S}^\nu$ topology and proving relative versions of such an $\mathcal{S}^\nu$ approximation result. The previous results can be extended to approximate $\mathcal{S}^\nu$ maps $f : S \to N$ from a locally compact semialgebraic set $S \subset \mathbb{R}^m$ to a Nash manifold $N \subset \mathbb{R}^n$ by Nash maps $g : S \to N$ in the $\mathcal{S}^\nu$ topology, making use of a suitable Nash tubular neighborhood of N in $\mathbb{R}^n$ (see [42, Lem.I. 3.2]). We can even go further and obtain approximation results for $\mathcal{S}^\nu$ maps between Nash sets with monomial singularities (see [4, Thm. 1.7]). For further applications of this type of approximation results we refer the reader to [4,17,19].

Nash maps by regular maps

The problem of approximating (smooth or) Nash maps between real algebraic manifolds by regular maps is an old and deep question in real algebraic geometry. Let X and Y be real algebraic manifolds of positive dimension such that X is compact. The set $\mathcal{R}(X, Y)$ of regular maps from X to Y turns out to be dense in the corresponding space $\mathcal{N}(X, Y)$ of Nash maps endowed with the $\mathcal{C}^\infty$ compact-open topology only in exceptional cases. Besides the Stone–Weierstrass theorem (quoted above) for which Y is an Euclidean space, the density of $\mathcal{R}(X, Y)$ in $\mathcal{N}(X, Y)$ is known only when the target space Y is one of the spheres $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^4$ or a grassmannian. For a general rational real algebraic manifold Y, we must restrict hardly the possible domains of definition X to some special types. A remarkable example is the density of $\mathcal{R}(\mathbb{S}^m, \mathbb{S}^n)$ in $\mathcal{N}(\mathbb{S}^m, \mathbb{S}^n)$ for each pair (m, n) when $n = 1, 2, 4$. If $n \neq 1, 2, 4$, the density of $\mathcal{R}(\mathbb{S}^m, \mathbb{S}^n)$ in $\mathcal{N}(\mathbb{S}^m, \mathbb{S}^n)$ remains a fascinating mystery. For further details, we refer the reader to [5, Ch. 12 & §13.3] and the quoted references there, to the survey [11] and to the articles [6–10,12,22,30]. If Y is 'generic' in a suitable way, $\mathcal{R}(X, Y)$ is an 'extremely small' closed subset of $\mathcal{N}(X, Y)$, see [20,21]. This lack of regular maps between real algebraic manifolds seems to be the main obstruction for an extension of the Nash-Tognoli algebraization techniques from smooth manifolds to singular polyhedral spaces and, in particular, to compact Nash sets, see [2,23].

Continuous maps by continuous rational maps

Kucharz has studied deeply approximation results of continuous maps between a compact algebraic manifold X and a sphere $\mathbb{S}^n$ by continuous rational maps. As X is compact the author considers on the space $\mathcal{C}(X, \mathbb{S}^n)$ of continuous maps from X to $\mathbb{S}^n$

the compact-open topology. In case X has dimension n, the space $\mathcal{R}_0(X, \mathbb{S}^n)$ of (nice) continuous rational maps from X to $\mathbb{S}^n$ is dense in $\mathcal{C}(X, \mathbb{S}^n)$ (see [31, Thm. 1.2, Cor. 1.3]). In addition, for any pair (m, n) of non-negative integers, the set $\mathcal{R}_0(\mathbb{S}^m, \mathbb{S}^n)$ is dense in $\mathcal{C}(\mathbb{S}^m, \mathbb{S}^n)$ (see [31, Thm. 1.5]). In general $\mathcal{R}_0(X, \mathbb{S}^n)$ needs not to be dense in $\mathcal{C}(X, \mathbb{S}^n)$. Simple obstructions can be expressed in terms of homology or cohomology classes representable by algebraic subsets. We refer the reader to [29,31] for further details.

Homeomorphisms between smooth manifolds by diffeomorphisms

There are many and celebrated approximation results concerning homeomorphisms between smooth manifolds by diffeomorphisms in the literature. The obstruction theory originated from the problem of smoothing a continuous map with good properties or smoothing a combinatorial manifold M deserves special attention because it is in the core of differential topology. Such a theory was mainly developed by Milnor, Thom, Munkres and Hirsch. They found that the obstructions concentrate in certain homology classes belonging to the homology groups of the combinatorial manifold M relative to its boundary ∂M with coefficients in the quotient groups of the orientation-preserving diffeomorphisms of the unit spheres $\mathbb{S}^p$ modulo the orientation-preserving diffeomorphisms of the unit balls $\mathbb{B}^{p+1}$. We refer the reader to [26,27,32,36–38,47] for further details concerning the mentioned foundational obstruction results and to [14,24,28,34,35] for some recent developments.

In addition, Milnor [33] discovered two compact polyhedra which are homeomorphic but not piecewise linearly homeomorphic (a counterexample to the Hauptvermutung). In the case in which the homeomorphism is semialgebraic the situation is completely different. In fact, Shiota and Yokoi [45] proved, using approximation techniques, that two semialgebraically homeomorphic compact polyhedra in $\mathbb{R}^n$ are also piecewise linearly homeomorphic. Shiota [44] improved the previous result and he obtained a PL homeomorphism by a constructive procedure, involving more sophisticated approximation techniques, that starts from the original homeomorphism. He proved that, for any ordered field R equipped with any o-minimal structure, two definably homeomorphic compact polyhedra in R^n are PL homeomorphic (the o-minimal Hauptvermutung). Together with the fact that any compact definable set is definably homeomorphic to a compact polyhedron, he concludes that o-minimal topology is 'tame'.

1.2 Our approximation results in the semialgebraic setting

Let $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ be semialgebraic sets such that S is compact. In this case Whitney's $\mathcal{S}^0$ topology of the space $\mathcal{S}^0(S, T)$ coincides with its compact-open topology. In fact, $\mathcal{S}^0(S, T)$ is a metrizable space with respect to the distance $d(f, g) := \max\{\|f(x) - g(x)\|_n : x \in S\}$. We will use freely this fact along this work.

Problem 1.1 *Given an integer* $\nu \geq 1$ *or* $\nu = \infty$*, is* $\mathcal{S}^\nu(S, T)$ *dense in* $\mathcal{S}^0(S, T)$*?*

It is well-known that the answer is affirmative if the target space is an $\mathcal{S}^\nu$ manifold.

Known Result 1.2 *For each integer $\nu \geq 1$ or for $\nu = \infty$, if T is an $\mathcal{S}^\nu$ manifold, then $\mathcal{S}^\nu(S, T)$ is dense in $\mathcal{S}^0(S, T)$.*

The reason why the latter result works is that each $\mathcal{S}^0$ map $f : S \to T \subset \mathbb{R}^n$ can be uniformly approximated by a polynomial map $g : S \to \mathbb{R}^n$ (recall that S is compact) and then one can use an $\mathcal{S}^\nu$ (bent) tubular neighborhood $\rho : U \to T$ of T in $\mathbb{R}^n$ (see [42, I.3.5 & II.6.1]) to obtain the desired approximating map $\rho \circ g$. However, an arbitrary semialgebraic set $T \subset \mathbb{R}^n$ does not have $\mathcal{S}^\nu$ tubular neighborhoods in $\mathbb{R}^n$ if $\nu \geq 1$ (see Example 1.10 below). It has only $\mathcal{S}^0$ tubular neighborhoods in $\mathbb{R}^n$, provided it is locally compact [15].

First main result

Our first result gives a complete affirmative solution to Problem 1.1 for $\nu = 1$.

Theorem 1.3 *Let $S \subset \mathbb{R}^m$ be a compact semialgebraic set and let $T \subset \mathbb{R}^n$ be a semialgebraic set. Then $\mathcal{S}^1(S, T)$ is dense in $\mathcal{S}^0(S, T)$. More precisely, given any $n \in \mathbb{N}$, the following assertion holds: for each $f \in \mathcal{S}^0(S, \mathbb{R}^n)$ and each $\varepsilon > 0$, there exists $g \in \mathcal{S}^1(S, \mathbb{R}^n)$ such that $g(S) \subset f(S)$ and $\|g(x) - f(x)\|_n < \varepsilon$ for every $x \in S$.*

For an arbitrary positive integer $\nu \geq 2$ we obtain density results in two very significant situations we are going to present.

Second main result

Let us recall the definition of locally $\mathcal{S}^\nu$ polyhedral semialgebraic set, which represents a polyhedral counterpart of the concept of $\mathcal{S}^\nu$ manifold. Let ν be an integer ≥ 1. A semialgebraic set $T \subset \mathbb{R}^n$ is called *locally $\mathcal{S}^\nu$ equivalent to a polyhedron*, or *locally $\mathcal{S}^\nu$ polyhedral* for short, if for each point $x \in T$ there exist two open semialgebraic neighborhoods U_x and V_x of x in $\mathbb{R}^n$, an $\mathcal{S}^\nu$ diffeomorphism $\phi_x : U_x \to V_x$ and a compact polyhedron Q of $\mathbb{R}^n$ such that $\phi_x(U_x \cap T) = V_x \cap Q$. The term *compact polyhedron of $\mathbb{R}^n$* means the realization of a finite simplicial complex of $\mathbb{R}^n$ (see [39, §2]). The importance of locally $\mathcal{S}^\nu$ polyhedral semialgebraic sets is that if they are in addition compact, they admit $\mathcal{S}^\nu$ triangulations (see Sect. 2.1 below).

Our second main result asserts that Known Result 1.2 extends to the case in which the target space T is an arbitrary locally $\mathcal{S}^\nu$ polyhedral compact semialgebraic set.

Theorem 1.4 *Let $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ be compact semialgebraic sets. If T is locally $\mathcal{S}^\nu$ polyhedral for some integer $\nu \geq 1$, then $\mathcal{S}^\nu(S, T)$ is dense in $\mathcal{S}^0(S, T)$.*

As an immediate consequence we obtain:

Corollary 1.5 *Let $S \subset \mathbb{R}^m$ be a compact semialgebraic set and let $T \subset \mathbb{R}^n$ be a compact polyhedron. Then $\mathcal{S}^\nu(S, T)$ is dense in $\mathcal{S}^0(S, T)$ for each integer $\nu \geq 1$.*

Theorem 1.4 and Corollary 1.5 are sharp in the sense that they are false if the approximating maps are Nash, that is, if $\nu = \infty$.

Example 1.6 Let $S := [-1, 1]$ and $T := \{(x, y) \in [-2, 2] \times \mathbb{R} : x^2 - y^2 = 0\}$. Observe that T is a compact polyhedron and $T' := T \cap ((-2, 2) \times \mathbb{R})$ is a Nash set whose irreducible components are $T'_\pm := \{(x, y) \in (-2, 2) \times \mathbb{R} : x \pm y = 0\}$. Consider the $\mathcal{S}^0$ map $f : S \to T' \subset T,\ t \mapsto (t, |t|)$ and suppose there exists a Nash map $g : S \to T$ close to f in $\mathcal{S}^0(S, T)$. Since g is close to f and $f(S) \subset T'$, we may assume $g(S) \subset T'$. As S is an irreducible semialgebraic set in the sense of [18, §3], the image $g(S) \subset T'$ must be an irreducible semialgebraic set [18, 3.1(iv)], so $g(S)$ must be contained either in T'_+ or in T'_-, which is impossible because g is close to f and $\dim(f(S) \cap T'_\pm) = 1$. This proves that $\mathcal{N}(S, T)$ is not dense in $\mathcal{S}^0(S, T)$. □

A by-product of the argument we will use to prove Theorem 1.4 is the following.

Corollary 1.7 *Let K be a finite simplicial complex of $\mathbb{R}^p$ and let $P \subset \mathbb{R}^p$ be the corresponding compact polyhedron $|K|$. Then, for each integer $\nu \geq 1$, there exists a sequence $\{\iota_n^\nu\}_{n\in\mathbb{N}}$ in $\mathcal{S}^\nu(P, P)$ with the following universal property: if f is a real-valued function in $\mathcal{S}^0(P)$ such that $f|_\sigma \in \mathcal{S}^\nu(\sigma)$ for each $\sigma \in K$, then the sequence $\{f \circ \iota_n^\nu\}_{n\in\mathbb{N}}$ is contained in $\mathcal{S}^\nu(P)$ and converges to f in $\mathcal{S}^0(P)$. In particular, the sequence $\{\iota_n^\nu\}_{n\in\mathbb{N}}$ converges to the identity map in $\mathcal{S}^0(P, P)$.*

Third main result

Consider the compact algebraic curve $T := \{y^2 - x^3(1 - x) = 0\} \subset \mathbb{R}^2$. It has a cusp at the origin, so it is not locally $\mathcal{S}^1$ polyhedral. Thus, Theorem 1.4 does not apply if T is the target space. However, Problem 1.1 continues to have an affirmative solution, because T is a Nash set. More precisely, we are able to prove the following result.

Theorem 1.8 *Let $S \subset \mathbb{R}^m$ be a compact semialgebraic set and let T be an open semialgebraic subset of a Nash set $Y \subset \mathbb{R}^n$. Then $\mathcal{S}^\nu(S, T)$ is dense in $\mathcal{S}^0(S, T)$ for each integer $\nu \geq 1$.*

Theorem 1.8 is again sharp in the sense that it is false if the approximating maps are Nash, that is, if $\nu = \infty$: consider the $\mathcal{S}^0$ map $S \to T',\ t \mapsto (t, |t|)$ of Example 1.6 and observe that it cannot be approximated by Nash maps between S and T'.

Remarks 1.9 (1) In the statements of Theorems 1.3, 1.4 and 1.8 and of Corollary 1.5, we may assume that the $\mathcal{S}^\nu$ approximating maps are $\mathcal{S}^0$ homotopic to the original $\mathcal{S}^0$ maps. This holds because in Theorem 1.3 $f(S)$ is compact and in the remaining results T is locally compact. Hence $f(S)$ and T admit by [15] $\mathcal{S}^0$ tubular neighborhoods in $\mathbb{R}^n$.

(2) Our results are also sharp in the following sense. Let ℓ, ν be integers such that $1 \leq \ell < \nu$, let S be a compact $\mathcal{S}^\nu$ manifold and let $T \subset \mathbb{R}^n$ be a semialgebraic set. If T is not an $\mathcal{S}^\nu$ manifold, then one cannot expect that $\mathcal{S}^\nu(S, T)$ is dense in $\mathcal{S}^\ell(S, T)$ (equipped with its Whitney's $\mathcal{S}^\ell$ topology defined in the obvious way). An easy example is the following. Let $S := \mathbb{S}^1$ be the circumference of $\mathbb{R}^2$ with center the origin and radius 1 and let $T := \{(x, y, z) \in S \times \mathbb{R} : z^3 - y^{3\ell+1} = 0\}$. The $\mathcal{S}^\ell$ map $f : S \to T$ defined by $f(x, y) := (x, y, y^{\ell+1/3})$ cannot be $\mathcal{S}^\ell$ approximated by a map $g := (g_1, g_2, g_3) : S \to T$ in $\mathcal{S}^\nu(S, T)$. Otherwise, by the inverse function

theorem, the map $G := (g_1, g_2) : S \to S$ would be an $\mathcal{S}^\nu$-diffeomorphism and $(g_3 \circ G^{-1})(x, y) = y^{\ell+1/3}$ would belong to $\mathcal{S}^\nu(S)$, which is a contradiction. □

Involved tools

The proofs of Theorems 1.3 and 1.4 are based on the use of $\mathcal{S}^1$ triangulations of arbitrary compact semialgebraic sets [40, Thm. 1.1] and $\mathcal{S}^\nu$ triangulations of locally $\mathcal{S}^\nu$ polyhedral compact semialgebraic sets [43, Prop.I. 3.13 & Rmk.I. 3.22] combined with simplicial approximation of continuous maps between compact polyhedra [39, Ch.2] and with a 'shrink-widen' approximation technique introduced in Sect. 3. Corollary 1.7 is a consequence of such techniques. The proof of Theorem 1.8 involves the use of $\mathcal{S}^\nu$ *weak retractions* that are developed in Sect. 4. Let $M \subset \mathbb{R}^m$ be a Nash manifold and let $X \subset M$ be a Nash normal-crossings divisor. Let $W \subset M$ be an open semialgebraic neighborhood of X. An $\mathcal{S}^\nu$ weak retraction is an $\mathcal{S}^\nu$ map $\rho : W \to X$ whose restriction to X is arbitrary close to the identity map on X. In Proposition 4.2 we prove the existence of $\mathcal{S}^\nu$ weak retractions. We combine this tool with a strategy employed in [3, Lem. 2.2] that involves resolution of singularities. The use of $\mathcal{S}^\nu$ weak retractions instead of usual retractions is justified by the following example.

Example 1.10 There exists no $\mathcal{S}^1$ retraction from a semialgebraic neighborhood U of $T := \{xy = 0\} \subset \mathbb{R}^2$ onto T. Suppose that $\rho : U \to T$ is such an $\mathcal{S}^1$ retraction. As $\rho|_T$ is equal to the identity map id_T on T, we deduce that $d_0\rho = \mathrm{id}_{\mathbb{R}^2}$. Consequently, there exist open semialgebraic neighborhoods U_1 and U_2 of the origin in $\mathbb{R}^2$ such that the restriction $\rho|_{U_1} : U_1 \to U_2 \subset T$ is an $\mathcal{S}^1$ diffeomorphism, which is a contradiction. □

Structure of the article

All basic notions and preliminary results used in this paper are presented in Sect. 2. The reader can proceed directly to Sect. 3 and refer to Sect. 2 when needed. In Sect. 3 we describe our 'shrink-widen' approximation technique and we prove Theorems 1.3 and 1.4, and Corollary 1.7. Section 4 is devoted to the proof of the existence of $\mathcal{S}^\nu$ weak retractions, that we use in Sect. 5 to prove Theorem 1.8.

2 Preliminaries

In this section we introduce many concepts and notations needed in the article. Let us recall some general properties of semialgebraic sets. Semialgebraic sets are closed under Boolean combinations and, by means of quantifier elimination, they are also closed under projections. Any set $S \subset \mathbb{R}^m$ defined by a first order formula in the language of ordered fields is a semialgebraic set [5, pp. 28, 29]. Thus, the basic topological constructions as the closure of S, the interior of S and the boundary of S in $\mathbb{R}^m$ (denoted by $\mathrm{Cl}(S)$, $\mathrm{Int}(S)$ and ∂S respectively) are semialgebraic if S is. Also images and preimages of semialgebraic sets by semialgebraic maps are again semialgebraic. The *dimension* $\dim(S)$ of a semialgebraic set S is the dimension of its Zariski closure in $\mathbb{R}^m$ [5, §2.8]. The *local dimension* $\dim(S_x)$ *of* S *at a point* $x \in \mathrm{Cl}(S)$

is the dimension of $U \cap S$ for a small enough open semialgebraic neighborhood U of x in $\mathbb{R}^m$. The dimension of S coincides with the maximum of these local dimensions. For any fixed integer k the set of points $x \in S$ such that $\dim(S_x) = k$ is a semialgebraic subset of S. If the function $S \to \mathbb{N},\ x \mapsto \dim(S_x)$ is constant, then S is said to be *pure dimensional*.

2.1 Simplicial approximation and $\mathscr{S}^\nu$ triangulations

Let K be a finite simplicial complex of $\mathbb{R}^p$. Given any simplex $\sigma \in K$, we denote $\mathrm{Bd}(\sigma)$ the *relative boundary of* σ defined as the union of proper faces of σ and $\sigma^0 := \sigma \setminus \mathrm{Bd}(\sigma)$ the *relative interior of* σ, which is equal to the set of points of σ whose barycentric coordinates are all strictly positive. The sets σ^0 are called *open simplexes of* K. We indicate $|K|$ the subset $\bigcup_{\sigma \in K} \sigma$ of $\mathbb{R}^p$ equipped with the topology inherited from the Euclidean one of $\mathbb{R}^p$. Let K_* be the set of vertices of K and for each $v \in K_*$ let $\mathrm{Star}(v, K)$ be the star of v in K, that is, the open neighborhood $\bigcup_{\sigma \in K, v \in \sigma} \sigma^0$ of v in $|K|$. For each positive integer k denote $K^{(k)}$ the k^{th}-iterated barycentric subdivision of K. If we fix $\varepsilon > 0$ and pick k large enough, then every simplex $\tau \in K^{(k)}$ has diameter $\mathrm{diam}_p(\tau) := \max_{x,y \in \tau}\{\|x - y\|_p\} < \varepsilon$ (see [35, Thm. 15.4]).

Let L be a finite simplicial complex of some $\mathbb{R}^q$ and let $g : K_* \to L_*$ be a map between the sets of vertices of K and L satisfying the following condition: *if* $v_1, \ldots, v_r$ *are vertices of* K *that span a simplex of* K, *then* $g(v_1), \ldots, g(v_r)$ *are vertices of* L *that span a simplex of* L. Then, g extends uniquely to a continuous map from $|K|$ to $|L|$ whose restriction to each simplex σ of K is the restriction to σ of an affine map $\mathbb{R}^p \to \mathbb{R}^q$. We denote this extension again $g : |K| \to |L|$ and we say that g is a *simplicial map*. Let $f : |K| \to |L|$ be a continuous map. A simplicial map $g : |K| \to |L|$ is called a *simplicial approximation of* f if $f(\mathrm{Star}(v, K)) \subset \mathrm{Star}(g(v), L)$ for each vertex $v \in K_*$. If g is a simplicial approximation of f, then for each $x \in |K|$ there exists $\xi_x \in L$ such that $\{g(x), f(x)\} \subset \xi_x$, so $\|g(x) - f(x)\|_q \leq \mathrm{diam}_q(\xi_x)$ (see [35, Cor.14.2]). The finite simplicial approximation theorem [35, Thm. 16.1] assures that: *given a continuous map* $f : |K| \to |L|$, *there exists a positive integer* k *and a simplicial approximation* $g : |K^{(k)}| \to |L|$ *of* f.

Theorem 2.1 ([35, §14, 15 & 16]) *Let* K *and* L *be two finite simplicial complexes and let* $f : |K| \to |L|$ *be a continuous map. Suppose* $|L| \subset \mathbb{R}^q$. *Then, for each* $\varepsilon > 0$, *there exist two positive integers* k, ℓ *and a simplicial map* $g : |K^{(k)}| \to |L^{(\ell)}|$ *such that* $\|g(x) - f(x)\|_q < \varepsilon$ *for each* $x \in |K^{(k)}| = |K|$.

Proof Choose an integer $\ell \geq 1$ such that $\mathrm{diam}_q(\xi) < \varepsilon$ for each $\xi \in L^{(\ell)}$. Now, apply the finite simplicial approximation theorem to the continuous function $f : |K| \to |L^{(\ell)}| = |L|$ and the proof is concluded. □

In the semialgebraic setting we have the following triangulation result, see [5, Thm. 9.2.1 & Rmk. 9.2.3.a)].

Theorem 2.2 *Given any compact semialgebraic set* $S \subset \mathbb{R}^m$, *there exist a finite simplicial complex* K *and a semialgebraic homeomorphism* $\Phi : |K| \to S$. *In addi-*

tion, for each $\sigma \in K$ *the set* $\Phi(\sigma^0) \subset \mathbb{R}^m$ *is a Nash manifold and the restriction* $\Phi|_{\sigma^0} : \sigma^0 \to \Phi(\sigma^0)$ *is a Nash diffeomorphism.*

Recently Ohmoto and Shiota proved a remarkable global $\mathcal{S}^1$ version of the latter theorem. We state their result in the compact case only, see [40, Thm. 1.1].

Theorem 2.3 *Given any compact semialgebraic set* $T \subset \mathbb{R}^n$*, there exist a finite simplicial complex* L *and a semialgebraic homeomorphism* $\Psi : |L| \to T$ *such that* $\Psi \in \mathcal{S}^1(|L|, T)$.

It is not known if the semialgebraic homeomorphism Ψ can be chosen of class $\mathcal{C}^2$ (see [40, Sect.1]). However, for a locally $\mathcal{S}^\nu$ polyhedral semialgebraic set we have in addition the following result, see [43, Prop.I. 3.13 & Rmk.I. 3.22].

Theorem 2.4 *Let* $T \subset \mathbb{R}^n$ *be a compact semialgebraic set. If* T *is locally* $\mathcal{S}^\nu$ *polyhedral for some integer* $\nu \geq 1$*, then there exist a finite simplicial complex* L *and a semialgebraic homeomorphism* $\Psi : |L| \to T$ *such that the restriction of* Ψ *to* ξ *belongs to* $\mathcal{S}^\nu(\xi, T)$ *for each* $\xi \in L$.

2.2 Sets of regular and singular points of a semialgebraic set

Let $Z \subset \mathbb{C}^n$ be a complex algebraic set and let $I_{\mathbb{C}}(Z)$ be the ideal of all polynomials $F \in \mathbb{C}[\mathrm{x}]$ such that $F(z) = 0$ for each $z \in Z$. A point $z \in Z$ is *regular* if the localization of the polynomial ring $\mathbb{C}[\mathrm{x}]/I_{\mathbb{C}}(Z)$ at the maximal ideal $\mathfrak{M}_z$ associated to z is a regular local ring. In this complex setting the Jacobian criterion and Hilbert's Nullstellensatz imply that $z \in Z$ is regular if and only if there exists an open neighborhood $U \subset \mathbb{C}^n$ of z such that $U \cap Z$ is an analytic manifold. We denote $\operatorname{Reg}(Z)$ the set of regular points of Z and it is an open dense subset of Z. If Z is irreducible, it is pure dimensional and $\operatorname{Reg}(Z)$ is a connected analytic manifold. In case Z is not irreducible, then the connected components of $\operatorname{Reg}(Z)$ are finitely many analytic manifolds (possibly of different dimensions).

Let $X \subset \mathbb{R}^n$ be a (real) algebraic set and let $I_{\mathbb{R}}(X)$ be the ideal of all polynomials $f \in \mathbb{R}[\mathrm{x}]$ such that $f(x) = 0$ for each $x \in X$. A point $x \in X$ is *regular* if the localization of $\mathbb{R}[\mathrm{x}]/I_{\mathbb{R}}(X)$ at the maximal ideal $\mathfrak{m}_x$ associated to x is a regular local ring [5, §3.3]. In addition, $x \in X$ is *smooth* if there exists an open neighborhood $U \subset \mathbb{R}^n$ such that $U \cap X$ is a Nash manifold. It holds that each regular point is a smooth point, but in the real case the converse is not always true as it shows the following example.

Example 2.5 Let $X := \{(x^2 + y^2)xz - y^4 = 0\} \subset \mathbb{R}^3$. The set of regular points of X is $\operatorname{Reg}(X) = X \setminus \{x = 0, y = 0\}$. However, the set of smooth points of X is $X \setminus \{0\}$. To prove this fact it suffices to observe that the maps $\varphi_\varepsilon : \{(t, s) \in \mathbb{R}^2 : t > 0\} \to \mathbb{R}^3$ for $\varepsilon = \pm 1$ defined by $\varphi_\varepsilon(s, t) := \varepsilon((s^2 + t^2)s^2, (s^2 + t^2)st, t^4)$ are Nash embeddings, whose images cover $X \setminus \{z = 0\}$. It follows that each point $(0, 0, a) \in X$ with $a \neq 0$ is smooth. □

Let $\widetilde{X} \subset \mathbb{C}^n$ be the complex algebraic set that is the zero set of the extended ideal $I_{\mathbb{R}}(X)\mathbb{C}[\mathrm{x}]$. We call $\widetilde{X}$ the *complexification of* X. The ideal $I_{\mathbb{C}}(\widetilde{X})$ coincides with the tensored ideal $I_{\mathbb{R}}(X)\otimes_{\mathbb{R}}\mathbb{C}$, so $\widetilde{X}$ is the smallest complex algebraic subset of $\mathbb{C}^n$ that contains X and

$$\mathbb{C}[\mathrm{x}]/I_{\mathbb{C}}(\widetilde{X})\cong(\mathbb{R}[\mathrm{x}]/I_{\mathbb{R}}(X))\otimes_{\mathbb{R}}\mathbb{C}.$$

It holds that the localization $(\mathbb{R}[\mathrm{x}]/I_{\mathbb{R}}(X))_{\mathfrak{m}_x}$ is a regular local ring if and only if so is its complexification

$$(\mathbb{R}[\mathrm{x}]/I_{\mathbb{R}}(X))_{\mathfrak{m}_x}\otimes_{\mathbb{R}}\mathbb{C}\cong(\mathbb{C}[\mathrm{x}]/I_{\mathbb{C}}(\widetilde{X}))_{\mathfrak{M}_x}.$$

Consequently, the *set of regular points* of X is $\operatorname{Reg}(X)=\operatorname{Reg}(\widetilde{X})\cap X$ and its *set of singular points* is $\operatorname{Sing}(X):=X\setminus\operatorname{Reg}(X)$. The connected components of the open subset $\operatorname{Reg}(X)$ of X is a finite union of Nash manifolds (possibly of different dimensions).

We turn out next to Nash sets. Let $X\subset\mathbb{R}^n$ be a Nash set. A point $x\in X$ is *regular* if the localization $\mathcal{N}(X)_{\mathfrak{n}_x}$ at the maximal ideal $\mathfrak{n}_x$ of $\mathcal{N}(X)$ associated to x is a regular local ring. Denote $\operatorname{Reg}(X)$ the set of regular points of X. Again a point $x\in X$ is *smooth* if there exists an open neighborhood $U\subset\mathbb{R}^n$ of x such that $U\cap X$ is a Nash manifold. As before each regular point is a smooth point but Example 2.5 shows that the converse is not true in general. The Nash set $X\subset\mathbb{R}^n$ is said to be *non-singular* if $X=\operatorname{Reg}(X)$. Assume that X is irreducible. It holds that X is a non-singular Nash set if and only if it is a connected Nash manifold [42, Def.II. 1.12 and Prop.II. 5.6]. Alternatively, this can be shown as an application of Artin-Mazur's Theorem [5, Thm. 8.4.4].

A natural question arises when confronting the definitions of regular point of a real algebraic set $X\subset\mathbb{R}^n$ from the algebraic and Nash viewpoints. Using the properties of completions and henselization [1, Prop.VII. 2.2 and Prop.VII. 3.1] one shows that a point $x\in X$ is regular from the algebraic point of view if and only if it is regular from the Nash point of view.

Note in addition that if the irreducible components of a Nash set X are non-singular, then a point $x\in X$ is regular if and only if it is smooth.

2.3 Desingularization of algebraic sets

Let $X\subset Y\subset\mathbb{R}^n$ be algebraic sets such that Y is non-singular. Recall that X is a *normal-crossings divisor of* Y if for each point $x\in X$ there exists a regular system of parameters $x_1,\dots,x_d$ such that X is given in a Zariski neighborhood of x in Y by the equation $x_1\cdots x_k=0$ for some $k=1,\dots,d$. In particular, the irreducible components of X are non-singular and have codimension 1 in Y. A map $f:=(f_1,\dots,f_n):Z\to\mathbb{R}^n$ on a (non-empty) subset Z of $\mathbb{R}^m$ is said to be *regular* if its components are quotients of polynomials $f_k:=\frac{g_k}{h_k}$ such that $Z\cap\{h_k=0\}=\varnothing$.

The following is a version of Hironaka's desingularization theorems [25] we will use fruitfully in the sequel.

Theorem 2.6 (Desingularization) *Let $X\subset\mathbb{R}^n$ be an algebraic set. Then there exist a non-singular algebraic set $X'\subset\mathbb{R}^m$ and a proper regular map $\phi:X'\to X$ such that $\phi^{-1}(\operatorname{Sing}(X))$ is a normal-crossings divisor of X' and*

$$\phi|_{X'\setminus\phi^{-1}(\mathrm{Sing}(X))} : X'\setminus\phi^{-1}(\mathrm{Sing}(X)) \to X\setminus \mathrm{Sing}(X)$$

is a Nash diffeomorphism whose inverse map is regular.

Remark 2.7 If X is pure dimensional, $X\setminus \mathrm{Sing}(X) = \mathrm{Reg}(X)$ is dense in X. Consequently, as ϕ is proper, ϕ is also surjective. Furthermore X' is a real algebraic manifold, that is, it is non-singular and pure dimensional. □

3 Proof of Theorems 1.3 and 1.4

The purpose of this section is to prove Theorems 1.3 and 1.4, and Corollary 1.7. We begin developing our tools to approach those proofs.

3.1 The 'shrink-widen' covering and approximation lemmas

Let σ be a simplex of $\mathbb{R}^p$, let σ^0 be the simplicial interior of σ and let b_σ be the barycenter of σ. Given $\varepsilon \in (0,1)$ denote $h_\varepsilon : \mathbb{R}^p \to \mathbb{R}^p,\ x \mapsto b_\sigma + (1-\varepsilon)(x-b_\sigma)$ the homothety of $\mathbb{R}^p$ of center b_σ and ratio $1-\varepsilon$. We define the $(1-\varepsilon)$-*shrinking* σ^0_ε *of* σ^0 by $\sigma^0_\varepsilon := h_\varepsilon(\sigma^0)$. Observe that $\mathrm{Cl}(\sigma^0_\varepsilon) = h_\varepsilon(\sigma) \subset \sigma^0$ for every $\varepsilon \in (0,1)$ and σ^0_ε tends to σ_0 when $\varepsilon \to 0$. In addition, $\sigma^0 = \bigcup_{\varepsilon\in(0,1)} \sigma^0_\varepsilon$ and $\sigma^0_{\varepsilon_2} \subset \sigma^0_{\varepsilon_1}$ if $0 < \varepsilon_1 \leq \varepsilon_2 < 1$.

We fix the following notations for the rest of the subsection. Let $S \subset \mathbb{R}^m$ be a compact semialgebraic set, let K be a finite simplicial complex of $\mathbb{R}^p$ and let $\Phi : |K| \to S$ be a semialgebraic homeomorphism such that for each open simplex σ^0 of K the set $\Phi(\sigma^0) \subset \mathbb{R}^m$ is a Nash manifold and the restriction $\Phi|_{\sigma^0} : \sigma^0 \to \Phi(\sigma^0)$ is a Nash diffeomorphism. Define $\mathcal{K}^0 := \{\Phi(\sigma^0)\}_{\sigma\in K}$ and $\mathcal{K} := \{\Phi(\sigma)\}_{\sigma\in K}$. To lighten the notation the elements of $\mathcal{K}$ will be denoted with the letters $\mathsf{s}, \mathsf{t}, \ldots$ while those of $\mathcal{K}^0$ with the letters $\mathsf{s}^0, \mathsf{t}^0, \ldots$ in such a way that $\mathrm{Cl}(\mathsf{s}^0) = \mathsf{s}$ and s^0 is the interior of s as a semialgebraic manifold-with-boundary. In other words, if $\mathsf{s} = \Phi(\sigma)$, then $\mathsf{s}^0 = \Phi(\sigma^0)$. Moreover, we indicate s^0_ε the shrinking of $\mathsf{s}^0 = \Phi(\sigma^0)$ corresponding to σ^0_ε via Φ, that is, $\mathsf{s}^0_\varepsilon := \Phi(\sigma^0_\varepsilon)$.

Consider a Nash tubular neighborhood $\rho_{\mathsf{s}^0} : T_{\mathsf{s}^0} \to \mathsf{s}^0$ of s^0 in $\mathbb{R}^m$ and the family of open semialgebraic sets $T_{\mathsf{s}^0,\delta} := \{x \in T_{\mathsf{s}^0} :\ \|x - \rho_{\mathsf{s}^0}(x)\|_m < \delta\}$ where $\delta > 0$. We write $\mathsf{s}^0_{\varepsilon,\delta}$ to denote the δ-*widening of* s^0_ε *with respect to* ρ_{s^0}, which is the open neighborhood $\mathsf{s}^0_{\varepsilon,\delta} := (\rho_{\mathsf{s}^0})^{-1}(\mathsf{s}^0_\varepsilon) \cap T_{\mathsf{s}^0,\delta}$ of s^0_ε in $\mathbb{R}^m$. If C is a closed subset of $\mathbb{R}^m$ such that $C \cap \mathrm{Cl}(\mathsf{s}^0_\varepsilon) = \varnothing$, there exists $\delta > 0$ such that $C \cap \mathrm{Cl}(\mathsf{s}^0_{\varepsilon,\delta}) = \varnothing$. Denote $\rho_{\mathsf{s}^0,\varepsilon,\delta} := \rho_{\mathsf{s}^0}|_{\mathsf{s}^0_{\varepsilon,\delta}} : \mathsf{s}^0_{\varepsilon,\delta} \cap S \to \mathsf{s}^0_\varepsilon$ the Nash retraction obtained restricting ρ_{s^0} from $\mathsf{s}^0_{\varepsilon,\delta} \cap S$ to s^0_ε.

Lemma 3.1 *Fix* $\delta > 0$. *Then, for each* $\mathsf{s}^0 \in \mathcal{K}^0$ *there exist a non-empty open semialgebraic subset* V_{s^0} *of* s^0 *(a 'shrinking' of* s^0*), an open semialgebraic neighborhood* U_{s^0} *of* V_{s^0} *in* S *(a 'widening' of* V_{s^0}*) satisfying* $V_{\mathsf{s}^0} = U_{\mathsf{s}^0} \cap \mathsf{s}^0$ *and a Nash retraction* $r_{\mathsf{s}^0} : U_{\mathsf{s}^0} \to V_{\mathsf{s}^0}$ *such that:*

(i) $\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0\in\mathcal{K}^0}$ *is an open covering of* S.

(ii) $\mathrm{Cl}(U_{\mathsf{s}^0}) \cap \mathsf{t} = \varnothing$ *for each pair* $(\mathsf{s}^0, \mathsf{t}) \in \mathcal{K}^0 \times \mathcal{K}$ *satisfying* $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$.
(iii) $\sup_{x \in U_{\mathsf{s}^0}} \{\|x - r_{\mathsf{s}^0}(x)\|_m\} < \delta$ *for each* $\mathsf{s}^0 \in \mathcal{K}^0$.

Proof Write $d := \dim(S)$ and $\mathcal{K}^0_e := \{\mathsf{s}^0 \in \mathcal{K}^0 : \dim(\mathsf{s}^0) \leq e\}$ for $e = 0, \ldots, d$. Let us prove by induction on $e \in \{0, 1, \ldots, d\}$ that for each $\mathsf{s}^0 \in \mathcal{K}^0_e$ there exist an open semialgebraic subset $U^e_{\mathsf{s}^0}$ of S and a Nash retraction $r^e_{\mathsf{s}^0} : U^e_{\mathsf{s}^0} \to V^e_{\mathsf{s}^0} := U^e_{\mathsf{s}^0} \cap \mathsf{s}^0 \neq \varnothing$ such that:

(1) $\bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_e} \mathsf{s}^0 \subset \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_e} U^e_{\mathsf{s}^0}$.
(2) $\mathrm{Cl}(U^e_{\mathsf{s}^0}) \cap \mathsf{t} = \varnothing$ for each pair $(\mathsf{s}^0, \mathsf{t}) \in \mathcal{K}^0_e \times \mathcal{K}$ satisfying $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$.
(3) $\sup_{x \in U^e_{\mathsf{s}^0}} \{\|x - r^e_{\mathsf{s}^0}(x)\|_m\} < \delta$ for each $\mathsf{s}^0 \in \mathcal{K}^0_e$.

Obviously, the sets $U_{\mathsf{s}^0} := U^d_{\mathsf{s}^0}$ and the maps $r_{\mathsf{s}^0} := r^d_{\mathsf{s}^0}$ with $\mathsf{s}^0 \in \mathcal{K}^0_d = \mathcal{K}^0$ will be the desired open semialgebraic sets and Nash retractions.

Consider first the case $e = 0$. Choose $\delta' \in (0, \delta)$ such that the open ball $B(v, 2\delta')$ of $\mathbb{R}^m$ of center v and radius $2\delta'$ does not meet $\bigcup_{\mathsf{t} \in \mathcal{K}, v \notin \mathsf{t}} \mathsf{t}$ for each $\{v\} \in \mathcal{K}^0_0$. Take $U^0_{\{v\}} := B(v, \delta') \cap S$, $V^0_{\{v\}} := \{v\}$ and $r^0_{\{v\}} : U^0_{\{v\}} \to V^0_{\{v\}}$, $x \mapsto v$ the constant map for each $\{v\} \in \mathcal{K}^0_0$.

Fix $e \in \{0, \ldots, d-1\}$ and suppose that the assertion is true for such an e. As

$$C_\sigma := \sigma \setminus \Phi^{-1}\left(\bigcup_{\tau \in K, \tau \subset \mathrm{Bd}(\sigma)} U^e_{\Phi(\tau^0)}\right)$$

is a compact subset of $\sigma^0 = \bigcup_{\varepsilon \in (0,1)} \sigma^0_\varepsilon$, there exists $\varepsilon \in (0, 1)$ such that $C_\sigma \subset \sigma^0_\varepsilon$ for each $\sigma \in K$ of dimension $e + 1$. We have

$$\bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_{e+1}} \mathsf{s}^0 \subset \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_e} U^e_{\mathsf{s}^0} \cup \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e} \mathsf{s}^0_\varepsilon.$$

If $(\mathsf{s}^0, \mathsf{t}) \in (\mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e) \times \mathcal{K}$ satisfies $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$, then $\mathrm{Cl}(\mathsf{s}^0_\varepsilon) \cap \mathsf{t} = \varnothing$ because $\mathrm{Cl}(\mathsf{s}^0_\varepsilon) \subset \mathsf{s}^0$. Pick $\delta' \in (0, \delta)$ such that $\mathrm{Cl}(\mathsf{s}^0_{\varepsilon,\delta'} \cap S) \cap \mathsf{t} = \varnothing$ for each pair $(\mathsf{s}^0, \mathsf{t}) \in (\mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e) \times \mathcal{K}$ satisfying $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$. For each $\mathsf{s}^0 \in \mathcal{K}^0_{e+1}$ define:

- $V^{e+1}_{\mathsf{s}^0} := V^e_{\mathsf{s}^0}$, $U^{e+1}_{\mathsf{s}^0} := U^e_{\mathsf{s}^0}$ and $r^{e+1}_{\mathsf{s}^0} := r^e_{\mathsf{s}^0}$ if $\mathsf{s}^0 \in \mathcal{K}^0_e$, and
- $V^{e+1}_{\mathsf{s}^0} := \mathsf{s}^0_\varepsilon$, $U^{e+1}_{\mathsf{s}^0} := \mathsf{s}^0_{\varepsilon,\delta'} \cap S$ and $r^{e+1}_{\mathsf{s}^0} := \rho_{\mathsf{s}^0,\varepsilon,\delta'}$ if $\mathsf{s}^0 \in \mathcal{K}^0_{e+1} \setminus \mathcal{K}^0_e$ (recall that the retraction $\rho_{\mathsf{s}^0,\varepsilon,\delta'}$ was defined above the statement of this lemma).

The open semialgebraic sets $U^{e+1}_{\mathsf{s}^0}$, the non-empty semialgebraic sets $V^{e+1}_{\mathsf{s}^0}$ and the retractions $r^{e+1}_{\mathsf{s}^0} : U^{e+1}_{\mathsf{s}^0} \to V^{e+1}_{\mathsf{s}^0}$ for $\mathsf{s}^0 \in \mathcal{K}^0_{e+1}$ satisfy conditions (1) to (3), as required. □

As a consequence of the previous result we obtain the following approximation lemma.

Lemma 3.2 *Let L be a finite simplicial complex of $\mathbb{R}^q$, let $g \in \mathcal{S}^0(S, |L|)$ and let $\nu \geq 1$ be a positive integer. Suppose that for each $\mathsf{t} \in \mathcal{K}$ the restriction $g|_{\mathsf{t}^0}$ belongs to $\mathcal{S}^\nu(\mathsf{t}^0, |L|)$ and there exists $\xi_\mathsf{t} \in L$ such that $g(\mathsf{t}) \subset \xi_\mathsf{t}$. Fix $\eta > 0$. Then there exists $h \in \mathcal{S}^\nu(S, |L|)$ with the following properties:*

(i) *For each* $\mathsf{t} \in \mathcal{K}$*, there exists an open semialgebraic neighborhood* W_{t} *of* t *in* S *such that* $h(W_{\mathsf{t}}) \subset \xi_{\mathsf{t}}$.
(ii) $\|h(x) - g(x)\|_q < \eta$ *for each* $x \in S$.

Proof As g is uniformly continuous, there exists $\delta > 0$ such that

$$\|g(x) - g(x')\|_q < \eta \text{ for each pair } x, x' \in S \text{satisfying } \|x - x'\|_m < \delta. \quad (3.1)$$

By Lemma 3.1 for each $\mathsf{s}^0 \in \mathcal{K}^0$ there exist an open semialgebraic subset U_{s^0} of S with $V_{\mathsf{s}^0} := U_{\mathsf{s}^0} \cap \mathsf{s}^0 \neq \varnothing$ and a Nash retraction $r_{\mathsf{s}^0} : U_{\mathsf{s}^0} \to V_{\mathsf{s}^0}$ such that $\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0 \in \mathcal{K}^0}$ is a covering of S satisfying:

$$\mathrm{Cl}(U_{\mathsf{s}^0}) \cap \mathsf{t} = \varnothing \text{ for each pair } (\mathsf{s}^0, \mathsf{t}) \in \mathcal{K}^0 \times \mathcal{K} \text{ satisfying } \mathsf{s}^0 \cap \mathsf{t} = \varnothing \text{ and} \quad (3.2)$$

$$\sup_{x \in U_{\mathsf{s}^0}} \{\|x - r_{\mathsf{s}^0}(x)\|_m\} < \delta \text{ for each } \mathsf{s}^0 \in \mathcal{K}^0. \quad (3.3)$$

Let $\{\theta_{\mathsf{s}^0} : S \to [0, 1]\}_{\mathsf{s}^0 \in \mathcal{K}^0}$ be an $\mathcal{S}^\nu$ partition of unity subordinated to the finite open semialgebraic covering $\{U_{\mathsf{s}^0}\}_{\mathsf{s}^0 \in \mathcal{K}^0}$ of S. For each $\mathsf{s}^0 \in \mathcal{K}^0$, the semialgebraic map

$$g \circ r_{\mathsf{s}^0} : U_{\mathsf{s}^0} \to V_{\mathsf{s}^0} \subset \mathsf{s}^0 \subset \mathsf{s} \to \xi_{\mathsf{s}}, \; x \mapsto r_{\mathsf{s}^0}(x) \mapsto g(r_{\mathsf{s}^0}(x))$$

is $\mathcal{S}^\nu$, so also the semialgebraic map $H_{\mathsf{s}^0} : S \to \mathbb{R}^q$ defined by

$$H_{\mathsf{s}^0}(x) := \begin{cases} \theta_{\mathsf{s}^0}(x) \cdot g(r_{\mathsf{s}^0}(x)) & \text{if } x \in U_{\mathsf{s}^0}, \\ 0 & \text{if } x \in S \setminus U_{\mathsf{s}^0}, \end{cases}$$

belongs to $\mathcal{S}^\nu(S, \mathbb{R}^q)$. Consider the $\mathcal{S}^\nu$ map $H := \sum_{\mathsf{s}^0 \in \mathcal{K}^0} H_{\mathsf{s}^0} : S \to \mathbb{R}^q$.

Fix $\mathsf{t} \in \mathcal{K}$ and define $W_{\mathsf{t}} := S \setminus \bigcup_{\mathsf{s}^0 \in \mathcal{K}^0, \, \mathsf{s}^0 \cap \mathsf{t} = \varnothing} \mathrm{Cl}(U_{\mathsf{s}^0})$, which is by (3.2) an open semialgebraic neighborhood of t in S. We claim: $H(W_{\mathsf{t}}) \subset \xi_{\mathsf{t}}$.

Pick $x \in W_{\mathsf{t}}$. If $\mathsf{s}^0 \in \mathcal{K}^0$ and $\mathsf{s}^0 \cap \mathsf{t} = \varnothing$, then $\theta_{\mathsf{s}^0}(x) = 0$ because the support of θ_{s^0} is contained in U_{s^0} and $x \notin \mathrm{Cl}(U_{\mathsf{s}^0})$. If $\mathsf{s}^0 \cap \mathsf{t} \neq \varnothing$, then $\mathsf{s}^0 \subset \mathsf{t}$, so we conclude

$$\sum_{\substack{\mathsf{s}^0 \in \mathcal{K}^0, \, \mathsf{s}^0 \subset \mathsf{t}, \\ x \in U_{\mathsf{s}^0}}} \theta_{\mathsf{s}^0}(x) = 1 \text{ and} \quad (3.4)$$

$$H(x) = \sum_{\substack{\mathsf{s}^0 \in \mathcal{K}^0, \, \mathsf{s}^0 \subset \mathsf{t}, \\ x \in U_{\mathsf{s}^0}}} \theta_{\mathsf{s}^0}(x) g(r_{\mathsf{s}^0}(x)). \quad (3.5)$$

If $\mathsf{s}^0 \in \mathcal{K}^0$ satisfies $\mathsf{s}^0 \subset \mathsf{t}$ and $x \in U_{\mathsf{s}^0}$, then $r_{\mathsf{s}^0}(x) \in V_{\mathsf{s}^0} \subset \mathsf{s}^0$, so $g(r_{\mathsf{s}^0}(x)) \in \xi_{\mathsf{t}}$. As ξ_{t} is a convex set and each $g(r_{\mathsf{s}^0}(x)) \in \xi_{\mathsf{t}}$ if $\mathsf{s}^0 \subset \mathsf{t}$ and $x \in U_{\mathsf{s}^0}$, we conclude by means of (3.4) and (3.5) that $H(x) \in \xi_{\mathsf{t}}$. Consequently, $H(W_{\mathsf{t}}) \subset \xi_{\mathsf{t}}$ as claimed.

As $S = \bigcup_{\mathsf{t} \in \mathcal{K}} \mathsf{t} = \bigcup_{\mathsf{t} \in \mathcal{K}} W_{\mathsf{t}}$, we deduce $H(S)$ is contained in $|L|$ and $h : S \to |L|, \; x \mapsto H(x)$ is an $\mathcal{S}^\nu$ map that satisfies property (i).

It remains to show that property (ii) holds for h. Pick $x \in S$ and observe that

$$\sum_{\mathsf{s}^0 \in \mathcal{K}^0,\, x \in U_{\mathsf{s}^0}} \theta_{\mathsf{s}^0}(x) = 1.$$

Using inequalities (3.1) and (3.3) we deduce

$$\begin{aligned} \|h(x) - g(x)\|_q &= \Big\| \sum_{\mathsf{s}^0 \in \mathcal{K}^0,\, x \in U_{\mathsf{s}^0}} \theta_{\mathsf{s}^0}(x)\big(g(r_{\mathsf{s}^0}(x)) - g(x)\big) \Big\|_q \\ &\leq \sum_{\mathsf{s}^0 \in \mathcal{K}^0,\, x \in U_{\mathsf{s}^0}} \theta_{\mathsf{s}^0}(x)\|g(r_{\mathsf{s}^0}(x)) - g(x)\|_q < \eta, \end{aligned}$$

as required. □

3.2 Proof of Theorem 1.4

Let $f : S \to T$ be an $\mathcal{S}^0$ map between compact semialgebraic sets $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$. Suppose T is locally $\mathcal{S}^\nu$ polyhedral for some integer $\nu \geq 1$. By Theorem 2.2 there exist a finite simplicial complex K and a semialgebraic homeomorphism $\Phi : |K| \to S$ such that the restriction $\Phi|_{\sigma^0} : \sigma^0 \to \Phi(\sigma^0)$ is a Nash diffeomorphism for each open simplex σ^0 of K. Define $\mathcal{K} := \{\Phi(\sigma)\}_{\sigma \in K}$ and $\mathcal{K}^0 := \{\Phi(\sigma^0)\}_{\sigma \in K}$. Similarly, Theorem 2.4 implies the existence of a finite simplicial complex L and a semialgebraic homeomorphism $\Psi : |L| \to T$ such that $\Psi|_\xi \in \mathcal{S}^\nu(\xi, T)$ for each $\xi \in L$. Suppose the realization $|L|$ of L belongs to $\mathbb{R}^q$.

Choose an arbitrary $\varepsilon > 0$. We will prove *the existence of a map* $H \in \mathcal{S}^\nu(S, T)$ *such that* $\|H(x) - f(x)\|_n < \varepsilon$ *for each* $x \in S$.

By the uniform continuity of Ψ, there exists $\delta > 0$ such that

$$\|\Psi(z) - \Psi(z')\|_n < \varepsilon \ \text{ for each pair } z, z' \in |L| \text{ satisfying } \|z - z'\|_q < \delta. \quad (3.6)$$

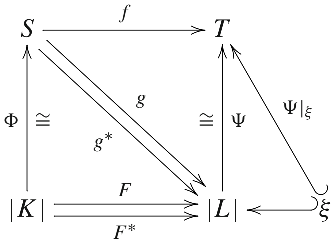
Consider the $\mathcal{S}^0$ map $F := \Psi^{-1} \circ f \circ \Phi : |K| \to |L|$. By Theorem 2.1 we know that after replacing K and L by suitable iterated barycentric subdivisions there exists a simplicial map $F^* : |K| \to |L|$ such that

$$\|F^*(y) - F(y)\|_q < \delta/2 \ \text{ for each } y \in |K|. \quad (3.7)$$

Define the $\mathcal{S}^0$ maps $g := \Psi^{-1} \circ f = F \circ \Phi^{-1} : S \to |L|$ and $g^* := F^* \circ \Phi^{-1} : S \to |L|$. For each $\mathsf{t} \in \mathcal{K}$ the restriction $\Phi^{-1}|_{\mathsf{t}^0} : \mathsf{t}^0 \to \Phi^{-1}(\mathsf{t}^0)$ is a Nash diffeomorphism. Thus, as $F^*|_{\Phi^{-1}(\mathsf{t}^0)}$ is an affine map, $g^*|_{\mathsf{t}^0} \in \mathcal{S}^\nu(\mathsf{t}^0, |L|)$. Moreover, there exists $\xi_{\mathsf{t}} \in L$ such that $g^*(\mathsf{t}) \subset \xi_{\mathsf{t}}$. By (3.7) we have:

$$\|g^*(x) - g(x)\|_q < \delta/2 \ \text{ for each } x \in S. \quad (3.8)$$

The following commutative diagram summarizes the situation we have achieved until the moment.



Now, we approximate g^* by a suitable $\mathcal{S}^\nu$ map h^* between S and $|L|$ that will provide, after composing with Ψ, the required approximating $\mathcal{S}^\nu$ map $H := \Psi \circ h^* : S \to T$.

Indeed, by Lemma 3.2 there exist $h^* \in \mathcal{S}^\nu(S, |L|)$ and for each $\mathsf{t} \in \mathcal{K}$ an open semialgebraic neighborhood W_t of t in S satisfying:

$$h^*(W_\mathsf{t}) \subset \xi_\mathsf{t} \text{ for each } \mathsf{t} \in \mathcal{K} \text{ and} \tag{3.9}$$

$$\|h^*(x) - g^*(x)\|_q < \delta/2 \text{ for each } x \in S. \tag{3.10}$$

We define $H := \Psi \circ h^* : S \to T$ and claim: $H \in \mathcal{S}^\nu(S, T)$.

Recall that $\{W_\mathsf{t}\}_{\mathsf{t} \in \mathcal{K}}$ is an open semialgebraic covering of S. Thanks to (3.9) the restriction $h^*|_{W_\mathsf{t}} : W_\mathsf{t} \to \xi_\mathsf{t}$ is a well-defined $\mathcal{S}^\nu$ map for each $\mathsf{t} \in \mathcal{K}$. In addition, $H|_{W_\mathsf{t}} = \Psi|_{\xi_\mathsf{t}} \circ h^*|_{W_\mathsf{t}}$. As both $\Psi|_{\xi_\mathsf{t}}$ and $h^*|_{W_\mathsf{t}}$ are $\mathcal{S}^\nu$ maps, $H|_{W_\mathsf{t}}$ is also an $\mathcal{S}^\nu$ map. Consequently, $H \in \mathcal{S}^\nu(S, T)$, as claimed.

Next, by (3.8) and (3.10) we have $\|h^*(x) - g(x)\|_q < \delta$ for each $x \in S$. Combining the latter inequality with (3.6), we conclude

$$\|H(x) - f(x)\|_n = \|\Psi(h^*(x)) - \Psi(g(x))\|_n < \varepsilon \quad \text{for each } x \in S,$$

as required. □

3.3 Proof of Theorem 1.3

Let S and $\Phi : |K| \to S$ be as above. Consider an $\mathcal{S}^0$ map $f : S \to \mathbb{R}^n$ and define $T := f(S)$. By Theorem 2.3 there exist a finite simplicial complex L and a semialgebraic homeomorphism $\Psi : |L| \to T$ such that $\Psi \in \mathcal{S}^1(|L|, T)$. Repeating the preceding argument with $\nu = 1$, we obtain that for every $\varepsilon > 0$ there exists $H \in \mathcal{S}^1(S, T)$ such that $\|H(x) - f(x)\|_n < \varepsilon$. □

3.4 Proof of Corollary 1.7

Let K and $P \subset \mathbb{R}^p$ satisfy the conditions in the statement. Apply Lemma 3.2 to $S := P$, $\Phi := \mathrm{id}_P$, $L := K$, $g := \mathrm{id}_P$ and $\eta := 2^{-n}$ for each $n \in \mathbb{N}$. We obtain a map $\iota_n^\nu \in \mathcal{S}^\nu(P, P)$ and an open semialgebraic neighborhood W_σ of σ in P for each $\sigma \in K$ such that:

- $\iota_n^\nu(W_\sigma) \subset \sigma$ for each $\sigma \in K$ and
- $\|x - \iota_n^\nu(x)\|_m < 2^{-n}$ for each $x \in S$.

Thus, the sequence $\{\iota_n^\nu\}_n$ converges to the identity map in $\mathcal{S}^0(P, P)$. Consequently, if $f \in \mathcal{S}^0(P)$, the sequence $\{f \circ \iota_n^\nu\}_{n\in\mathbb{N}}$ converges to f in $\mathcal{S}^0(P)$. In addition, if $f|_\sigma \in \mathcal{S}^\nu(\sigma)$ for each $\sigma \in K$, each function $f \circ \iota_n^\nu$ is an $\mathcal{S}^\nu$ function because so is the restriction $(f \circ \iota_n^\nu)|_{W_\sigma} = f|_\sigma \circ \iota_n^\nu|_{W_\sigma}$ for each $\sigma \in K$, as required. □

4 $\mathcal{S}^\nu$ weak retractions

In this section we construct $\mathcal{S}^\nu$ weak retractions $\rho : W \to X$ of open semialgebraic neighborhoods W of a Nash normal-crossings divisor X of a Nash manifold M (Proposition 4.2). Recall that $\mathcal{S}^\nu$ weak retractions $\rho : W \to X$ are $\mathcal{S}^\nu$ maps whose restrictions to X are close to the identity map on X. Their construction requires a preliminary result concerning compatible Nash retractions (Proposition 4.1), which is of independent interest. The $\mathcal{S}^\nu$ weak retractions will be used in Sect. 5 to prove Theorem 1.8.

4.1 Compatible Nash retractions

Let $M \subset \mathbb{R}^m$ be a d-dimensional Nash manifold and let X be a Nash subset of M. We say that X is a *Nash normal-crossings divisor of M* (see [19]) if:

- for each point $x \in X$ there exists an open semialgebraic neighborhood $U \subset M$ of x and a Nash diffeomorphism $\varphi := (\mathrm{x}_1, \dots, \mathrm{x}_d) : U \to \mathbb{R}^d$ such that $\varphi(x) = 0$ and $\varphi(X \cap U) = \{\mathrm{x}_1 \cdots \mathrm{x}_r = 0\}$ for some $r = 1, \dots, d$, and
- the (Nash) irreducible components of X are Nash manifolds (of dimension $d - 1$).

Assume in the following that X is a Nash normal-crossings divisor of M. For each $\ell \geq 2$ define inductively $\mathrm{Sing}_\ell(X) := \mathrm{Sing}(\mathrm{Sing}_{\ell-1}(X))$ and $\mathrm{Sing}_1(X) := \mathrm{Sing}(X)$. The irreducible components of $\mathrm{Sing}_\ell(X)$ are Nash manifolds for each $\ell \geq 1$ such that $\mathrm{Sing}_\ell(X) \neq \varnothing$. In fact, if $Y_{\ell,1}, \dots, Y_{\ell,s_\ell}$ are the irreducible components of $\mathrm{Sing}_\ell(X)$, then $\mathrm{Sing}_{\ell+1}(X) = \bigcup_{i\neq j}(Y_{\ell,i} \cap Y_{\ell,j})$. For simplicity we write $\mathrm{Sing}_0(X) = X$ and $\mathrm{Sing}_{-1}(X) = M$.

The reader checks directly: *If* $\mathrm{Sing}_\ell(X) \neq \varnothing$, *then* $\dim(\mathrm{Sing}_\ell(X)) = d - \ell - 1$.

We assume that M is irreducible, that is, it is a connected Nash manifold. Let $r \geq 0$ be such that $\mathrm{Sing}_r(X) \neq \varnothing$ but $\mathrm{Sing}_{r+1}(X) = \varnothing$. Let Z be an irreducible component of $\mathrm{Sing}_t(X) \neq \varnothing$ for some $0 \leq t \leq r$. A Nash retraction $\rho : W \to Z$, where $W \subset M$ is an open semialgebraic neighborhood of Z, is *compatible with* X if $\rho(X_i \cap W) = X_i \cap Z$ for each irreducible component X_i of X such that $X_i \cap Z \neq \varnothing$.

Proposition 4.1 (Compatible Nash retractions) *There exist an open semialgebraic neighborhood $W \subset M$ of Z and a Nash retraction $\rho : W \to Z$ that is compatible with X. In addition, $\rho(Y \cap W) = Y \cap Z$ for each irreducible component Y of* $\mathrm{Sing}_\ell(X)$ *where* $\ell \geq 1$.

Proof Fix $\ell \geq 0$ such that $\operatorname{Sing}_\ell(X) \neq \varnothing$ and let Y be one of its irreducible components. As X is a Nash normal-crossings divisor of M, the intersection $Y \cap Z$ is a Nash manifold. If $Y \cap Z \neq \varnothing$, we consider an open tubular semialgebraic neighborhood $N_Y \subset \mathbb{R}^m$ of $Y \cap Z$ endowed with a Nash retraction $\rho_Y : N_Y \to Y \cap Z$, see [5, 8.9.5]. Assume $N_Y \cap N_{Y'} = \varnothing$ if Y' is an irreducible component of $\operatorname{Sing}_\ell(X)$ such that $Y' \cap Z \neq \varnothing$ and $Y \cap Y' \cap Z = \varnothing$. Denote $Y_{\ell,1}, \dots, Y_{\ell,s_\ell}$ the irreducible components of $\operatorname{Sing}_\ell(X)$ for each $-1 \leq \ell \leq r$. In particular, $M = \operatorname{Sing}_{-1}(X) = Y_{-1,1}$.
Claim: *There exist open semialgebraic neighborhoods*

$$W_{-1} \subset W_0 \subset \cdots \subset W_\ell \subset W_{\ell+1} \subset \cdots \subset W_r$$

of Z in M such that $Y_{\ell,j} \cap W_\ell = \varnothing$ if $Y_{\ell,j} \cap Z = \varnothing$ and Nash retractions $\rho_{\ell,j} : Y_{\ell,j} \cap W_\ell \to Y_{\ell,j} \cap Z$ whenever $Y_{\ell,j} \cap Z \neq \varnothing$ satisfying the following compatibility conditions:

$$\rho_{\ell,j_1}|_{Y_{\ell,j_1} \cap Y_{\ell,j_2} \cap W_\ell} = \rho_{\ell,j_2}|_{Y_{\ell,j_1} \cap Y_{\ell,j_2} \cap W_\ell} \quad \text{if } 1 \leq j_1, j_2 \leq s_\ell, \tag{4.1}$$

$$\rho_{\ell,j}|_{Y_{\ell',k} \cap W_\ell} = \rho_{\ell',k}|_{Y_{\ell',k} \cap W_\ell} \quad \text{for each } \ell', k, j \text{ such that } Y_{\ell',k} \subset Y_{\ell,j}. \tag{4.2}$$

Assume the Claim proved for a while. Define $W := W_{-1} \subset M$, which is an open semialgebraic neighborhood of Z, and $\rho := \rho_{-1,1} : W = Y_{-1,1} \cap W \to Z$, which is a Nash retraction such that $\rho(X_i \cap W) = X_i \cap Z$ for each irreducible component X_i of X satisfying $X_i \cap Z \neq \varnothing$, that is, ρ is compatible with X. In addition, $\rho(Y \cap W) = Y \cap Z$ for each irreducible component Y of $\operatorname{Sing}_\ell(X)$ where $\ell \geq 1$. Thus, we are reduced to prove the Claim above by inverse induction on ℓ.
Step 1: If $\ell = r$, then $\operatorname{Sing}_{r+1}(X) = \varnothing$, so $\operatorname{Sing}_r(X)$ is a Nash manifold and its irreducible components $Y_{r,j}$ are its connected components.

Define $W_r := M \setminus \bigcup_{Y_{r,j} \cap Z = \varnothing} Y_{r,j}$. We claim: *if $Y_{r,j} \cap Z \neq \varnothing$, it holds $Y_{r,j} \subset Z$.*

Pick a point $x \in Y_{r,j} \cap Z$. Let $U \subset M$ be an open semialgebraic neighborhood of x endowed with a Nash diffeomorphism $\varphi := (\mathrm{x}_1, \dots, \mathrm{x}_d) : U \to \mathbb{R}^d$ such that $\varphi(x) = 0$ and $\varphi(U \cap X) = \{\mathrm{x}_1 \cdots \mathrm{x}_\alpha = 0\}$ for some $\alpha = 1, \dots, d$. Both $\varphi(U \cap Z)$ and $\varphi(U \cap Y_{r,j})$ are linear coordinate varieties contained in $\{\mathrm{x}_1 \cdots \mathrm{x}_\alpha = 0\}$ that contain the linear coordinate variety $\{\mathrm{x}_1 = 0, \dots, \mathrm{x}_\alpha = 0\}$. As $\operatorname{Sing}_{r+1}(X) = \varnothing$, we deduce $\alpha = r + 1$ and $\varphi(U \cap Y_{r,j}) = \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_\alpha = 0\} \subset \varphi(U \cap Z)$. Therefore, $U \cap Y_{r,j} \subset U \cap Z$ and by the identity principle $Y_{r,j} \subset Z$, because $Y_{r,j}$ is irreducible.

Consequently, we define in this case $\rho_{r,j} := \mathrm{id}_{Y_{r,j}}$.
Step 2: Assume the Claim true for $\ell + 1, \dots, r$ and let us check that it is also true for ℓ. We have $\operatorname{Sing}(\operatorname{Sing}_\ell(X)) = \operatorname{Sing}_{\ell+1}(X)$. The irreducible components of $\operatorname{Sing}_{\ell+1}(X)$ are denoted $Y_{\ell+1,1}, \dots, Y_{\ell+1,s_{\ell+1}}$. By induction hypothesis there exist an open semialgebraic neighborhood $W_{\ell+1} \subset M$ of Z such that $Y_{\ell+1,k} \cap W_{\ell+1} = \varnothing$ if $Y_{\ell+1,k} \cap Z = \varnothing$ and Nash retractions $\rho_{\ell+1,k} : Y_{\ell+1,k} \cap W_{\ell+1} \to Y_{\ell+1,k} \cap Z$ if $Y_{\ell+1,k} \cap Z \neq \varnothing$ satisfying:

$$\rho_{\ell+1,k_1}|_{Y_{\ell+1,k_1} \cap Y_{\ell+1,k_2} \cap W_{\ell+1}} = \rho_{\ell+1,k_2}|_{Y_{\ell+1,k_1} \cap Y_{\ell+1,k_2} \cap W_{\ell+1}} \quad \text{if } 1 \leq k_1, k_2 \leq s_{\ell+1}, \tag{4.3}$$

$$\rho_{\ell+1,k}|_{Y_{\ell',i} \cap W_{\ell+1}} = \rho_{\ell',i}|_{Y_{\ell',i} \cap W_{\ell+1}} \quad \text{for each } \ell', i, k \text{ such that } Y_{\ell',i} \subset Y_{\ell+1,k}. \tag{4.4}$$

Pick $j = 1, \dots, s_\ell$ such that $Y_{\ell,j} \cap Z \neq \varnothing$. If $Y_{\ell,j} \subset Z$, we take $\rho_{\ell,j} := \mathrm{id}_{Y_{\ell,j}}$ and we are done, so let $J_\ell := \{j = 1, \dots, s_\ell : Y_{\ell,j} \cap Z \neq \varnothing$ and $Y_{\ell,j} \not\subset Z\}$ and assume $j \in J_\ell$.

We claim: *$Y_{\ell,j} \cap Z$ is a Nash manifold of dimension $d - \ell^* - 1$ for some $\ell + 1 \leq \ell^* \leq r$ and it is a union of irreducible components of* $\mathrm{Sing}_{\ell^*}(X) \subset \mathrm{Sing}_{\ell+1}(X)$.

As X is a Nash normal-crossings divisor of M, the intersection $Y_{\ell,j} \cap Z$ is a Nash manifold. Pick a point $x \in Y_{\ell,j} \cap Z$. Let $U \subset M$ be an open semialgebraic neighborhood of x endowed with a Nash diffeomorphism $\varphi : U \to \mathbb{R}^d$ such that $\varphi(x) = 0$ and $\varphi(U \cap X) = \{\mathrm{x}_1 \cdots \mathrm{x}_\alpha = 0\}$ as above. Both $\varphi(U \cap Z)$ and $\varphi(U \cap Y_{\ell,j})$ are linear coordinate varieties contained in $\{\mathrm{x}_1 \cdots \mathrm{x}_\alpha = 0\}$ that contain the linear coordinate variety $\{\mathrm{x}_1 = 0, \dots, \mathrm{x}_\alpha = 0\}$.

Assume $\varphi(U \cap Z) = \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_\beta = 0\}$ and $\varphi(U \cap Y_{\ell,j}) = \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_\gamma = 0, \mathrm{x}_{\beta+1} = 0, \dots, \mathrm{x}_{\beta-\gamma+\ell+1} = 0\}$ for some $\gamma \leq \beta \leq \alpha$ and $\beta - \gamma + \ell + 1 \leq \alpha$. Define $\ell^* := \beta - \gamma + \ell$. We have $\gamma < \beta$ because $Y_{\ell,j} \not\subset Z$, so $\ell + 1 \leq \ell^*$. As

$$\varphi(U \cap Y_{\ell,j} \cap Z) = \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_{\ell^*+1} = 0\},$$

we deduce $Y_{\ell,j} \cap Z$ is a union of irreducible components of $\mathrm{Sing}_{\ell^*}(X)$, as claimed.

Let $I_{\ell,j} := \{i = 1, \dots, s_{\ell^*} : Y_{\ell^*,i} \subset Y_{\ell,j} \cap Z\}$ and observe that $Y_{\ell,j} \cap Z = \bigcup_{i \in I_{\ell,j}} Y_{\ell^*,i}$. As $Y_{\ell,j} \cap Z$ is a Nash manifold, the $Y_{\ell^*,i}$ are pairwise disjoint for $i \in I_{\ell,j}$, so the tubular neighborhoods $N_{Y_{\ell^*,i}}$ are pairwise disjoint for $i \in I_{\ell,j}$. Thus, the Nash map

$$\rho^*_{\ell,j} : \bigcup_{i \in I_{\ell,j}} N_{Y_{\ell^*,i}} \to Y_{\ell,j} \cap Z = \bigcup_{i \in I_{\ell,j}} Y_{\ell^*,i}, \ x \mapsto \rho_{Y_{\ell^*,i}}(x) \quad \text{if } x \in N_{Y_{\ell^*,i}} \tag{4.5}$$

is well-defined and satisfies $\rho^*_{\ell,j}|_{Y_{\ell,j} \cap Z} = \mathrm{id}_{Y_{\ell,j} \cap Z}$, so it is a Nash retraction.

Define $K_{\ell,j} := \{k = 1, \dots, s_{\ell+1} : Y_{\ell+1,k} \subset Y_{\ell,j}\}$. Observe that $\bigcup_{k \in K_{\ell,j}} Y_{\ell+1,k}$ is a Nash normal-crossings divisor of the Nash manifold $Y_{\ell,j}$. We claim:

$$\mathrm{Sing}_{\ell+1}(X) \cap Y_{\ell,j} = \bigcup_{k \in K_{\ell,j}} Y_{\ell+1,k}. \tag{4.6}$$

Pick a point $x \in \mathrm{Sing}_{\ell+1}(X) \cap Y_{\ell,j}$. Let $U \subset M$ be an open semialgebraic neighborhood of x endowed with a Nash diffeomorphism $\varphi : U \to \mathbb{R}^d$ such that $\varphi(x) = 0$ and $\varphi(U \cap X) = \{\mathrm{x}_1 \cdots \mathrm{x}_\alpha = 0\}$. Both $\varphi(\mathrm{Sing}_{\ell+1}(X) \cap U)$ and $\varphi(U \cap Y_{\ell,j})$ are linear coordinate varieties contained in $\{\mathrm{x}_1 \cdots \mathrm{x}_\alpha = 0\}$ that contain the linear coordinate variety $\{\mathrm{x}_1 = 0, \dots, \mathrm{x}_\alpha = 0\}$. As $x \in \mathrm{Sing}_{\ell+1}(X)$, it holds $\alpha \geq \ell + 2$. We may assume $\varphi(U \cap Y_{\ell,j}) = \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_{\ell+1} = 0\}$. Observe that $\{\mathrm{x}_1 = 0, \dots, \mathrm{x}_{\ell+2} = 0\} \subset \varphi(\mathrm{Sing}_{\ell+1}(X) \cap U)$, so there exists $k = 1, \dots, s_{\ell+1}$ such that

$$\varphi(Y_{\ell+1,k} \cap U) = \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_{\ell+2} = 0\} \subset \{\mathrm{x}_1 = 0, \dots, \mathrm{x}_{\ell+1} = 0\} = \varphi(U \cap Y_{\ell,j}).$$

As $Y_{\ell+1,k}$ is irreducible, $Y_{\ell+1,k} \subset Y_{\ell,j}$, so $k \in K_{\ell,j}$. Thus, $x \in \bigcup_{k\in K_{\ell,j}} Y_{\ell+1,k}$. The converse inclusion $\bigcup_{k\in K_{\ell,j}} Y_{\ell+1,k} \subset \operatorname{Sing}_{\ell+1}(X) \cap Y_{\ell,j}$ is clear.

Fix $\nu \geq \dim(M) = d$. Combining (4.3), (4.6) and [4, Thm. 1.6 & Prop. 7.6], we deduce the existence of a Nash extension $f_{\ell,j} : Y_{\ell,j} \cap W_{\ell+1} \to \mathbb{R}^n$ such that $f_{\ell,j}|_{Y_{\ell+1,k}\cap W_{\ell+1}} = \rho_{\ell+1,k}|_{Y_{\ell+1,k}\cap W_{\ell+1}}$ for each $k \in K_{\ell,j}$.

As $\rho_{\ell+1,k}|_{Y_{\ell+1,k}\cap Z} = \operatorname{id}_{Y_{\ell+1,k}\cap Z}$ for each $k \in K_{\ell,j}$ and

$$Y_{\ell,j} \cap Z = \bigcup_{i\in I_{\ell,j}} Y_{\ell^*,i} \subset \operatorname{Sing}_{\ell+1}(X) \cap Y_{\ell,j} \cap Z = \bigcup_{k\in K_{\ell,j}} (Y_{\ell+1,k} \cap Z),$$

we deduce $f_{\ell,j}|_{Y_{\ell,j}\cap Z} = \operatorname{id}_{Y_{\ell,j}\cap Z}$. The semialgebraic set $U_{\ell,j} := f_{\ell,j}^{-1}(\bigcup_{i\in I_{\ell,j}} N_{Y_{\ell^*,i}})$ is an open semialgebraic subset of $Y_{\ell,j}$ that contains $Y_{\ell,j} \cap Z = \bigcup_{i\in I_{\ell,j}} Y_{\ell^*,i}$.

The semialgebraic set $C_\ell := \bigcup_{j\in J_\ell}(Y_{\ell,j}\setminus U_{\ell,j})$ is a closed semialgebraic subset of M and $Z \cap C_\ell = \varnothing$, because $Y_{\ell,j} \cap Z \subset U_{\ell,j}$ for each $j \in J_\ell$ and

$$Z \cap C_\ell = \bigcup_{j\in J_\ell} ((Y_{\ell,j} \cap Z)\setminus U_{\ell,j}) = \varnothing.$$

Define

$$W_\ell := W_{\ell+1}\setminus\Big(C_\ell \cup \bigcup_{Y_{\ell,k}\cap Z=\varnothing} Y_{\ell,k}\Big) \subset W_{\ell+1}.$$

Observe that $Z \subset W_\ell$ and $Y_{\ell,j} \cap W_\ell \subset Y_{\ell,j}\setminus C_\ell \subset Y_{\ell,j}\setminus(Y_{\ell,j}\setminus U_{\ell,j}) \subset U_{\ell,j}$ for each $j \in J_\ell$.

Consider the composition

$$\rho_{\ell,j} := \rho^*_{\ell,j} \circ f_{\ell,j}|_{Y_{\ell,j}\cap W_\ell} : Y_{\ell,j} \cap W_\ell \subset U_{\ell,j} \xrightarrow{f_{\ell,j}|_{U_{\ell,j}}} \bigcup_{i\in I_{\ell,j}} N_{Y_{\ell^*,i}} \xrightarrow{\rho^*_{\ell,j}} Y_{\ell,j} \cap Z,$$

where $\rho^*_{\ell,j}$ is the Nash retraction defined in (4.5) for each $j \in J_\ell$.

Note that $\rho_{\ell,j} : Y_{\ell,j}\cap W_\ell \to Y_{\ell,j}\cap Z$ is a Nash retraction such that $\rho_{\ell,j}|_{Y_{\ell+1,k}\cap W_\ell} = \rho_{\ell+1,k}|_{Y_{\ell+1,k}\cap W_\ell}$ for each $k \in K_{\ell,j}$, because

$$f_{\ell,j}|_{Y_{\ell+1,k}\cap W_\ell} = \rho_{\ell+1,k}|_{Y_{\ell+1,k}\cap W_\ell}, \quad f_{\ell,j}(Y_{\ell+1,k} \cap W_\ell) = Y_{\ell+1,k} \cap Z \quad \text{and} \quad \rho^*_{\ell,j}|_{Y_{\ell,j}\cap Z} = \operatorname{id}_{Y_{\ell,j}\cap Z}.$$

Let ℓ', i, j be such that $Y_{\ell',i} \subset Y_{\ell,j}$ and $\ell' \geq \ell + 1$. Then

$$Y_{\ell',i} \subset \operatorname{Sing}_{\ell'}(X) \cap Y_{\ell,j} \subset \operatorname{Sing}_{\ell+1}(X) \cap Y_{\ell,j} = \bigcup_{k\in K_{\ell,j}} Y_{\ell+1,k}.$$

As $Y_{\ell',i}$ is irreducible, there exists $k \in K_{\ell,j}$ such that $Y_{\ell',i} \subset Y_{\ell+1,k}$. We have by (4.4)

$$\rho_{\ell,j}|_{Y_{\ell',i}\cap W_\ell} = (\rho_{\ell,j}|_{Y_{\ell+1,k}\cap W_\ell})|_{Y_{\ell',i}\cap W_\ell} = (\rho_{\ell+1,k}|_{Y_{\ell+1,k}\cap W_\ell})|_{Y_{\ell',i}\cap W_\ell}$$
$$= \rho_{\ell',i}|_{Y_{\ell',i}\cap W_\ell}.$$

If $\ell' = \ell$ and $Y_{\ell,i} \subset Y_{\ell,j}$, we deduce $Y_{\ell,i} = Y_{\ell,j}$, that is, $i = j$ and there is nothing to prove. Consequently, the Nash retractions $\rho_{\ell,j}$ satisfy condition (4.2).

Let $1 \le j_1, j_2 \le s_\ell$ be such that $Y_{\ell,j_1} \cap Y_{\ell,j_2} \cap W_\ell \neq \varnothing$. Proceeding analogously to 4.1, one shows that the intersection $Y_{\ell,j_1} \cap Y_{\ell,j_2}$ is a Nash manifold whose connected components are (pairwise disjoint) irreducible components $Y_{\ell',i}$ of $\operatorname{Sing}_{\ell'}(X)$ for some $\ell' \ge \ell + 1$, which are obviously contained in Y_{ℓ,j_k} for $k = 1, 2$. As $\rho_{\ell,j_k}|_{Y_{\ell',i}\cap W_\ell} = \rho_{\ell',i}|_{Y_{\ell',i}\cap W_\ell}$ for each ℓ', i, ℓ such that $Y_{\ell',i} \subset Y_{\ell,j_k}$, we conclude

$$\rho_{\ell,j_1}|_{Y_{\ell,j_1}\cap Y_{\ell,j_2}\cap W_\ell} = \rho_{\ell,j_2}|_{Y_{\ell,j_1}\cap Y_{\ell,j_2}\cap W_\ell},$$

so the Nash retractions $\rho_{\ell,j}$ satisfy condition (4.1). This finishes the induction step and we are done. □

4.2 $\mathcal{S}^\nu$ weak retractions

Fix a Nash manifold $M \subset \mathbb{R}^m$ of dimension d, a Nash normal-crossings divisor X of M and an integer $\nu \ge 1$. The purpose of this subsection is to prove the existence of $\mathcal{S}^\nu$ weak retractions, that is, $\mathcal{S}^\nu$ maps $\rho : W \to X$, where $W \subset M$ is an open semialgebraic neighborhood of X, whose restrictions to X are arbitrarily close to the identity map id_X on X. Namely,

Proposition 4.2 ($\mathcal{S}^\nu$ weak retractions) *There exist $\mathcal{S}^\nu$ weak retractions $\rho : W \to X$ that are arbitrarily close to the identity map on X. More explicitly, there exists an open semialgebraic neighborhood $W \subset M$ of X with the following property: for each neighborhood $\mathcal{U}$ of id_X in $\mathcal{S}^0(X, X)$, there exists a map $\rho \in \mathcal{S}^\nu(W, X)$ such that $\rho|_X \in \mathcal{U}$.*

Before proving this proposition we state a preliminary result, whose proof is similar to the one of [17, Lem. 4.2] but easier.

Lemma 4.3 ($\mathcal{S}^\nu$ double collar) *Let $Y \subset M$ be a Nash submanifold of dimension $d - 1$ that is closed in M. Let $V \subset M$ be an open semialgebraic neighborhood of Y and $\pi : V \to Y$ an $\mathcal{S}^\nu$ retraction. Let $h : V \to \mathbb{R}$ be an $\mathcal{S}^\nu$ function such that $Y \subset \{h = 0\}$ and $d_x h : T_x M \to \mathbb{R}$ is surjective for each $x \in Y$. Consider the $\mathcal{S}^\nu$ map $\varphi := (\pi, h) : V \to Y \times \mathbb{R}$. Then there exist an open semialgebraic neighborhood $W \subset V$ of Y and a strictly positive $\mathcal{S}^\nu$ function $\varepsilon : Y \to \mathbb{R}$ such that $\varphi(W) = \{(x, t) \in Y \times \mathbb{R} : |t| < \varepsilon(x)\}$ and $\varphi|_W : W \to \varphi(W)$ is an $\mathcal{S}^\nu$ diffeomorphism.*

Proof of Proposition 4.2 Let $X_1, \dots, X_s$ be the irreducible components of X and fix $j = 1, \dots, s$. By [42, II.6.2] there exists a Nash vector subbundle $(\mathscr{E}_j, \theta_j, X_j)$ of the trivial Nash vector bundle $(X_j \times \mathbb{R}^m, \vartheta_j, X_j)$, a (strictly) positive Nash function δ_j on X_j and a Nash diffeomorphism $\chi_j : V_j \to \mathscr{E}_{j,\delta_j}$ from a semialgebraic open neighborhood $V_j \subset M$ of X_j onto

$$\mathscr{E}_{j,\delta_j} := \{(x, y) \in \mathscr{E}_j : \|y\|_m < \delta_j(x)\}$$

such that $\chi_j|_{X_j} = (\mathrm{id}_{X_j}, 0)$. The tuple $(V_j, \chi_j, \mathscr{E}_j, \theta_j, X_j, \delta_j)$ is a Nash tubular neighborhood of X_j in M and the composition $\Theta_j := \theta_j|_{\mathscr{E}_{j,\delta_j}} \circ \chi_j : V_j \to X_j$ is a Nash retraction. As X_j is a Nash hypersurface of M, the Nash subbundle $(\mathscr{E}_j, \theta_j, X_j)$ has rank 1, that is, it is a line bundle. Shrinking V_j if necessary, we may assume in addition $V_j \cap X_k = \varnothing$ whenever $X_j \cap X_k = \varnothing$. We refer the reader to [41, §III.10] for the orientability of vector bundles. We distinguish two cases.

CASE 1. *Assume first that $\mathscr{E}_j$ is a trivial Nash line bundle* (or equivalently, it is an orientable Nash line bundle [41, Def. III.10.4]). Then, $(\mathscr{E}_j, \theta_j, X_j)$ is a Nash diffeomorphic to the trivial bundle $(X_j \times \mathbb{R}, \pi_j, X_j)$. This means that there exists a Nash diffeomorphism

$$\mu_j := (\lambda_j, h_j) : V_j \to \{(x, y) \in X_j \times \mathbb{R} : |y| < \delta_j(x)\},\ z \mapsto (\lambda_j(z), h_j(z))$$

such that $\mu_j|_{X_j} = (\mathrm{id}_{X_j}, 0)$. Thus, $X_j = h_j^{-1}(0)$, $d_x h_j : T_x M \to \mathbb{R}$ is surjective for each $x \in X_j$ and $\ker(d_x h_j) = T_x X_j$. Consequently, if $x \in X_j \cap X_k$ for some $k \neq j$, then $d_x(h_j|_{X_k}) : T_x X_k \to \mathbb{R}$ is also surjective because $\ker(d_x h_j) = T_x X_j$ and X_j and X_k are transverse at x in M.

By Proposition 4.1 there exist an open semialgebraic neighborhood $W_j \subset V_j$ of X_j together with a Nash retraction $\rho_j : W_j \to X_j$ compatible with X. Observe that $W_j \cap X_k \subset V_j \cap X_k = \varnothing$ whenever $X_j \cap X_k = \varnothing$. After shrinking W_j there exists by Lemma 4.3 a strictly positive $\mathcal{S}^\nu$ function $\varepsilon_j : X_j \to \mathbb{R}$ such that the $\mathcal{S}^\nu$ map

$$\phi_j := (\rho_j, h_j) : W_j \to \Omega_j := \{(x, t) \in X_j \times \mathbb{R} : |t| < \varepsilon_j(x)\}$$

is an $\mathcal{S}^\nu$ diffeomorphism. As $\rho_j(W_j \cap X_k) = X_j \cap X_k$, we can also assume by Lemma 4.3 $\phi_j(W_j \cap X_k) = \Omega_j \cap ((X_j \cap X_k) \times \mathbb{R})$ for each $k = 1, \dots, s$ (recall that $d_x(h_j|_{X_k}) : T_x X_k \to \mathbb{R}$ is surjective at each $x \in X_k \cap X_j$ for $k \neq j$). Let $\eta_j : X_j \to \mathbb{R}$ be a strictly positive $\mathcal{S}^\nu$ function such that $\eta_j < \varepsilon_j$ on X_j and let $f : \mathbb{R} \to [0, 1]$ be an $\mathcal{S}^\nu$ function such that $f(t) = 0$ for $|t| \leq \frac{1}{3}$ and $f(t) = 1$ for $|t| \geq \frac{1}{2}$. Consider the $\mathcal{S}^\nu$ map

$$\varphi_j : \Omega_j \to \Omega_j,\ (x, t) \mapsto (x, f(t/\eta_j(x))t).$$

Define $\psi_j := (\phi_j^{-1} \circ \varphi_j \circ \phi_j) : W_j \to W_j$ and observe that $\psi_j(W_j \cap X_k) \subset W_j \cap X_k$ for each $k = 1, \dots, s$. We extend ψ_j by the identity to the whole M and obtain an $\mathcal{S}^\nu$ map $\Psi_j : M \to M$ such that:

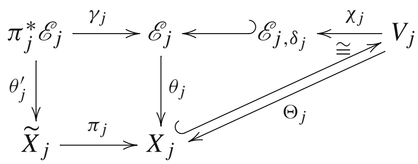
- $\Psi_j(W_j^*) = X_j$ for $W_j^* := \phi_j^{-1}(\{(x, t) \in X_j \times \mathbb{R} : |t| \leq \eta_j(x)/3\})$.
- $\Psi_j(y) = y$ if $y \in M \setminus \phi_j^{-1}(\{(x, t) \in X_j \times \mathbb{R} : |t| < \eta_j(x)/2\})$.

- $\Psi_j(X_k) \subset X_k$ for each $k = 1, \dots, s$.
- Ψ_j is arbitrarily close to the identity map on X if η_j is small enough.

Only the last assertion requires a further comment. As Ψ_j is the identity map on the difference $M \setminus \phi_j^{-1}(\{(x,t) \in X_j \times \mathbb{R} : |t| < \eta_j(x)/2\})$ and ϕ_j is a Nash diffeomorphism (in particular a proper map), it is enough to prove by [42, Rem.II.1.5] that φ_j is close to the identity map on $U := \{(x,t) \in X_j \times \mathbb{R} : |t| < \eta_j(x)/2\}$. If $(x,t) \in U$, we have $|f(t/\eta_j(x)) - 1| \leq 1$ and

$$\|\varphi_j(x,t) - (x,t)\|_{m+1} = |f(t/\eta_j(x)) - 1||t| \leq |t| < \eta_j(x)/2 < \eta_j(x),$$

so Ψ_j is arbitrarily close to the identity map on X if η_j is small enough.
Case 2. *Assume next that $\mathscr{E}_j$ is not orientable*. Let $\pi_j : \widetilde{X}_j \to X_j$ be a Nash double cover such that the pull-back $(\pi_j^*\mathscr{E}_j, \theta'_j, \widetilde{X}_j)$ is an orientable Nash line bundle [41, Cor.III.10.6], hence $\pi_j^*\mathscr{E}_j \cong \widetilde{X}_j \times \mathbb{R}$ is a trivial Nash line bundle over $\widetilde{X}_j$. We can take $\widetilde{X}_j := \{(x,y) \in \mathscr{E}_j : \|y\|_m = 1\}$ (which is the unit sphere bundle in $\mathscr{E}_j$ with respect to the metric induced by that of $X_j \times \mathbb{R}^m$) and $\pi_j := \theta_j|_{\widetilde{X}_j} : \widetilde{X}_j \to X_j$. Consider the Nash morphism of Nash line bundles $\gamma_j : \pi_j^*\mathscr{E}_j \to \mathscr{E}_j$ that makes the following diagram commutative

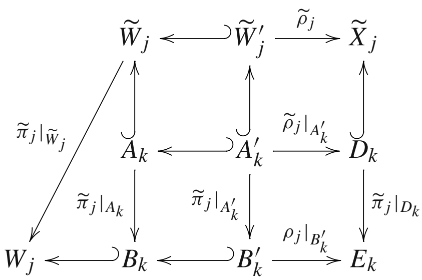


Notice that $\gamma_j : \pi_j^*\mathscr{E}_j \to \mathscr{E}_j$ is a Nash doble cover. Denote $\widetilde{V}_j := \gamma_j^{-1}(\mathscr{E}_{j,\delta_j})$, consider the Nash double cover $\widetilde{\pi}_j := \chi_j^{-1} \circ \gamma_j|_{\widetilde{V}_j} : \widetilde{V}_j \to V_j$ and identify $\widetilde{X}_j$ with $\widetilde{X}_j \times \{0\} \subset \pi_j^*\mathscr{E}_j$. Define $\widetilde{X} := \widetilde{\pi}_j^{-1}(X \cap V_j)$, which is a Nash normal-crossings divisor of the Nash manifold $\widetilde{V}_j$, and $\widetilde{X}_k := \widetilde{\pi}_j^{-1}(X_k \cap V_j)$ for $k = 1, \dots, s$. Each $\widetilde{X}_k$ is a finite union of disjoint irreducible components of $\widetilde{X}$. Denote the regular involution that generate the $\mathbb{Z}_2$ deck transformation group of the Nash double cover $\gamma_j : \pi_j^*\mathscr{E}_j \to \mathscr{E}_j$ with $\sigma_j : \pi_j^*\mathscr{E}_j \to \pi_j^*\mathscr{E}_j$. Recall that σ_j has no fixed points, it inverts the orientation and $\gamma_j \circ \sigma_j = \gamma_j$. Consequently, the same happens with $\sigma_j|_{\widetilde{V}_j} : \widetilde{V}_j \to \widetilde{V}_j$, that is, it has no fixed points, it inverts the orientation and $\widetilde{\pi}_j \circ \sigma_j|_{\widetilde{V}_j} = \widetilde{\pi}_j$. For simplicity we denote $\sigma_j|_{\widetilde{V}_j}$ with σ_j.

By Proposition 4.1 there exists a Nash retraction $\rho_j : W_j \to X_j$ compatible with X where $W_j \subset V_j$ is an open semialgebraic neighborhood of X_j. Define $\widetilde{W}_j := \widetilde{\pi}_j^{-1}(W_j)$, which is invariant under σ_j. We claim: *The Nash retraction $\rho_j : W_j \to X_j$ lifts to a Nash retraction $\widetilde{\rho}_j : \widetilde{W}'_j \to \widetilde{X}_j$ compatible with $\widetilde{X}$ for some open semialgebraic neighborhood $\widetilde{W}'_j \subset \widetilde{W}_j$ of $\widetilde{X}_j$.*

As $\widetilde{\pi}_j|_{\widetilde{W}_j} : \widetilde{W}_j \to W_j$ is a local Nash diffeomorphism, there exists by [5, 9.3.9] a finite open semialgebraic covering $\widetilde{W}_j = \bigcup_{k=1}^{\ell} A_k$ such that $\widetilde{\pi}_j|_{A_k} : A_k \to B_k :=$

$\widetilde{\pi}_j(A_k)$ is a Nash diffeomorphism. If we consider the covering $\{A_k, \sigma_j(A_k) : k = 1, \dots, \ell\}$ of $\widetilde{W}_j$ we may assume in addition $\sigma_j(A_k) = A_{i(k)}$, $B_k = B_{i(k)}$ and $\widetilde{\pi}_j|_{A_k} = \widetilde{\pi}_j|_{A_{i(k)}} \circ \sigma_j|_{A_k}$ for each $k = 1, \dots, \ell$ and for some $i(k) = 1, \dots, \ell$. Observe that $W_j = \bigcup_{k=1}^{\ell} B_k$ is a finite open semialgebraic covering. Let $E_k := B_k \cap X_j$ and observe that $X_j = \bigcup_{k=1}^{\ell} E_k$ is an open semialgebraic covering. Define $B'_k := B_k \cap \rho_j^{-1}(E_k)$, $W'_j := \bigcup_{k=1}^{\ell} B'_k$, $A'_k := A_k \cap \widetilde{\pi}_j^{-1}(B'_k)$, $\widetilde{W}'_j := \bigcup_{k=1}^{\ell} A'_k$ and $D_k := A'_k \cap \widetilde{X}_j$. Observe that $\widetilde{X}_j = \bigcup_{k=1}^{\ell} D_k$, the restriction map $\widetilde{\pi}_j|_{D_k} : D_k \to E_k$ is a Nash diffeomorphism, $E_k = B'_k \cap X_j$, $\rho_j|_{B'_k} : B'_k \to E_k$ is a Nash retraction, $E_k = E_{i(k)}$, $B'_k = B'_{i(k)}$, $\sigma_j(A'_k) = A'_{i(k)}$ and $\sigma_j(\widetilde{W}'_j) = \widetilde{W}'_j$. Consider the commutative diagram



where $\widetilde{\rho}_j : \widetilde{W}'_j \to \widetilde{X}_j,\ x \mapsto (\widetilde{\pi}_j|_{D_k})^{-1}((\rho_j \circ \widetilde{\pi}_j)(x)$ if $x \in A'_k$.

The map $\widetilde{\rho}_j$ is well-defined, it is a Nash retraction (because ρ_j is a Nash retraction) *and $\rho_j \circ \widetilde{\pi}_j = \widetilde{\pi}_j \circ \widetilde{\rho}_j$ on $\widetilde{W}'_j$*. In addition, *$\widetilde{\rho}_j$ is compatible with $\widetilde{X}$* (because ρ_j is compatible with X) *and $\widetilde{\rho}_j \circ \sigma_j = \sigma_j \circ \widetilde{\rho}_j$ on $\widetilde{W}'_j$*.

To prove that $\widetilde{\rho}_j$ is well-defined pick $x \in A'_k \cap A'_i$. Then $\widetilde{\pi}_j(x) \in \widetilde{\pi}_j(A'_k) \cap \widetilde{\pi}_j(A'_i) = B'_k \cap B'_i$, so $\rho_j(\widetilde{\pi}_j(x)) \in E_k \cap E_i = B'_k \cap B'_i \cap X_j$. As $D_k \cap D_i = A'_k \cap A'_i \cap \widetilde{X}_j$ and $\widetilde{\pi}_j|_{D_k \cap D_i} : D_k \cap D_i \to E_k \cap E_i$ is a Nash diffeomorphism, there exists a unique $y \in D_k \cap D_i$ such that $(\rho_j \circ \widetilde{\pi}_j)(x) = \widetilde{\pi}_j(y)$, so $(\widetilde{\pi}_j|_{D_k})^{-1}((\rho_j \circ \widetilde{\pi}_j)(x)) = (\widetilde{\pi}_j|_{D_i})^{-1}((\rho_j \circ \widetilde{\pi}_j)(x))$ and $\widetilde{\rho}_j$ is well-defined.

Let us check that $\widetilde{\rho}_j \circ \sigma_j = \sigma_j \circ \widetilde{\rho}_j$ on $\widetilde{W}'_j$. It is enough to prove this property locally. As $\widetilde{\pi}_j \circ \sigma_j = \widetilde{\pi}_j$ and σ_j is an involution, we have $\widetilde{\pi}_j|_{D_k} \circ \sigma_j|_{D_{i(k)}} = \widetilde{\pi}_j|_{D_{i(k)}}$ and $\sigma_j|_{D_{i(k)}}^{-1} = \sigma_j|_{D_k}$, so $\sigma_j|_{D_k} \circ (\widetilde{\pi}_j|_{D_k})^{-1} = (\widetilde{\pi}_j|_{D_{i(k)}})^{-1}$. Thus,

$$\begin{aligned}
(\sigma_j \circ \widetilde{\rho}_j)|_{A'_k} &= \sigma_j|_{D_k} \circ \left((\widetilde{\pi}_j|_{D_k})^{-1} \circ (\rho_j \circ \widetilde{\pi}_j)|_{A'_k}\right) \\
&= (\widetilde{\pi}_j|_{D_{i(k)}})^{-1} \circ \rho_j|_{B'_k} \circ \widetilde{\pi}_j|_{A'_k} \\
&= (\widetilde{\pi}_j|_{D_{i(k)}})^{-1} \circ \rho_j|_{B'_{i(k)}} \circ \left(\widetilde{\pi}_j|_{A'_{i(k)}} \circ \sigma_j|_{A'_k}\right) \\
&= \left((\widetilde{\pi}_j|_{D_{i(k)}})^{-1} \circ (\rho_j \circ \widetilde{\pi}_j)|_{A'_{i(k)}}\right) \circ \sigma_j|_{A'_k} = (\widetilde{\rho}_j \circ \sigma_j)|_{A'_k}
\end{aligned}$$

for each k. The fact that $\widetilde{\rho}_j$ is compatible with $\widetilde{X}$ is clear by construction.

After shrinking $\widetilde{W}'_j$ we may assume (as in Case 1) that there exists an $\mathcal{S}^{\nu}$ function $\widetilde{h}_j : \widetilde{V}_j \to \mathbb{R}$ such that:

(i) $\widetilde{X}_j \subset \{\widetilde{h}_j = 0\}$.
(ii) $d_x\widetilde{h}_j : T_x\widetilde{V}_j \to \mathbb{R}$ is surjective for each $x \in \widetilde{X}_j$.

Substituting $\widetilde{h}_j$ by $\widetilde{h}'_j := \widetilde{h}_j - \widetilde{h}_j \circ \sigma_j$, we may assume in addition $\widetilde{h}_j \circ \sigma_j = -\widetilde{h}_j$. Let us check that such change keeps properties (i) and (ii). As $\sigma_j(\widetilde{X}_j) = \widetilde{X}_j$, we have $\widetilde{X}_j \subset \{\widetilde{h}'_j = 0\}$ (so property (i) holds for $\widetilde{h}'_j$). Pick $x \in \widetilde{X}_j$. The isomorphism $d_x\sigma_j : T_x\widetilde{V}_j \to T_{\sigma_j(x)}\widetilde{V}_j$ inverts the orientation and in fact if $v \in T_x\widetilde{V}_j \setminus T_x\widetilde{X}_j$ satisfies $d_x\widetilde{h}_j(v) > 0$, then $d_x\sigma_j(v) \in T_{\sigma_j(x)}\widetilde{V}_j$ satisfies $d_{\sigma_j(x)}\widetilde{h}_j(d_x\sigma_j(v)) < 0$. Consequently,

$$d_x\widetilde{h}'_j(v) = d_x\widetilde{h}_j(v) - d_{\sigma_j(x)}\widetilde{h}_j(d_x\sigma_j(v)) > 0$$

and $d_x\widetilde{h}'_j : T_x\widetilde{V}_j \to \mathbb{R}$ is surjective (so property (ii) holds for $\widetilde{h}'_j$).

After shrinking $\widetilde{W}'_j$ there exists by Lemma 4.3 a strictly positive $\mathcal{S}^\nu$ function $\varepsilon_j : \widetilde{X}_j \to \mathbb{R}$ such that the $\mathcal{S}^\nu$ map

$$\widetilde{\phi}_j := (\widetilde{\rho}_j, \widetilde{h}_j) : \widetilde{W}'_j \to \widetilde{\Omega}_j := \{(x,t) \in \widetilde{X}_j \times \mathbb{R} : \ |t| < \varepsilon_j(x)\}$$

is an $\mathcal{S}^\nu$ diffeomorphism and $\varepsilon_j \circ \sigma_j = \varepsilon_j$, so $\sigma_j(\widetilde{W}'_j) = \widetilde{W}'_j$. In addition, as $\widetilde{\phi}_j|_{\widetilde{X}_j} = (\mathrm{id}_{\widetilde{X}_j}, 0)$, we have $\widetilde{X}_j = \widetilde{h}_j^{-1}(0)$, $d_x\widetilde{h}_j : T_x\widetilde{V}_j \to \mathbb{R}$ is surjective for each $x \in \widetilde{X}_j$ and $\ker(d_x\widetilde{h}_j) = T_x\widetilde{X}_j$. Thus, if $x \in \widetilde{X}_j \cap \widetilde{X}_k$ for some $k \neq j$, then $d_x(\widetilde{h}_j|_{\widetilde{X}_k}) : T_x\widetilde{X}_k \to \mathbb{R}$ is also surjective because $\ker(d_x\widetilde{h}_j) = T_x\widetilde{X}_j$ and $\widetilde{X}_j$ and $\widetilde{X}_k$ are transverse at x in $\widetilde{V}_j$. Consequently, by Lemma 4.3 we may assume $\widetilde{\phi}_j(\widetilde{W}'_j \cap \widetilde{X}_k) = \widetilde{\Omega}_j \cap ((\widetilde{X}_j \cap \widetilde{X}_k) \times \mathbb{R})$ for each $k = 1, \dots, s$. Let $\eta_j : \widetilde{X}_j \to \mathbb{R}$ be a strictly positive $\mathcal{S}^\nu$ function such that $\eta_j < \varepsilon_j$ and $\eta_j \circ \sigma_j = \eta_j$ on $\widetilde{X}_j$. Let $f : \mathbb{R} \to [0,1]$ be an even $\mathcal{S}^\nu$ function such that $f(t) = 0$ for $|t| \le \frac{1}{3}$ and $f(t) = 1$ for $|t| \ge \frac{1}{2}$. Consider the $\mathcal{S}^\nu$ map

$$\widetilde{\varphi}_j : \widetilde{\Omega}_j \to \widetilde{\Omega}_j, \ (x,t) \mapsto (x, f(t/\eta_j(x))t).$$

Define $\widetilde{\psi}_j := (\widetilde{\phi}_j^{-1} \circ \widetilde{\varphi}_j \circ \widetilde{\phi}_j) : \widetilde{W}'_j \to \widetilde{W}'_j$ and observe that $\widetilde{\psi}_j(\widetilde{W}'_j \cap \widetilde{X}_k) \subset \widetilde{W}'_j \cap \widetilde{X}_k$ for each $k = 1, \dots, s$. We extend $\widetilde{\psi}_j$ by the identity to the whole $\widetilde{V}_j$ and obtain an $\mathcal{S}^\nu$ map $\widetilde{\Psi}_j : \widetilde{V}_j \to \widetilde{V}_j$ such that:

- $\widetilde{\Psi}_j(\widetilde{W}^*_j) = \widetilde{X}_j$ for $\widetilde{W}^*_j := \widetilde{\phi}_j^{-1}(\{(x,t) \in \widetilde{X}_j \times \mathbb{R} : \ |t| \le \eta_j(x)/3\})$.
- $\widetilde{\Psi}_j(y) = y$ if $y \in \widetilde{V}_j \setminus \widetilde{\phi}_j^{-1}(\{(x,t) \in \widetilde{X}_j \times \mathbb{R} : \ |t| < \eta_j(x)/2\})$.
- $\widetilde{\Psi}_j(\widetilde{X}_k) \subset \widetilde{X}_k$ for each $k = 1, \dots, s$.
- $\widetilde{\Psi}_j$ is arbitrarily close to the identity map on $\widetilde{X}$ if η_j is small enough.
- $\sigma_j \circ \widetilde{\Psi}_j = \widetilde{\Psi}_j \circ \sigma_j$.

Only the latter equality $\sigma_j \circ \widetilde{\Psi}_j = \widetilde{\Psi}_j \circ \sigma_j$ requires some comment. It is enough to prove that $\widetilde{\psi}_j = \sigma_j \circ \widetilde{\psi}_j \circ \sigma_j|_{\widetilde{W}'_j}$. Consider the involution $\tau : \widetilde{\Omega}_j \to \widetilde{\Omega}_j, \ (x,t) \mapsto (\sigma_j(x), -t)$ and observe that $\widetilde{\phi}_j \circ \sigma_j|_{\widetilde{W}'_j} = \tau \circ \widetilde{\phi}_j$ and $\tau \circ \widetilde{\varphi}_j \circ \tau = \widetilde{\varphi}_j$ (because $\eta_j \circ \sigma_j = \eta_j$ and f is an even function). Consequently,

$$\begin{aligned}\sigma_j \circ \widetilde{\psi}_j \circ \sigma_j|_{\widetilde{W}'_j} &= \sigma_j \circ (\widetilde{\phi}_j^{-1} \circ \widetilde{\varphi}_j \circ \widetilde{\phi}_j) \circ \sigma_j|_{\widetilde{W}'_j} \\ &= (\widetilde{\phi}_j \circ \sigma_j|_{\widetilde{W}'_j})^{-1} \circ \widetilde{\varphi}_j \circ (\widetilde{\phi}_j \circ \sigma_j|_{\widetilde{W}'_j}) \\ &= \widetilde{\phi}_j^{-1} \circ \tau \circ \widetilde{\varphi}_j \circ \tau \circ \widetilde{\phi}_j = \widetilde{\phi}_j^{-1} \circ \widetilde{\varphi}_j \circ \widetilde{\phi}_j = \widetilde{\psi}_j.\end{aligned}$$

Thus, $\sigma_j \circ \widetilde{\Psi}_j = \widetilde{\Psi}_j \circ \sigma_j$.

We conclude that there exists an $\mathcal{S}^\nu$ map $\Psi_j : M \to M$ such that:

- $\Psi_j(W_j^*) = X_j$ for $W_j^* := \widetilde{\pi}_j(\widetilde{W}_j^*)$.
- $\Psi_j(y) = y$ for each $y \in M \setminus \widetilde{\pi}_j(\widetilde{W}'_j)$.
- $\Psi_j(X_k) \subset X_k$ for $k = 1, \dots, s$.
- Ψ_j is arbitrarily close to the identity map on X if η_j is small enough.

Final construction. The composition $\rho := \Psi_1 \circ \cdots \circ \Psi_s$ is an $\mathcal{S}^\nu$ map close to the identity map on X and maps the closed semialgebraic neighborhood

$$W := \bigcup_{j=1}^{s} (\Psi_{j+1} \circ \cdots \circ \Psi_s)^{-1}(W_j^*) \subset M$$

of X onto X, where $\Psi_{j+1} \circ \cdots \circ \Psi_s$ denotes id_M if $j = s$. Thus, $\rho : W \to X$ is an $\mathcal{S}^\nu$ weak retraction that is arbitrarily close to the identity map on X, as required. □

5 Proof of Theorem 1.8

We begin this section with a semialgebraic version of a well-known result for continuous maps. For the sake of completeness we include a short proof that only involves standard arguments.

Proposition 5.1 *Let $S \subset \mathbb{R}^m$, $T \subset \mathbb{R}^n$ and $T' \subset \mathbb{R}^p$ be semialgebraic sets, and let $f : T \to T' \subset \mathbb{R}^p$ be a continuous semialgebraic map. Assume S is compact and T is locally compact. Then the map $f_* : \mathcal{S}^0(S, T) \to \mathcal{S}^0(S, T')$, $g \mapsto f \circ g$ is continuous.*

Proof Let $g_0 \in \mathcal{S}^0(S, T)$ and let $\varepsilon > 0$. As $g_0(S) \subset T$ is compact and T is locally compact, there exists a compact neighborhood L of $g_0(S)$ in T. The restriction of f to L is uniformly continuous, so there exists $\delta > 0$ such that $\|f(y) - f(y')\|_p < \varepsilon$ for each $y, y' \in L$ satisfying $\|y - y'\|_n < \delta$. If g is close enough to g_0 in $\mathcal{S}^0(S, T)$, then $g(S) \subset \mathrm{Int}(L)$ and $\|g - g_0\|_n < \delta$ on S. Thus, $\|f \circ g - f \circ g_0\|_p < \varepsilon$ on S, as required. □

We are ready to present the proof of Theorem 1.8, which is inspired by some techniques developed in [3].

Proof of Theorem 1.8 First, by [5, 2.7.5] we may assume T is closed in $\mathbb{R}^n$. Thus, by [13], T is a Nash subset of $\mathbb{R}^n$ (see also [46]). Let $F \in \mathcal{N}(\mathbb{R}^n)$ be a Nash equation of T. By Artin-Mazur description of Nash functions [5, 8.4.4] there exists a non-singular irreducible algebraic subset V of some $\mathbb{R}^{n+p}$ of dimension n, a connected component

M' of V, a Nash diffeomorphism $\sigma : \mathbb{R}^n \to M'$ (whose inverse is the restriction to M' of the projection $\Pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$ onto the first n coordinates) and a polynomial function $G : V \to \mathbb{R}$ such that $G(\sigma(x)) = F(x)$ for each $x \in \mathbb{R}^n$. In particular,

$$\{G = 0\} \cap M' = \{F \circ \Pi|_{M'} = 0\} = \sigma(T).$$

Thus, the Zariski closure $\overline{\sigma(T)}^{\text{zar}}$ of $\sigma(T)$ satisfies $\overline{\sigma(T)}^{\text{zar}} \cap M' = \sigma(T)$. Consequently, we may assume from the beginning that T is a finite union of connected components of an algebraic set $Y \subset \mathbb{R}^n$.

Denote $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ the projection onto the first $n+1$ coordinates. By [3, Lem. 2.2] there exists an irreducible algebraic set $Z \subset \mathbb{R}^{n+2}$ such that $\operatorname{Sing}(Z) = Y \times \{(0,0)\} \subset \{x_{n+1} = 0, x_{n+2} = 0\}$ and the restriction $\psi := \pi|_Z : Z \to \mathbb{R}^{n+1}$ is a semialgebraic homeomorphism.

By Theorem 2.6 there exists a non-singular real algebraic set $Z' \subset \mathbb{R}^q$ and a proper regular map $\phi : Z' \to Z$ such that the restriction

$$\phi|_{Z' \setminus \phi^{-1}(\operatorname{Sing}(Z))} : Z' \setminus \phi^{-1}(\operatorname{Sing}(Z)) \to Z \setminus \operatorname{Sing}(Z)$$

is a Nash diffeomorphism whose inverse map is also regular and $Y' := \phi^{-1}(\operatorname{Sing}(Z))$ is an (algebraic) normal-crossings divisor of Z'. As $T' := \phi^{-1}(T \times \{(0,0)\})$ is an open and closed subset of Y', it is a union of connected components of Y', so it is a Nash normal-crossings divisor of Z'.

Let $f \in \mathcal{S}^0(S, T)$, fix a real number $\varepsilon > 0$ and let $K_0 := f(S)$, which is a compact semialgebraic subset of $\mathbb{R}^n$. Let $V_1 \subset V_2 \subset \mathbb{R}^{n+1}$ be open semialgebraic neighborhoods of $K_0 \times \{0\}$ whose closures $K_i := \operatorname{Cl}(V_i)$ are compact and $K_1 \subset V_2$. As $\psi : Z \to \mathbb{R}^{n+1}$ is a semialgebraic homeomorphism and $\phi : Z' \to Z$ is a proper regular map, we deduce $K_i' := (\psi \circ \phi)^{-1}(K_i)$ is a compact semialgebraic subset of Z' and $K_1' \subset V_2' := (\psi \circ \phi)^{-1}(V_2)$. The restriction $(\psi \circ \phi)|_{K_2'} : K_2' \to \mathbb{R}^{n+1}$ is a uniformly continuous map, so there exists $\eta > 0$ such that if $z, z' \in K_2'$ and $\|z - z'\|_q < \eta$, then

$$\|(\psi \circ \phi)(z) - (\psi \circ \phi)(z')\|_{n+1} < \frac{\varepsilon}{3}.$$

By Proposition 4.2 there exists a small open semialgebraic neighborhood $W \subset Z'$ of T' and an $\mathcal{S}^\nu$ weak retraction $\rho : W \to T'$ that is arbitrarily close to the identity map on T'. Shrinking the Nash manifold W, we may assume in addition $W \cap Y' = T'$, $\|\rho(y) - y\|_q < \eta$ for each $y \in W$ and $\rho(\operatorname{Cl}(W \cap K_1')) \subset V_2' \subset K_2'$. Thus, if $y \in W \cap K_1'$,

$$\|(\psi \circ \phi \circ \rho)(y) - (\psi \circ \phi)(y)\|_{n+1} < \frac{\varepsilon}{3}. \tag{5.1}$$

Denote $V_1' := (\psi \circ \phi)^{-1}(V_1)$. As $\psi \circ \phi$ is proper and $T' \subset W$, the semialgebraic set

$$C := (\psi \circ \phi)(Z' \setminus (W \cap V_1')) = (\psi \circ \phi)(Z' \setminus W) \cup (\psi \circ \phi)(Z' \setminus V_1')$$

is a closed semialgebraic subset of $\mathbb{R}^{n+1}$ that does not meet $(T \times \{0\}) \cap V_1$.

Suppose by contradiction that $y \in C \cap (T \times \{0\}) \cap V_1$. There exists $z \in Z' \setminus (W \cap V_1')$ such that $(\psi \circ \phi)(z) = y$, so $z \in (\psi \circ \phi)^{-1}(T \times \{0\}) \cap (\psi \circ \phi)^{-1}(V_1) = T' \cap V_1' \subset W \cap V_1'$, which is a contradiction.

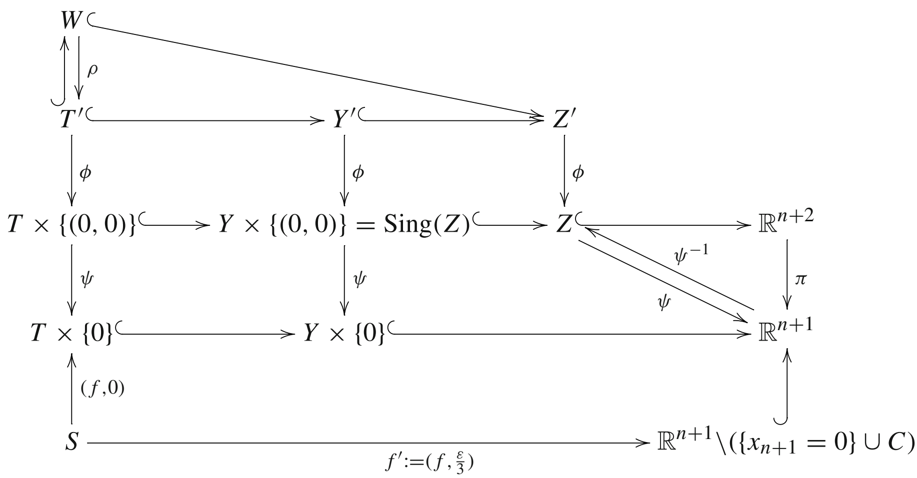
Consider the distance function $\delta : \mathbb{R}^{n+1} \to [0, +\infty),\ y \mapsto \operatorname{dist}(y, C)$ and observe that $\delta|_{(T \times \{0\}) \cap V_1}$ is strictly positive. The homomorphism $\delta_* : \mathcal{S}^0(S, \mathbb{R}^{n+1}) \to \mathcal{S}^0(S, \mathbb{R}),\ g \mapsto \delta \circ g$ is by Proposition 5.1 continuous. The continuous semialgebraic function $\delta \circ (f, 0)$ is strictly positive on S and attains a minimum $\delta_0 > 0$ over S. Let $\varepsilon' \in (0, \varepsilon)$ be such that if $g \in \mathcal{S}^0(S, \mathbb{R}^{n+1})$ and $\|g - (f, 0)\|_{n+1} < \varepsilon'$ on S then

$$|\delta \circ g - \delta \circ (f, 0)| < \frac{\delta_0}{2} \quad \text{on } S,$$

so in particular $\delta \circ g$ is strictly positive and $\operatorname{Im}(g) \subset \mathbb{R}^{n+1} \setminus C$. Consider the continuous semialgebraic function $f' := (f, \frac{\varepsilon'}{3}) : S \to \mathbb{R}^{n+1}$. We have

$$\|(f, 0) - f'\| = \frac{\varepsilon'}{3} < \frac{\varepsilon}{3} \quad \text{on } S \tag{5.2}$$

and $\operatorname{Im}(f') \cap \{x_{n+1} = 0\} = \varnothing$, so $\operatorname{Im}(f') \cap (\{x_{n+1} = 0\} \cup C) = \varnothing$. The following commutative diagram summarizes the situation we have achieved until the moment.



The $\mathcal{S}^0$ map $\psi^{-1} \circ f'$ satisfies $\operatorname{Im}(\psi^{-1} \circ f') \cap (\operatorname{Sing}(Z) \cup \psi^{-1}(C)) = \varnothing$ (recall that $\psi(\operatorname{Sing}(Z)) = Y \times \{0\} \subset \{x_{n+1} = 0\}$), whereas the $\mathcal{S}^0$ map $f'' := (\phi|_{Z' \setminus Y'})^{-1} \circ \psi^{-1} \circ f' : S \to Z'$ is well-defined and $\operatorname{Im}(f'') \cap (\psi \circ \phi)^{-1}(C) = \varnothing$. Thus, $\operatorname{Im}(f'') \cap (Z' \setminus (W \cap V_1')) = \varnothing$, so $\operatorname{Im}(f'') \subset W \cap V_1'$. Write $f'' : S \to W \cap V_1'$ and note that $f' = \psi \circ \phi \circ f''$. By (5.1)

$$\begin{aligned} &\|\psi \circ \phi \circ \rho \circ f'' - f'\|_{n+1} \\ &\quad = \|\psi \circ \phi \circ \rho \circ f'' - \psi \circ \phi \circ f''\|_{n+1} < \frac{\varepsilon}{3} \quad \text{on } S. \end{aligned} \tag{5.3}$$

By [15, Thm. 1] there exists an open semialgebraic neighborhood $U \subset \mathbb{R}^m$ of S such that f'' extends to a continuous semialgebraic map $F'' : U \to W \cap V_1'$ between the Nash manifolds U and $W \cap V_1'$. Let $H_0 : U \to W \cap V_1'$ be a Nash map close to F'' (use [42, Thm.II. 4.1]). The restriction $h_0 := H_0|_S : S \to W \cap V_1'$ is a Nash map close to f''. By Proposition 5.1 the homomorphism

$$(\psi \circ \phi \circ \rho)_* : \mathcal{S}^0(S, W \cap V_1') \to \mathcal{S}^0(S, \mathbb{R}^{n+1}),\ g \mapsto (\psi \circ \phi \circ \rho) \circ g$$

is continuous, so $(h, 0) := \psi \circ \phi \circ \rho \circ h_0 : S \to T \times \{0\}$ is close to $\psi \circ \phi \circ \rho \circ f'' : S \to \mathbb{R}^{n+1}$. Thus, we may assume

$$\|\psi \circ \phi \circ \rho \circ h_0 - \psi \circ \phi \circ \rho \circ f''\|_{n+1} < \frac{\varepsilon}{3}. \tag{5.4}$$

By (5.2), (5.3) and (5.4) we deduce

$$\begin{aligned}\|h - f\|_n = \|(h, 0) - (f, 0)\|_{n+1} &\le \|\psi \circ \phi \circ \rho \circ h_0 - \psi \circ \phi \circ \rho \circ f''\|_{n+1} \\ &+ \|\psi \circ \phi \circ \rho \circ f'' - f'\|_{n+1} + \|f' - (f, 0)\|_{n+1} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

In addition, $(h, 0) = \psi \circ \phi \circ \rho \circ h_0$ is an $\mathcal{S}^\nu$ map, because it is a composition of $\mathcal{S}^\nu$ maps. Thus, we have found an $\mathcal{S}^\nu$ map $h : S \to T$ that is close to f, as required. □

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