Contemporary Mathematics
Volume **697**, 2017
http://dx.doi.org/10.1090/conm/697/14050

Polynomial, regular and Nash images of Euclidean spaces

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Dedicated to Murray Marshall, in memoriam.

Abstract. Let $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a map. We say that f is *polynomial* if its components f_k are polynomials. The map f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials g_k, h_k such that h_k never vanishes on $\mathbb{R}^n$. More generally, the map f is *Nash* if each component f_k is a Nash function, that is, an analytic function whose graph is a *semialgebraic set*. Recall that a subset $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* if it has a description as a finite boolean combination of polynomial equalities and inequalities. By Tarski-Seidenberg's principle the image of a map whose graph is a semialgebraic set is a semialgebraic set. Consequently, the images of polynomial, regular and Nash maps are semialgebraic sets. In 1990 *Oberwolfach reelle algebraische Geometrie* week, the second author proposed a kind of converse problem: *To characterize the semialgebraic sets in* $\mathbb{R}^m$ *that are either polynomial or regular images of some* $\mathbb{R}^n$. In the same period Shiota formulated a conjecture that characterizes Nash images of $\mathbb{R}^n$, that has been recently proved by the first author. In this survey we collect our main contributions to these problems and present some new examples. We have approached our contributions along the last two decades in three directions:

(i) To construct explicitly polynomial and regular maps whose images are the members of large families of semialgebraic sets whose boundaries are piecewise linear.
(ii) To find obstructions to be polynomial/regular images of $\mathbb{R}^n$.
(iii) To prove Shiota's conjecture and some relevant consequences.

1. First examples and obstructions

We will be concerned, except for the last section, with polynomial and regular images of $\mathbb{R}^n$. To ease the presentation we introduce the following two invariants. Given a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$, we define

$$\mathrm{p}(\mathcal{S}) := \inf\{n \geq 1 : \exists\ f : \mathbb{R}^n \to \mathbb{R}^m \text{ polynomial such that } f(\mathbb{R}^n) = \mathcal{S}\},$$
$$\mathrm{r}(\mathcal{S}) := \inf\{n \geq 1 : \exists\ f : \mathbb{R}^n \to \mathbb{R}^m \text{ regular such that } f(\mathbb{R}^n) = \mathcal{S}\}.$$

2010 *Mathematics Subject Classification.* Primary 14P10, 54C30; Secondary 12D15.

Key words and phrases. Semialgebraic set, polynomial map, polynomial image, regular map, regular image, Nash map, Nash image, well-welded set.

The first and second authors were supported by Spanish GR MTM2014-55565 and Grupos UCM 910444. Third author is an external collaborator of Spanish GR MTM2014-55565.

If $\mathcal{S}$ is not representable as a polynomial or regular image of some $\mathbb{R}^n$, then $\mathrm{p}(\mathcal{S}) := +\infty$ and/or $\mathrm{r}(\mathcal{S}) := +\infty$. By [**BCR**, 2.8.8] one has $\dim(\mathcal{S}) \le \mathrm{r}(\mathcal{S}) \le \mathrm{p}(\mathcal{S})$. We will see that these inequalities may be strict, although we do not know if there exists a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ such that $\dim(\mathcal{S}) < \mathrm{r}(\mathcal{S}) < \mathrm{p}(\mathcal{S}) < \infty$. Our feeling is that the previous invariants only take the values $\dim(\mathcal{S})$, $\dim(\mathcal{S})+1$ or $+\infty$.

1.A. Potential applications. There are certain problems in Real Geometry that are reduced, for semialgebraic sets that are either polynomial or regular images of $\mathbb{R}^n$, to their analysis for $\mathbb{R}^n$. This has the advantage that no contour conditions appear and there are more powerful tools. Let us discuss two of them.

1.A.1. *Optimization.* Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is either a polynomial or a regular map and let $\mathcal{S} := f(\mathbb{R}^n)$. Then the optimization of a given regular function $g : \mathcal{S} \to \mathbb{R}$ is equivalent to the optimization of the composition $g \circ f$ on $\mathbb{R}^n$. In this way one can avoid contour conditions (see for instance [**NDS**, **PS**, **Sc**] for relevant tools concerning optimization of polynomial functions on $\mathbb{R}^n$). The weakness of this construction is that complexity of the composition $g \circ f$ is much higher than the one of g. For a regular function $g : \mathcal{S} \to \mathbb{R}$ the problem that we have to solve is

$$\Big(\frac{\partial g}{\partial x_1} \circ f, \cdots, \frac{\partial g}{\partial x_n} \circ f\Big) \cdot \Big(\frac{\partial f_i}{\partial x_j}\Big)_{1 \le i,j \le n} = (0, \dots, 0)$$

where the matrix $J_f := (\frac{\partial f_i}{\partial x_j})_{1 \le i,j \le n}$ depends only on f. We expect that a "good knowledge" of the jacobian matrix J_f will be useful if we have to device optimization problems for many regular functions on $\mathcal{S}$, because, although complexity increases, the matrix J_f is the same for all the maximization problems.

Recall that if $\mathcal{S} \subset \mathbb{R}^n$ is a compact semialgebraic set there is a doubly exponential algorithm (in the number n of variables describing $\mathcal{S}$) triangulating $\mathcal{S}$ (see [**BCR**, Ch.9,§2] and [**HRR**]). Thus, semialgebraic compact sets can be considered as finite simplicial complexes, but we remark that the known algorithm can produce a doubly exponential number of simplices. The algorithms we have developed to show that certain semialgebraic sets with piecewise linear boundary are polynomial or regular images of $\mathbb{R}^n$ are constructive, but the degrees of the involved maps are very high. It will be interesting to estimate the smallest degree for which there is a suitable polynomial or regular map and to compare its complexity with the doubly exponential one for the triangulations of semialgebraic sets.

1.A.2. *Positivstellensätze.* Another classical problem is the algebraic characterization of those regular functions $g : \mathbb{R}^n \to \mathbb{R}$ that are either strictly positive or positive semidefinite on $\mathcal{S}$. In case $\mathcal{S}$ is a basic closed semialgebraic set these problems were solved by Stengle [**St**], see also [**BCR**, 4.4.3]. Note that g is strictly positive (resp. positive semidefinite) on $\mathcal{S}$ if and only if $g \circ f$ is strictly positive (resp. positive semidefinite) on $\mathbb{R}^n$ and both questions are decidable using [**St**]. Thus, this provides an algebraic characterization of positiveness for polynomial and regular functions on semialgebraic sets that are either polynomial or regular images of $\mathbb{R}^n$. These semialgebraic sets need not be either closed, as it happens with the interior of a convex polyhedron, or basic, as it happens with the complement of a convex polyhedron. Thus, this provides a large class of semialgebraic sets (neither closed nor basic), which are out of the scope of the classical Positivstellensätze, for which there is a certificate of positiveness for polynomial and regular functions. Again, the weakness of this strategy arises from the complexity of the composition $g \circ f$.

1.B. Obstruction for the representation. To state some elementary obstructions for the finiteness of $\mathrm{p}(\mathcal{S})$ we recall some terminology. Jelonek called in [**J1**] *parametric semilines* the non-trivial polynomial images of $\mathbb{R}$. By analogy we call *regular semilines* the non-trivial regular images of $\mathbb{R}$. The *exterior boundary* of $\mathcal{S}\subset\mathbb{R}^m$ is defined as $\delta\mathcal{S}:=\mathrm{Cl}(\mathcal{S})\setminus\mathcal{S}$ where $\mathrm{Cl}(\mathcal{S})$ denotes the closure of $\mathcal{S}$ in $\mathbb{R}^m$ in the Euclidean topology of $\mathbb{R}^m$. We denote $\overline{\mathcal{S}}^{\mathrm{zar}}$ the Zariski closure of $\mathcal{S}$ in $\mathbb{R}^m$, that is, the smallest algebraic subset of $\mathbb{R}^m$ that contains $\mathcal{S}$. A continuous map $f:\mathbb{R}^n\to\mathbb{R}^m$ is *proper at a point* $p\in\mathbb{R}^m$ if there exists a compact neighborhood $K\subset\mathbb{R}^m$ of p such that $f^{-1}(K)$ is compact. *The exterior boundary $\delta\mathcal{S}$ of $\mathcal{S}:=f(\mathbb{R}^n)$ is a subset of the set $\mathcal{N}(f)$ of points $p\in\mathbb{R}^n$ at which f is not proper.* Indeed, if $p\in\delta\mathcal{S}\setminus\mathcal{N}(f)$ there exists a compact neighborhood K of p such that the restriction $f^{-1}(K)$ is compact. Thus, $K\cap\mathcal{S}=f(f^{-1}(K))$ is a closed subset of K, so $p\in K\cap\mathrm{Cl}(\mathcal{S})=K\cap\mathcal{S}$, which is a contradiction. Consequently, $\delta\mathcal{S}\subset\mathcal{N}(f)$. The invariant $\mathrm{p}(\mathcal{S})$ was firstly studied in [**FG1**], where some of its properties are stated:

PROPOSITION 1.1. *Suppose that* $\mathrm{p}(\mathcal{S})<+\infty$. *The following conditions hold:*

(i) *$\mathcal{S}$ is connected, pure dimensional and its Zariski closure is irreducible.*
(ii) *For each semialgebraic function $f:\mathbb{R}^m\to\mathbb{R}$ the image $f(\mathcal{S})$ is a singleton or an unbounded set. In particular, $\mathcal{S}$ is unbounded or a singleton.*
(iii) *If $\mathcal{S}\subset\mathbb{R}^2$ is open and $\mathrm{p}(\mathcal{S})=2$, then $\overline{\delta\mathcal{S}}^{\mathrm{zar}}$ is a finite union of parametric semilines.*

The previous result provides us some tools to analyze the following examples:

EXAMPLES 1.2. (i) Let $\mathcal{S}:=\{x^2-zy^2=0\}$ be the Whitney umbrella. Then $\mathrm{p}(\mathcal{S})=+\infty$ because $\mathcal{S}$ is not pure dimensional.

(ii) The union $\mathcal{S}\subset\mathbb{R}^2$ of the lines $\{x=0\}$ and $\{y=0\}$ is a reducible algebraic set, so $\mathrm{p}(\mathcal{S})=+\infty$.

(iii) The exterior boundary of $\mathcal{S}:=\{x^2+y^2>1\}\subset\mathbb{R}^2$ is the unit circumference, which is not a finite union of parametric semilines since it is a bounded set. Consequently, $\mathrm{p}(\mathcal{S})>2$.

(iv) Both $\mathcal{S}:=\{xy<1\}$ and $\mathcal{T}:=\{x>0,\ xy>1\}$ are semialgebraic subsets of $\mathbb{R}^2$ such that $\mathrm{p}(\mathcal{S})>2$ and $\mathrm{p}(\mathcal{T})>2$. This is so because the common Zariski closure of their exterior boundaries is the hyperbola $\{xy=1\}$, which is not a finite union of parametric semilines.

(v) The punctured plane $\mathcal{S}:=\mathbb{R}^2\setminus\{(0,0)\}$ has $\mathrm{p}(\mathcal{S})=2$ because it is the image of the polynomial map $\mathbb{R}^2\to\mathbb{R}^2$, $(x,y)\mapsto(xy-1,(xy-1)x^2-y)$.

(vi) An open half-plane $\mathcal{S}$ has $\mathrm{p}(\mathcal{S})=2$. It is enough to check that the upper half-plane $\mathcal{H}:=\{y>0\}\subset\mathbb{R}^2$ is the image of the polynomial map

$$\mathbb{R}^2\to\mathbb{R}^2,\ (x,y)\mapsto(y(xy-1),(xy-1)^2+x^2).$$

Another elementary but very useful result appears in [**FU3**, Thm.2.1]. Recall that a subset $\mathcal{S}\subset\mathbb{R}^m$ is *basic semialgebraic* if there exist $g_1,\dots,g_r\in\mathbb{R}[\mathtt{x}_1,\dots,\mathtt{x}_m]$ such that $\mathcal{S}:=\{g_1*_1 0,\dots,g_r*_r 0\}$, where each $*_i$ stands for either the symbol $>$ or $\geq$. We fix a basic semialgebraic set $\mathcal{S}\subsetneq\mathbb{R}^m$.

THEOREM 1.3. *Then* $\mathrm{p}(\mathbb{R}^{m+1}\setminus(\mathcal{S}\times\{0\}))=m+1$.

Proof. Write $x := (x_1, \dots, x_m)$. First one check that if $\Lambda \subset \mathbb{R}^m$ and $g \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_m]$ the polynomial maps

$$f_1 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1},\ (x, x_{m+1}) \mapsto (x, x_{m+1}(1 + x_{m+1}^2 g(x))),$$

$$f_2 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1},\ (x, x_{m+1}) \mapsto (x, x_{m+1}(1 + x_{m+1}^2 g(x))(g^2(x) + (x_{m+1} - 1)^2)),$$

satisfy the equalities

$$f_1(\mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})) = \mathbb{R}^{m+1} \setminus ((\Lambda \cap \{g \geq 0\}) \times \{0\}),$$
$$f_2(\mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})) = \mathbb{R}^{m+1} \setminus ((\Lambda \cap \{g > 0\}) \times \{0\}).$$

Let $\mathcal{S} := \{g_1 *_1 0, \dots, g_r *_r 0\}$ be a proper basic semialgebraic subset of $\mathbb{R}^m$ where each $*_i \in \{>, \geq\}$. We proceed by induction on the number r of inequalities needed to describe $\mathcal{S}$. As $\mathcal{S} \subsetneq \mathbb{R}^m$, we may assume $\mathbf{0} \notin \mathcal{S}$. If $r = 1$, we have $\mathcal{S} := \{g_1 *_1 0\}$. Consider the polynomial map

$$f_0 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}, \quad (x_1, \dots, x_{m+1}) \mapsto (x_1 x_{m+1}, \dots, x_m x_{m+1}, x_{m+1}),$$

whose image is $\mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})$, where $\Lambda := \mathbb{R}^m \setminus \{\mathbf{0}\}$. Write $g := g_1$ and choose

$$f := \begin{cases} f_1 & \text{if } \{g *_1 0\} = \{g \geq 0\}, \\ f_2 & \text{if } \{g *_1 0\} = \{g > 0\}. \end{cases}$$

As $\mathbf{0} \notin \mathcal{S}$, we have $\Lambda \cap \mathcal{S} = \mathcal{S}$. Thus, the composition $f \circ f_0$ satisfies

$$(f \circ f_0)(\mathbb{R}^{m+1}) = f(\mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})) = \mathbb{R}^{m+1} \setminus ((\Lambda \cap \mathcal{S}) \times \{0\}) = \mathbb{R}^{m+1} \setminus (\mathcal{S} \times \{0\}).$$

Suppose now that the result holds for each proper basic semialgebraic subset of $\mathbb{R}^m$ described by $r - 1 \geq 1$ inequalities and let $\mathcal{S} := \{g_1 *_1 0, \dots, g_r *_r 0\}$ be a proper basic semialgebraic subset of $\mathbb{R}^m$. We may assume that $\Lambda := \{g_1 *_1 0, \dots, g_{r-1} *_{r-1} 0\} \subsetneq \mathbb{R}^m$ because otherwise $\mathcal{S} = \{g_r *_r 0\}$ can be described using just one inequality and this case has already been studied.

By the inductive hypothesis there exists a polynomial map $f_0 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ such that $f_0(\mathbb{R}^{m+1}) = \mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})$. Denote $g := g_r$ and choose

$$f := \begin{cases} f_1 & \text{if } \{g *_r 0\} = \{g \geq 0\}, \\ f_2 & \text{if } \{g *_r 0\} = \{g > 0\}. \end{cases}$$

Then $f(\mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})) = \mathbb{R}^{m+1} \setminus ((\Lambda \cap \{g_r *_r 0\}) \times \{0\}) = \mathbb{R}^{m+1} \setminus (\mathcal{S} \times \{0\})$ and the composition $f \circ f_0$ provides us

$$(f \circ f_0)(\mathbb{R}^{m+1}) = f(\mathbb{R}^{m+1} \setminus (\Lambda \times \{0\})) = \mathbb{R}^{m+1} \setminus (\mathcal{S} \times \{0\}),$$

as required. □

A useful consequence, alternatively proved in [**FG1**, Thm.1.5], is the following.

Corollary 1.4. *$\mathrm{p}(\mathbb{R}^{m+1} \setminus \mathcal{F}) = m + 1$ for each finite set $\mathcal{F} \subset \mathbb{R}^{m+1}$.*

Proof. As $\dim(\mathbb{R}^{m+1} \setminus \mathcal{F}) = m+1$, it is enough to check that $\mathbb{R}^{m+1} \setminus \mathcal{F}$ is a polynomial image of $\mathbb{R}^{m+1}$. Let us construct a polynomial bijection $f : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ such that $f(\mathcal{F}) \subset \mathbb{R}^m \times \{0\}$ and its inverse is polynomial. Write $\mathcal{F} := \{p_1, \dots, p_r\}$ and $p_i := (p_{i1}, \dots, p_{im}, p_{i,m+1})$. After a linear change of coordinates, we may assume that the projection $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m,\ (x_1, \dots, x_m, x_{m+1}) \to (x_1, \dots, x_m)$ induces a bijection between $\mathcal{F}$ and $\mathcal{S} := \pi(\mathcal{F})$. Let $P \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_m]$ be an interpolating polynomial such that $P(p_{i1}, \dots, p_{im}) = p_{i,m+1}$ for $i = 1, \dots, r$. Define

$$f(\mathtt{x}_1, \dots, \mathtt{x}_{m+1}) := (\mathtt{x}_1, \dots, \mathtt{x}_m, \mathtt{x}_{m+1} - P(\mathtt{x}_1, \dots, \mathtt{x}_m)).$$

Then $f(\mathcal{F}) = \mathcal{S} \times \{0\}$. Write $p'_i := (p_{i1}, \dots, p_{im})$ and

$$g(\mathtt{x}_1, \dots, \mathtt{x}_m) := \prod_{j=1}^{r} \sum_{i=1}^{m} (\mathtt{x}_i - p_{ij})^2 \in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_m].$$

Note that $\mathcal{S}$ is a basic semialgebraic set because $\mathcal{S} = \{g = 0\} = \{g \geq 0, -g \geq 0\}$, so there exists by Thm. 1.3 a polynomial map $h : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ such that

$$h(\mathbb{R}^{m+1}) = \mathbb{R}^{m+1} \setminus (\mathcal{S} \times \{0\}) = \mathbb{R}^{m+1} \setminus f(\mathcal{F}).$$

As the inverse map

$$f^{-1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}, \ (\mathtt{x}_1, \dots, \mathtt{x}_{m+1}) \to (\mathtt{x}_1, \dots, \mathtt{x}_m, \mathtt{x}_{m+1} + P(\mathtt{x}_1, \dots, \mathtt{x}_m))$$

is a polynomial map and $(f^{-1} \circ h)(\mathbb{R}^{m+1}) = \mathbb{R}^{m+1} \setminus \mathcal{F}$, we conclude that $\mathbb{R}^{m+1} \setminus \mathcal{F}$ is a polynomial image of $\mathbb{R}^{m+1}$, as required. □

1.C. The open quadrant. Fix a semialgebraic set $\mathcal{S} \subset \mathbb{R}^2$ of dimension 2. Condition (iii) in Prop. 1.1 shows that the exterior boundary $\delta\mathcal{S}$, which is empty if $\mathcal{S}$ is closed, plays a significant role to determine if $\mathcal{S}$ is a polynomial image of $\mathbb{R}^2$ in case $\mathcal{S}$ is open. This partially explains why it is more difficult to compute $\mathrm{p}(\mathcal{Q})$ than to calculate $\mathrm{p}(\mathcal{Q} \cup \{(0,0)\})$ and $\mathrm{p}(\mathrm{Cl}(\mathcal{Q}))$ for the open first quadrant $\mathcal{Q} := \{x > 0, y > 0\} \subset \mathbb{R}^2$. One can check that $\mathrm{p}(\mathcal{Q} \cup \{(0,0)\}) = \mathrm{p}((\mathrm{Cl}(\mathcal{Q})) = 2$ because $\mathcal{Q} \cup \{(0,0)\} = f(\mathbb{R}^2)$ and $\mathrm{Cl}(\mathcal{Q}) = g(\mathbb{R}^2)$ where

$$f : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x^4y^2, x^2y^4) \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x^2, y^2).$$

1.C.1. *First proof: computational.* Also $\mathrm{p}(\mathcal{Q}) = 2$, but the proof of this result constituted a challenge for many years. The first one appeared in [**FG1**, Thm. 1.7]. Choose $h_1 : \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (h_{11}(x,y), h_{12}(x,y))$ where

$$h_{11} := (1 - \mathtt{x}^3\mathtt{y} + \mathtt{y} - \mathtt{x}\mathtt{y}^2)^2 + (\mathtt{x}^2\mathtt{y})^2 \quad \text{and} \quad h_{12} := (1 - \mathtt{x}\mathtt{y} + \mathtt{x} - \mathtt{x}^4\mathtt{y})^2 + (\mathtt{x}^2\mathtt{y})^2.$$

It holds that $h_1(\mathbb{R}^2) \subset \mathcal{Q} \cup K$ where $K := \{(1,0), (0,1)\}$. To prove that this inclusion is an equality we needed Sturm's algorithm applied to a high degree univariate polynomial. As $h_1^{-1}(K) = \{(-1,0),(0,-1)\}$ is a finite set there exists, by Cor. 1.4, a polynomial map $h_0 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h_0(\mathbb{R}^2) = \mathbb{R}^2 \setminus h_1^{-1}(K)$ and consequently $(h_1 \circ h_0)(\mathbb{R}^2) = \mathcal{Q}$.

It is worthwhile mentioning that the map $h := h_1 \circ h_0$ proposed in [**FG1**] has total degree 56 (the sum of the degrees of its two components) and its total number of monomials is 168. The reading of [**FG1**] can become rather disappointing because a part of the proof of the main result requires computer assistance and it is a tedious task to verify that all the involved computations are correct.

1.C.2. *Second proof: algebraic.* We have wondered whether a less technical and less demanding approach was possible. It was proved in [**FU1**] without the aid of computers that $\mathcal{Q}$ is the image of the polynomial map $g := H \circ G \circ F$ where

$$\begin{aligned} &F : \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto ((xy-1)^2 + x^2, (xy-1)^2 + y^2), \\ &G : \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (x, y(xy-2)^2 + x(xy-1)^2), \\ &H : \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (x(xy-2)^2 + \tfrac{1}{2}xy^2, y). \end{aligned} \tag{1.1}$$

The polynomial maps F, G and H have small degree, but the total degree of its composition g is 72 and its total number of monomials is 350.

1.C.3. *Third proof: topological.* Looking for a more conceptual proof with less complexity we have found in [**FGU2**] a third map $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with $f_2(\mathbb{R}^2) = \mathcal{Q}$. We consider the polynomial map $f := (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ where $f_1, f_2 \in \mathbb{R}[\mathtt{x}, \mathtt{y}]$ are the polynomials

$$f_1 := (\mathtt{x}^2\mathtt{y}^4 + \mathtt{x}^4\mathtt{y}^2 - \mathtt{y}^2 - 1)^2 + \mathtt{x}^6\mathtt{y}^4 \quad \text{and} \quad f_2 := (\mathtt{x}^6\mathtt{y}^2 + \mathtt{x}^2\mathtt{y}^2 - \mathtt{x}^2 - 1)^2 + \mathtt{x}^6\mathtt{y}^4.$$

The polynomial map f has total degree 28 and its total number of monomials is 22. The proof we have developed in [**FGU2**] involves arguments of algebraic topology.

The equality $\mathrm{p}(\mathcal{Q}) = 2$ implies the following:

Corollary 1.5. (i) *Let $\mathcal{S} := \mathbb{R}^2 \setminus \{x \geq 0,\ y \leq 0\}$. Then $\mathrm{p}(\mathcal{S}) = 2$.*

(ii) *Let $m \geq 2$ and $\mathcal{Q}_{r,m} := \{h_1 > 0, \dots, h_r > 0\} \subset \mathbb{R}^m$ where $h_1, \dots, h_r$ are independent linear forms on $\mathbb{R}^m$. Then $\mathrm{p}(\mathcal{S}) = m$.*

Proof. (i) It is enough to observe that $\mathcal{S} = g(\mathcal{Q})$ where

$$g : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \equiv z := (x + iy) \mapsto z^3 \equiv (x^3 - 3xy^2, 3x^2y - y^3)$$

is written in complex coordinates.

(ii) We may assume $h_1 := \mathtt{x}_1, \dots, h_r := \mathtt{x}_r$. Observe that:

(1) $\mathrm{p}(\mathcal{H}) = \mathrm{p}(\mathcal{Q}) = 2$ where $\mathcal{H} := \{x_1 > 0\}$ and $\mathcal{Q} := \{x_1 > 0,\ x_2 > 0\}$ are the open half-plane and the open quadrant of $\mathbb{R}^2$ (see 1.2 (vi) and 1.C).

(2) The set $\mathcal{Q}_3 := \{x_1 > 0,\ x_2 > 0,\ x_3 > 0\} \subset \mathbb{R}^3$ is a polynomial image of $\mathbb{R}^3$. To show this, we proceed as follows: let $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be polynomial maps whose respective images are $\mathcal{H}$ and $\mathcal{Q}$. Now consider the polynomial maps:

$$(h_1, \mathrm{id}_{\mathbb{R}}) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R} \quad \text{and} \quad (\mathrm{id}_{\mathbb{R}}, h_2) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2.$$

Then $\mathcal{Q}_3$ is the image of the polynomial map $h := (\mathrm{id}_{\mathbb{R}}, h_2) \circ (h_1, \mathrm{id}_{\mathbb{R}}) : \mathbb{R}^3 \to \mathbb{R}^3$. From the previous facts it follows statement (ii). □

Examples 1.6. We are ready to show the different behavior of the invariants p and r over some examples.

(i) The disc $\mathcal{D} := \{x^2 + y^2 < 1\} \subset \mathbb{R}^2$ satisfies $\mathrm{p}(\mathcal{D}) = +\infty$ because it is a bounded set, but $\mathrm{r}(\mathcal{D}) = 2$.

The upper half-plane $\mathcal{H} := \{y > 0\}$ is by Ex. 1.2 (vi) the image of a polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Consider the Möbius transformation $g : \mathcal{H} \to \mathbb{C} \equiv \mathbb{R}^2$, $(x, y) \equiv z := x + iy \mapsto \frac{z-i}{z+i}$ that maps $\mathcal{H}$ onto $\mathcal{D}$. Thus $g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$ is a regular map, whose image is $\mathcal{D}$.

(ii) The open half-band $\mathbb{B} := \{x > 0, -1 < y < 1\}$ has $\mathrm{p}(\mathbb{B}) = +\infty$ by Prop. 1.1 (ii). However, $\mathrm{r}(\mathbb{B}) = 2$ because $\mathbb{B} = h(\mathcal{Q}_1)$, where $\mathcal{Q}_1$ is the open quadrant $\{x - y > 0, x + y > 0\}$, which has $\mathrm{r}(\mathcal{Q}_1) = 2$ by 1.C, and h is the regular map $h : \mathcal{Q}_1 \to \mathbb{R}^2$, $(x, y) \mapsto \left(x, \frac{y}{x}\right)$.

2. One dimensional polynomial and regular images of $\mathbb{R}^n$

In [**Fe**] the first author obtained a full geometric characterization of the 1-dimensional semialgebraic sets $\mathcal{S}$ such that either $\mathrm{p}(\mathcal{S})$ or $\mathrm{r}(\mathcal{S})$ is finite. In addition, he computed the exact values for these invariants. Let us explain these results.

Lemma 2.1. *Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map with $\dim(f(\mathbb{R}^n)) = 1$. Then there exist polynomial maps $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}^m$ such that $f = h \circ g$.*

Proof. Let $\mathbb{F} := \mathbb{R}(f_1, \ldots, f_m)$ and observe that $\operatorname{tr.deg}(\mathbb{F}/\mathbb{R}) = \dim(f(\mathbb{R}^n)) = 1$, so we may assume f_1 is not constant. By [**Sch2**, Lem. 2] (see also [**Sch1**, Lem. 2, pag. 710-711]), $\mathbb{F} = \mathbb{R}(g)$ for some $g \in \mathbb{R}[\mathtt{x}_1, \ldots, \mathtt{x}_n]$. We seek $h_1, \ldots, h_m \in \mathbb{R}[\mathtt{t}]$ such that $h_i(g) = f_i$. As $f_i \in \mathbb{R}(g)$, we have $f_i = \frac{p_i(g)}{q_i(g)}$ for some relatively prime polynomials $p_i, q_i \in \mathbb{R}[\mathtt{t}]$. By Bezout's lemma, $1 = p_i u_i + q_i v_i$ for some $u_i, v_i \in \mathbb{R}[\mathtt{t}]$. Substituting $\mathtt{t}$ by g we get

$$1 = p_i(g)u_i(g) + q_i(g)v_i(g) = q_i(g)f_iu_i(g) + q_i(g)v_i(g) = q_i(g)\big(f_iu_i(g) + v_i(g)\big),$$

so $q_i(g)$ is a nonzero constant. Thus, the polynomials $h_i(\mathtt{t}) := \frac{p_i(\mathtt{t})}{q_i(g)}$ satisfy $h_i(g) = f_i$. The polynomial maps $h := (h_1, \ldots, h_m) : \mathbb{R} \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}$ satisfy $f = h \circ g$, as required. □

Corollary 2.2. (i) *Let* $\mathcal{S} \subset \mathbb{R}^m$ *be a* 1*-dimensional polynomial image of* $\mathbb{R}^n$*. Then* $\operatorname{Cl}(\mathcal{S})$ *is a parametric semiline and either coincides with* $\mathcal{S}$ *or* $\operatorname{Cl}(\mathcal{S}) \setminus \mathcal{S}$ *is a singleton.*
(ii) *If* $\mathrm{p}(\mathcal{S}) < +\infty$ *is finite, then* $\mathrm{p}(\mathcal{S}) \leq 2$*. In addition,* $\mathrm{p}(\mathcal{S}) = 1$ *if and only if* $\mathcal{S}$ *is closed in* $\mathbb{R}^m$*.*

Proof. (i) By Lem. 2.1, there exist polynomial maps $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}^m$ such that $\mathcal{S} = h(g(\mathbb{R}^n))$. By Prop. 1.1 $g(\mathbb{R}^n)$ is an unbounded interval of $\mathbb{R}$. Without loss of generality, we may assume that $g(\mathbb{R}^n)$ is one of the following sets: $\mathbb{R}$, $[0, +\infty)$ or $(0, +\infty)$. If $\mathcal{S}$ is closed, $h^{-1}(\mathcal{S})$ is a closed subset of $\mathbb{R}$ that contains $g(\mathbb{R}^n)$, so $\mathcal{S} = h(\operatorname{Cl}(g(\mathbb{R}^n)))$. Thus, $\mathcal{S}$ is either $h(\mathbb{R})$ or $h([0, +\infty))$. As $[0, +\infty) = f_0(\mathbb{R})$ where $f_0 : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^2$, we conclude that $\mathcal{S}$ is a parametric semiline. Next, if $\mathcal{S}$ is not closed, $g(\mathbb{R}^n) = (0, +\infty)$ and $\operatorname{Cl}(\mathcal{S}) = \mathcal{S} \cup \{h(0)\}$ is a parametric semiline.

(ii) We have shown that if $\mathcal{S}$ is closed in $\mathbb{R}^m$ and $\mathrm{p}(\mathcal{S})$ is finite, then $\mathrm{p}(\mathcal{S}) = 1$. If $\mathcal{S}$ is not closed, $\mathcal{S} = h(g(\mathbb{R}^n)) = h((0, +\infty))$. In addition, $(0, +\infty) = f_0(\mathbb{R}^2)$ where $f_0 : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \to (xy - 1)^2 + y^2$, so $\mathcal{S} = (h \circ f_0)(\mathbb{R}^2)$. This proves that $\mathrm{p}(\mathcal{S}) \leq 2$ and in fact this is an equality because each non constant polynomial map $h : \mathbb{R} \to \mathbb{R}^m$ is proper [**GU**], so its image is a closed subset of $\mathbb{R}^m$. □

As $1 \leq \mathrm{r}(\mathcal{S}) \leq \mathrm{p}(\mathcal{S}) \leq +\infty$ for every $\mathcal{S} \subset \mathbb{R}^m$, there are only three possible values in the 1-dimensional case for both invariants p and r, which are $1, 2$ or $+\infty$. All possibilities satisfying the above restriction are attained except for the pair $\mathrm{r}(\mathcal{S}) = 1$ and $\mathrm{p}(\mathcal{S}) = 2$, which is not attainable. The geometric characterization of the 1-dimensional semialgebraic sets $\mathcal{S}$ such that either $\mathrm{r}(\mathcal{S})$ or $\mathrm{p}(\mathcal{S})$ is finite involves the concept of irreducibility of a semialgebraic set introduced in [**FG3**, 3.1] and the projective Zariski closure of either $\mathcal{S}$ or its complexification.

Definitions 2.3. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set. A function $f : \mathcal{S} \to \mathbb{R}$ is *Nash* if there exist an open semialgebraic neighborhood $\mathcal{V}$ of $\mathcal{S}$ in $\mathbb{R}^m$ and a Nash function $F : \mathcal{V} \to \mathbb{R}$ such that $F|_{\mathcal{V}} = f$. A *Nash function on an open semialgebraic set* $\mathcal{U} \subset \mathbb{R}^m$ is a smooth function on $\mathcal{U}$ whose graph is a semialgebraic set. The semialgebraic set $\mathcal{S}$ is *irreducible* if the ring $\mathcal{N}(\mathcal{S})$ of Nash functions on $\mathcal{S}$ is an integral domain. By [**FG3**, 3.1(iv)] regular images of $\mathbb{R}^n$ are irreducible semialgebraic sets.

Write $\mathbb{K}$ to refer indistinctly to $\mathbb{R}$ or $\mathbb{C}$ and denote the hyperplane of infinity of the projective space $\mathbb{KP}^m$ with $\mathsf{H}_\infty(\mathbb{K}) := \{x_0 = 0\}$. As usual, we manipulate the homeomorphism

$$\mathbb{K}^m \to \mathbb{KP}^m \setminus \mathsf{H}_\infty(\mathbb{K}) = \{x_0 = 1\},\ (x_1, \ldots, x_m) \mapsto [1 : x_1 : \cdots : x_m]$$

as an identity. For each $n \geq 1$ denote the complex conjugation with

$$\sigma_n : \mathbb{CP}^n \to \mathbb{CP}^n,\ z := [z_0 : z_1 : \cdots : z_n] \mapsto \overline{z} := [\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n}],$$

whose set of fixed points is $\mathbb{RP}^n$. A subset $A \subset \mathbb{CP}^n$ is called *invariant* if $\sigma_n(A) = A$.

For each $\mathcal{S} \subset \mathbb{R}^m \subset \mathbb{RP}^m \subset \mathbb{CP}^m$ we denote its Zariski closure in $\mathbb{KP}^m$ with $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{KP}^m}(\mathcal{S})$. The intersection $\overline{\mathcal{S}}^{\mathrm{zar}} := \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(\mathcal{S}) \cap \mathbb{R}^m$ is the *Zariski closure of* $\mathcal{S}$ *in* $\mathbb{R}^m$. A *complex rational curve* is the image of $\mathbb{CP}^1$ under a birational (and hence regular) map whereas a *real rational curve* is a real projective irreducible algebraic curve $\mathcal{C}$ such that the subset $\mathcal{C}_{(1)}$ of points in $\mathcal{C}$ with local dimension 1 (see [**BCR**, 2.8.12]) is the image of $\mathbb{RP}^1$ under a birational (and hence regular) map. We are ready to characterize the 1-dimensional semialgebraic sets $\mathcal{S}$ with finite $\mathrm{p}(\mathcal{S})$.

THEOREM 2.4. *Let* $\mathcal{S} \subset \mathbb{R}^m$ *be a* 1*-dimensional semialgebraic subset. Then* $\mathrm{p}(\mathcal{S})$ *is finite if and only if* $\mathcal{S}$ *is irreducible, unbounded,* $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S})$ *is an invariant rational curve such that* $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{C}) = \{p\}$ *is a singleton and the germ* $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{CP}^m}(S)_p$ *is analytically irreducible.*

The main differences between the invariants $\mathrm{p}(\mathcal{S})$ and $\mathrm{r}(\mathcal{S})$ in case $\dim(\mathcal{S}) = 1$ arise from the following elementary but enlightening examples.

EXAMPLES 2.5. (i) The circumference $\mathbb{S}^1$ and the real projective line $\mathbb{RP}^1$ are regular images of $\mathbb{R}$, but they are not polynomial images of $\mathbb{R}^n$ for any $n \in \mathbb{N}$. As $\mathbb{RP}^1$ is the image of $\mathbb{S}^1$ via the canonical projection $\pi : \mathbb{S}^1 \to \mathbb{RP}^1$, it is enough to prove that $\mathbb{S}^1$ is a regular image of $\mathbb{R}$. We may choose the regular map

$$f : \mathbb{R} \to \mathbb{S}^1,\ t \mapsto \Big(\Big(\frac{t^2-1}{t^2+1}\Big)^2 - \Big(\frac{2t}{t^2+1}\Big)^2,\ 2\Big(\frac{t^2-1}{t^2+1}\Big)\Big(\frac{2t}{t^2+1}\Big)\Big).$$

The previous map is the composition of the inverse of the stereographic projection of $\mathbb{S}^1$ with respect to $(1,0)$ with the polynomial map

$$g : \mathbb{C} \equiv \mathbb{R}^2 \to \mathbb{C} \equiv \mathbb{R}^2,\ z := x + \sqrt{-1}y \equiv (x,y) \mapsto z^2 \equiv (x^2 - y^2,\ 2xy).$$

(ii) The intervals $[0,1] = h_1(\mathbb{R})$ and $(0,1] = h_2(\mathbb{R})$ where

$$h_1 : \mathbb{R} \to \mathbb{R},\ t \mapsto \frac{t}{1+t^2} + \frac{1}{2} \quad \text{and} \quad h_2 : \mathbb{R} \to \mathbb{R},\ t \mapsto \frac{1}{1+t^2},$$

whereas the interval $(0,1) = h_3(\mathbb{R}^2)$ where

$$h_3 : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto \frac{(xy-1)^2 + x^2}{1 + (xy-1)^2 + x^2}.$$

Thus, we have $\mathrm{p}(I) = \infty$ and $\mathrm{r}(I) = 1$ for each bounded interval $I \subset \mathbb{R}$.

The counterpart of Cor. 2.2 and Thm. 2.4 in the regular setting consists of the full geometric characterization of the 1-dimensional regular images of Euclidean spaces and the description of those semialgebraic sets $\mathcal{S}$ with $\mathrm{r}(\mathcal{S}) = 1$.

THEOREM 2.6. *Let* $\mathcal{S} \subset \mathbb{R}^m$ *be a* 1*-dimensional semialgebraic set. Then,*

(i) $\mathrm{r}(\mathcal{S})$ *is finite if and only if* $\mathcal{S}$ *is irreducible and* $\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}(\mathcal{S})$ *is a rational curve.*

(ii) *Assume* $\mathrm{r}(\mathcal{S})$ *is finite. Then* $\mathrm{r}(\mathcal{S}) = 1$ *if and only if either* $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) = \mathcal{S}$ *or* $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \setminus \mathcal{S} = \{p\}$ *is a singleton and the analytic closure of the germ* $\mathcal{S}_p$ *is irreducible. In the remaining cases* $\mathrm{r}(\mathcal{S}) = 2$.

COROLLARY 2.7. *There is no* 1*-dimensional semialgebraic set* $\mathcal{S} \subset \mathbb{R}^m$ *with* $\mathrm{p}(\mathcal{S}) = 2$ *and* $\mathrm{r}(\mathcal{S}) = 1$.

PROOF. Suppose that there exists a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ with $\dim(\mathcal{S}) = \mathrm{r}(\mathcal{S}) = 1$ and $\mathrm{p}(\mathcal{S}) = 2$. By Cor. 2.2 (ii) and Thm. 2.4 the semialgebraic set $\mathcal{S}$ is unbounded and not closed in $\mathbb{R}^m$. Thus, $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \setminus \mathcal{S}$ has at least two elements: one point in $\mathsf{H}_\infty(\mathbb{R})$ because $\mathcal{S}$ is unbounded and another one in $\mathbb{R}^m$ since $\mathcal{S}$ is not closed in $\mathbb{R}^m$. This contradicts Thm. 2.6 (ii) because the difference $\mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \setminus \mathcal{S}$ has to be either empty or a singleton. □

The following table illustrates the situation.

$\mathcal{S}$	$\mathbb{R}$ or $[0, +\infty)$	$-$	$[0, 1)$	$(0, +\infty)$	$(0, 1)$	Any non-rational curve
$\mathrm{r}(\mathcal{S})$	1	1	1	2	2	$+\infty$
$\mathrm{p}(\mathcal{S})$	1	2	$+\infty$	2	$+\infty$	$+\infty$

This Table admits two different readings. The obvious one provides the value of $\mathrm{p}(\mathcal{S})$ and $\mathrm{r}(\mathcal{S})$ for some significant examples. A more subtle reading provides the values of $\mathrm{p}(\mathcal{S})$ and $\mathrm{r}(\mathcal{S})$ for a 1-dimensional semialgebraic set $\mathcal{S}$ according to the fact that $\mathcal{S}$ admits an either polynomial or regular parametrization whose domain is one of the above mentioned subsets of $\mathbb{R}$ or if $\mathrm{Cl}_{\mathbb{CP}^m}(\mathcal{S})$ is non-rational.

EXAMPLE 2.8. Consider the semialgebraic sets

$$\mathcal{S} := \{y^2 - x(x^2 - 1) = 0,\ x > 1\} \subset \mathbb{R}^2 \quad \text{and} \quad \mathcal{T} := \mathrm{Cl}(\mathcal{S}).$$

By Thm. 2.6 $\mathrm{r}(\mathcal{S}) = \mathrm{r}(\mathcal{T}) = +\infty$ because

$$\mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^2}(\mathcal{S}) = \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^2}(\mathcal{T}) = \{x_2^2 x_0 - x_1(x_1^2 - x_0^2) = 0\}$$

is an elliptic curve.

3. Examples and obstructions for dimension two

We only know a full answer to the problem of deciding the finiteness of $\mathrm{p}(\mathcal{S})$ and $\mathrm{r}(\mathcal{S})$ and computing their precise values for the 1-dimensional case. In what follows we implicitly assume that $\dim(\mathcal{S}) \geq 2$. We describe in this section some results concerning 2-dimensional convex semialgebraic sets whose boundary is piecewise linear. They constitute a natural precedent to the main results of Section 5. Recall the following improvement of Prop. 1.1 (iii) proposed in [**FG2**, 3.8].

PROPOSITION 3.1. *Let $\mathcal{S} \subset \mathbb{R}^2$ be a 2-dimensional semialgebraic set such that $\mathrm{p}(\mathcal{S}) = 2$ and let $\delta\mathcal{S}$ be its exterior boundary. Then $\delta\mathcal{S}$ is either empty or there exist a finite set $\mathcal{F} \subset \mathbb{R}^2$ and a finite family of parametric semilines $\mathcal{L}_1, \ldots, \mathcal{L}_r$ such that*

$$\delta\mathcal{S} \subset \mathcal{F} \cup \bigcup_{i=1}^{r} \mathcal{L}_i \subset \mathrm{Cl}(\mathcal{S}) \cap \overline{\delta\mathcal{S}}^{\mathrm{zar}}.$$

PROOF. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map such that $f(\mathbb{R}^2) = \mathcal{S}$. The set $\mathcal{N}(f) \subset \mathrm{Cl}(\mathcal{S})$ of points in $\mathbb{R}^m$ at which f is not proper is by [**J2**, 4.2] either empty or a finite union of parametric semilines $\mathcal{L}_1, \ldots, \mathcal{L}_s$. As $\delta\mathcal{S} \subset \mathcal{N}(f)$, we can assume $\mathcal{L}_i \cap \delta\mathcal{S}$ is a finite set for $r + 1 \leq i \leq s$. Thus, $\mathcal{F} := (\mathcal{L}_{r+1} \cup \cdots \cup \mathcal{L}_s) \cap \delta\mathcal{S}$ is a finite subset of $\delta\mathcal{S}$. Let us check that $\mathcal{L}_i \subset \mathrm{Cl}(\mathcal{S}) \cap \overline{\delta\mathcal{S}}^{\mathrm{zar}}$ for $i = 1, \ldots, r$. As $\overline{\mathcal{L}_i}^{\mathrm{zar}}$ is an irreducible algebraic set of dimension 1 and $\mathcal{L}_i \cap \delta\mathcal{S}$ is infinite, then $\overline{\mathcal{L}_i}^{\mathrm{zar}} \subset \overline{\delta\mathcal{S}}^{\mathrm{zar}}$, so $\mathcal{L}_i \subset \mathrm{Cl}(\mathcal{S}) \cap \overline{\delta\mathcal{S}}^{\mathrm{zar}}$. We conclude

$$\delta\mathcal{S} = \delta\mathcal{S} \cap \mathcal{N}(f) = \bigcup_{i=1}^{r}(\delta\mathcal{S} \cap \mathcal{L}_i) \cup \bigcup_{i=r+1}^{s}(\delta\mathcal{S} \cap \mathcal{L}_i) \subset \bigcup_{i=1}^{r}\mathcal{L}_i \cup \mathcal{F} \subset \mathrm{Cl}(\mathcal{S}) \cap \overline{\delta\mathcal{S}}^{\mathrm{zar}},$$

as required. □

Examples 3.2. (i) A *convex polygon* $\mathcal{K} \subset \mathbb{R}^2$ is the intersection of a finite family of closed half-planes, whose interior in $\mathbb{R}^2$ is nonempty. We denote $\mathrm{Int}(\mathcal{K})$ the interior of $\mathcal{K}$ as a topological manifold. It follows from 1.C and Prop. 3.1 that $\mathrm{p}(\mathrm{Int}(\mathcal{K})) = 2$ if and only if $\mathcal{K}$ has only two edges.

(ii) Let $\mathcal{S} := \{x > 0, y > 0, x + y > 1\} \subset \mathbb{R}^2$. Combining (i) with [**FG2**, 6.1] it follows that $\mathrm{r}(\mathcal{S}) = 2$ and $\mathrm{p}(\mathcal{S}) = 3$.

(iii) The first known examples of open polynomial images $\mathcal{S}$ of $\mathbb{R}^2$ obtained in [**FG1**, **FG2**] have connected complement $\mathbb{R}^2 \setminus \mathcal{S}$. The third author proved in [**U1**, Prop.1]: *the semialgebraic set* $\mathcal{S} := \{0 < y < x^2 + 1\} \subset \mathbb{R}^2$*, whose complement has two* 2*-dimensional connected components, is a polynomial image of* $\mathbb{R}^2$.

As the open half-plane $\mathcal{H} : \{y > 0\}$ has $\mathrm{p}(\mathcal{H}) = 2$, it is enough to prove the following: *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be the polynomial map defined by*

$$f(x, y) := (y + (y^2 + 1)x, (xy + 1)^2 + x^2).$$

Then, the restriction $g := f|_{\mathcal{H}} : \mathcal{H} \to \mathcal{S}$ *is bijective.*

Write $a := y + (y^2 + 1)x$ and $b := (1 + xy)^2 + x^2 > 0$. If $y > 0$,

$$a^2 + 1 - b = y^2((1 + xy)^2 + x^2) > 0,$$

so $g(\mathcal{H}) \subset \mathcal{S}$. Conversely, if $(a, b) \in \mathcal{S}$, the coordinates of the unique point $(x, y) \in \mathcal{H}$ such that $g(x, y) = (a, b)$ are $y := \sqrt{\frac{a^2+1-b}{b}}$ and $x := \frac{a-y}{y^2+1}$. Thus, g is a bijection between $\mathcal{H}$ and $\mathcal{S}$, as required.

The previous example contains the key to prove [**U1**, Thm.1].

Theorem 3.3. *For each positive integer* k *there exists an open semialgebraic subset* $\mathcal{S}$ *of* $\mathbb{R}^2$ *such that* $\mathrm{p}(\mathcal{S}) = 2$ *and whose complement* $\mathbb{R}^2 \setminus \mathcal{S}$ *has* $k + 1$ *two-dimensional connected components. In addition,* $\delta\mathcal{S}$ *has* $k+1$ *connected components and all of them are parametric semilines.*

Concerning regular images we proved in [**FGU3**, 4.5] the following.

Proposition 3.4. *Let* $\mathcal{S} \subset \mathbb{R}^2$ *be a* 2*-dimensional semialgebraic set such that* $\mathrm{r}(\mathcal{S}) = 2$ *and let* $\delta\mathcal{S}$ *be its exterior boundary. Then* $\delta\mathcal{S}$ *is either empty or there exist a finite set* $\mathcal{G} \subset \mathbb{R}^2$ *and a finite family of regular semilines* $\mathcal{M}_1, \ldots, \mathcal{M}_r$ *such that*

$$\delta\mathcal{S} \subset \mathcal{G} \cup \bigcup_{i=1}^{r}\mathcal{M}_i \subset \mathrm{Cl}(\mathcal{S}) \cap \overline{\delta\mathcal{S}}^{\mathrm{zar}}.$$

3.A. Convex polygons. The previous results led us to study distinguished families of sets $\mathcal{S} \subset \mathbb{R}^2$ such that $\dim(\mathcal{S}) = \mathrm{p}(\mathcal{S}) = 2$. The third author obtained in [**U2**, Thm.1 & Thm.2] conclusive results in this direction when $\mathcal{S}$ is either a convex polygon, its interior, its complement or the complement of its interior. A convex polygon $\mathcal{K} \subset \mathbb{R}^2$ is a *band* if it is affinely equivalent to $[-a, a] \times \mathbb{R}$ for some $a \geq 0$.

THEOREM 3.5. *Let $\mathcal{K} \subset \mathbb{R}^2$ be a convex polygon. We have:*

(i) *If $\mathcal{K}$ is not a band,* $\mathrm{p}(\mathbb{R}^2 \setminus \mathcal{K}) = \mathrm{p}(\mathbb{R}^2 \setminus \operatorname{Int}(\mathcal{K})) = 2$.
(ii) $\mathrm{r}(\mathcal{K}) = \mathrm{r}(\operatorname{Int}(\mathcal{K})) = 2$.

In addition, we showed in [**FG2**, §4] that $\mathcal{S} := \{x^2 + y^2 > 1\} \subset \mathbb{R}^2$ satisfies $\mathrm{p}(\mathcal{S}) = 3$, $\mathrm{r}(\mathcal{S}) = 2$ and $\mathrm{p}(\mathrm{Cl}(\mathcal{S})) = 2$.

4. Obstructions for the boundaries of polynomial images

In this section we find conditions that should satisfy $\delta\mathcal{S}$ in order to have $\mathrm{p}(\mathcal{S}) = \dim(\mathcal{S})$, $\mathrm{r}(\mathcal{S}) = \dim(\mathcal{S})$ or the finiteness of $\mathrm{p}(\mathcal{S})$ and $\mathrm{r}(\mathcal{S})$. Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set. For every integer $k \geq 0$ we denote $\mathcal{S}_{(k)}$ the set of points $p \in \mathcal{S}$ such that the local dimension of $\mathcal{S}$ at p equals k.

4.A. Exterior boundary and parametric semilines. The next result summarizes [**FG2**, 3.4] and the main results in [**FGU3**].

PROPOSITION 4.1. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d and let $\delta\mathcal{S}$ be its exterior boundary. We have:*

(i) *Let $\mathcal{Z} \subset \mathbb{R}^m$ be a $(d-1)$-dimensional algebraic set such that* $\dim(\mathcal{Z} \cap \delta\mathcal{S}) = d-1$ *and* $\mathcal{Z} \cap \mathrm{Cl}(\mathcal{S})$ *is bounded. Then* $\mathrm{p}(\mathcal{S}) > d$.
(ii) *Suppose* $\mathrm{p}(\mathcal{S}) = \dim(\mathcal{S})$. *Then, there exists a semialgebraic set $\mathcal{U}$ that is open and dense in $(\delta\mathcal{S})_{(d-1)}$ such that for each $p \in \mathcal{U}$ there exists a parametric semiline $\mathcal{L}$ through p satisfying* $\mathcal{L} \subset \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}((\delta\mathcal{S})_{(d-1)}) \cap \mathrm{Cl}(\mathcal{S})$.
(iii) *Suppose* $\mathrm{r}(\mathcal{S}) = \dim(\mathcal{S})$. *Then, there exists a semialgebraic set $\mathcal{V}$ that is open and dense in $(\delta\mathcal{S})_{(d-1)}$ such that for each $p \in \mathcal{V}$ there exists a regular semiline $\mathcal{M}$ through p satisfying* $\mathcal{M} \subset \mathrm{Cl}^{\mathrm{zar}}_{\mathbb{RP}^m}((\delta\mathcal{S})_{(d-1)}) \cap \mathrm{Cl}(\mathcal{S})$.

PROOF. Part (i) follows from (ii), so let us prove (ii). We assume $(\delta\mathcal{S})_{(d-1)} \neq \varnothing$ since otherwise there is nothing to prove. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map with $f(\mathbb{R}^n) = \mathcal{S}$ and $X := \operatorname{graph}(f) \subset \mathbb{R}^n \times \mathbb{R}^m \subset \mathbb{RP}^n \times \mathbb{RP}^m$. Denote $\mathcal{D} := \pi_2(\mathrm{Cl}_{\mathbb{RP}^n \times \mathbb{RP}^m}(X) \setminus X) \cap \mathbb{R}^m$ where $\pi_2 : \mathbb{RP}^n \times \mathbb{RP}^m \to \mathbb{RP}^m$, $(x, y) \to y$ is the projection onto the second space. As $\rho := \pi_2|_{\mathrm{Cl}_{\mathbb{RP}^n \times \mathbb{RP}^m}(X)} : \mathrm{Cl}_{\mathbb{RP}^n \times \mathbb{RP}^m}(X) \to \mathbb{RP}^m$ is proper and $\rho^{-1}(\mathsf{H}_\infty(\mathbb{R})) \subset \mathrm{Cl}_{\mathbb{RP}^n \times \mathbb{RP}^m}(X) \setminus X$, the restriction $f|_{\mathbb{R}^n \setminus f^{-1}(\mathcal{D})} : \mathbb{R}^n \setminus f^{-1}(\mathcal{D}) \to \mathbb{R}^m \setminus \mathcal{D}$ is also proper. In addition, notice that for each $y \in \mathcal{D}$, there exists a sequence $\{x_k\}_k \subset \mathbb{R}^n$ such that all its subsequences are unbounded and the sequence $\{f(x_k)\}_k$ converges to y.

By [**BCR**, 2.8.13] $\mathcal{D}$ is a semialgebraic set of dimension $\leq d-1$, so $\mathcal{R} := \mathrm{Cl}(\mathcal{D} \cap \mathcal{S}) \setminus (\mathcal{D} \cap \mathcal{S})$ has dimension $\leq d-2$. In particular,

$$\mathcal{T} := \delta\mathcal{S} \setminus \mathrm{Cl}(\mathcal{R}) = \mathrm{Cl}(\mathcal{S}) \setminus (\mathcal{S} \cup \mathrm{Cl}(\mathcal{R})) = \mathrm{Cl}(\mathcal{S}) \setminus (\mathrm{Cl}(\mathcal{D} \cap \mathcal{S}) \cup \mathcal{S}) = \delta\mathcal{S} \setminus \mathrm{Cl}(\mathcal{D} \cap \mathcal{S}) \tag{4.1}$$

has dimension $d-1$ and $\delta\mathcal{S} \setminus \mathcal{T} \subset \mathrm{Cl}(\mathcal{R})$ has dimension $\leq d-2$. Thus,

$$\mathcal{U} := ((\delta\mathcal{S})_{(d-1)} \cap \mathcal{T}) \setminus \overline{\delta\mathcal{S} \setminus (\delta\mathcal{S})_{(d-1)}}^{\mathrm{zar}} = (\delta\mathcal{S})_{(d-1)} \setminus (\mathrm{Cl}(\mathcal{D} \cap \mathcal{S}) \cup \overline{\delta\mathcal{S} \setminus (\delta\mathcal{S})_{(d-1)}}^{\mathrm{zar}})$$

is a dense open semialgebraic subset of $(\delta\mathcal{S})_{(d-1)}$. Pick a point $x \in (\delta\mathcal{S})_{(d-1)}$. By [**FU2**, Lem. 2.5] there exist, after a linear change of coordinates,

- integers r, k_i with $k_1 = \min\{k_1, \ldots, k_n\} < 0$, and
- polynomials $p_i \in \mathbb{R}[\mathtt{t}]$ with $p_i(0) \neq 0$ for $i = 2, \ldots, n$

such that $x = \lim_{t\to 0^+}(f\circ\alpha)(t)$ and $(f\circ\alpha)(t)\subset\mathcal{U}$ if $0<t\leq\varepsilon$ for some $\varepsilon>0$, where $\alpha(\mathtt{t}) := (\mathtt{t}^{k_1}+\mathtt{t}^r, \mathtt{t}^{k_2}p_2(\mathtt{t}),\dots,\mathtt{t}^{k_n}p_n(\mathtt{t}))$. After the substitution $\mathtt{t}\to\mathtt{t}^2$, we may assume that k_1, r are even. Write $h_1(\mathtt{x},\mathtt{y}) := \mathtt{y}^{|k_1|}+\mathtt{x}^r$ and for $i=2,\dots,n$

$$h_i(\mathtt{x},\mathtt{y}) := \begin{cases} \mathtt{y}^{|k_i|}p_i(\mathtt{x}) & \text{if } k_i<0,\\ \mathtt{x}^{k_i}p_i(\mathtt{x}) & \text{if } k_i\geq 0.\end{cases}$$

The map $h := (h_1,\dots,h_n):\mathbb{R}^2\to\mathbb{R}^n$ is proper, because if $\{z_k\}_k\subset\mathbb{R}^2$ is a such that all its subsequences are unbounded, then $\{h_1(z_k)\}_k\subset\mathbb{R}$ is an unbounded sequence, so $\{h(z_k)\}_k\subset\mathbb{R}^n$ is an unbounded sequence. In addition, $(f\circ h)(t,1/t) = (f\circ\alpha)(t)$, so $x = \lim_{t\to 0^+}(f\circ h)(t,1/t)$ in $\mathbb{RP}^n$. Denote $\mathcal{A} := \{(f\circ\alpha)(t):\ t\in(0,\varepsilon]\}\subset\mathcal{U}$.

Write $g := f\circ h$ and note that the restriction $g|_{\mathbb{R}^2\setminus g^{-1}(\mathcal{D})}:\mathbb{R}^2\setminus g^{-1}(\mathcal{D})\to\mathbb{R}^m\setminus\mathcal{D}$ is proper, because both h and the restriction $f|_{\mathbb{R}^n\setminus f^{-1}(\mathcal{D})}:\mathbb{R}^n\setminus f^{-1}(\mathcal{D})\to\mathbb{R}^m\setminus\mathcal{D}$ are proper. Write $\mathcal{S}_0 := g(\mathbb{R}^2)\subset\mathcal{S}$. By Prop. 3.1 there exist finitely many parametric semilines $\mathcal{L}_1,\dots,\mathcal{L}_r$ such that

$$\delta\mathcal{S}_0\subset\mathcal{D}_0 := \bigcup_{i=1}^r\mathcal{L}_i\subset\operatorname{Cl}(\mathcal{S}_0)\subset\operatorname{Cl}(\mathcal{S}) = \mathcal{S}\cup\delta\mathcal{S}$$

and the restriction $g:\mathbb{R}^2\setminus g^{-1}(\mathcal{D}_0)\to\mathbb{R}^m\setminus\mathcal{D}_0$ is proper and for each $z\in\mathcal{D}_0$ there is an unbounded sequence $\{x_k\}_k\subset\mathbb{R}^2$ such that the sequence $\{g(x_k)\}_k$ converges to z. As h is proper, $\mathcal{D}_0\subset\mathcal{D}$ and $\mathcal{D}_0\cap\mathcal{S}\subset\mathcal{D}\cap\mathcal{S}$. As $x\in\delta\mathcal{S}_0$, there exists an index $i=1,\dots,r$ such that $x\in\mathcal{L}_i$. Observe that

$$\mathcal{L}_i\subset(\mathcal{S}\cap\mathcal{D}_0)\cup(\delta\mathcal{S}\setminus(\delta\mathcal{S})_{(d-1)})\cup(\mathcal{L}_i\cap(\delta\mathcal{S})_{(d-1)}).$$

Suppose by contradiction that $\dim(\mathcal{L}_i\cap(\delta\mathcal{S})_{(d-1)}) = 0$. Then

$$x\in\mathcal{L}_i\subset\operatorname{Cl}(\mathcal{D}_0\cap\mathcal{S})\cup\operatorname{Cl}(\delta\mathcal{S}\setminus(\delta\mathcal{S})_{(d-1)})\subset\operatorname{Cl}(\mathcal{D}\cap\mathcal{S})\cup\overline{\delta\mathcal{S}\setminus(\delta\mathcal{S})_{(d-1)}}^{\mathrm{zar}},$$

which is false because $x\in\mathcal{U}$. Hence, $\dim(\mathcal{L}_i\cap(\delta\mathcal{S})_{(d-1)}) = 1$, so $\mathcal{L}_i\subset\overline{(\delta\mathcal{S})_{(d-1)}}^{\mathrm{zar}}$. The proof of (iii) is analogous to that of (ii) using Prop. 3.4 instead of Prop. 3.1. $\square$

Example 4.2. Let $*_i$ denotes either $>$ or $\geq$ and consider the m-dimensional semialgebraic set $\mathcal{S} := \{x_1 *_1 0,\dots,x_m *_m 0,\ x_1+\cdots+x_m>1\}\subset\mathbb{R}^m$. It follows from Prop. 4.1 (i) that $\mathrm{p}(\mathcal{S})>m$ since the algebraic set $\mathcal{Z} := \{x_1+\cdots+x_m = 1\}$ has dimension $m-1$ and $\mathcal{Z}\cap\operatorname{Cl}(\mathcal{S}) = \{x_1+\cdots+x_m = 1,\ x_1\geq 0,\dots,x_m\geq 0\} = \mathcal{Z}\cap\delta\mathcal{S}$ is bounded and has dimension $m-1$.

4.B. Few parametric semilines in the exterior boundary. It is natural to wonder about the number of parametric semilines through each point of the exterior boundary of a semialgebraic $\mathcal{S}$ of dimension d that is a polynomial image of $\mathbb{R}^d$. The following example, which is original of this survey, shows that this number can be essentially one. More precisely, we provide a semialgebraic set $\mathcal{S}$ that is a polynomial image of $\mathbb{R}^3$ and satisfies the following property:

4.B.1. *For each point $p\in\delta\mathcal{S} := \operatorname{Cl}(\mathcal{S})\setminus\mathcal{S}$ there exists a parametric semiline $\mathcal{L}_p\subset\delta\mathcal{S}$ through p such that if $\mathcal{L}'$ is another parametric semiline through p satisfying* $\dim(\mathcal{L}'\cap\delta\mathcal{S}) = 1$, *then $\mathcal{L}'\subset\mathcal{L}_p$.*

Example 4.3 (The exterior of a cylinder). Let $\mathcal{A} := \{x^2+y^2\geq 1\}\subset\mathbb{R}^3$ and let $\mathcal{S} =: \operatorname{Int}(\mathcal{A}) = \{x^2+y^2>1\}\subset\mathbb{R}^3$, which satisfies Property 4.B.1. Let us construct a polynomial map $f:\mathbb{R}^3\to\mathbb{R}^3$ such that $f(\mathbb{R}^3) = \mathcal{S}$.

Proof. It was proved in [**FG2**, Prop 4.1] that there exists a polynomial map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ whose image is $\mathcal{T} := \{x^2 + y^2 \geq 1\} \subset \mathbb{R}^2$. Thus, the polynomial map

$$f_1 : \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (\phi(x, y), z)$$

satisfies $f_1(\mathbb{R}^3) = \mathcal{A}$. Consider the polynomials

$$g := (\mathtt{x}^2 + \mathtt{y}^2)(1 + \mathtt{z}^2) - (2 + \mathtt{z}^2) \quad \text{and} \quad h := 1 + g^2$$

and the polynomial map

$$f_2 : \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (h(x, y, z)x, h(x, y, z)y, z).$$

4.B.2. We claim: $\mathcal{B} \subset f_2(\mathcal{A}) \subset \mathcal{S}$ *where* $\mathcal{B} := \{(1+z^2)(x^2+y^2) - (2+z^2) \geq 0\}$.

We prove first the second inclusion. If $(x, y, z) \in \mathcal{A}$ and $x^2 + y^2 > 1$, then $h(x, y, z)^2(x^2 + y^2) > 1$ because $h(x, y, z) \geq 1$, whereas if $x^2 + y^2 = 1$ then $g(x, y, z) = -1$, so $h(x, y, z) = 2$ and $h(x, y, z)^2(x^2 + y^2) = 4 > 1$. In order to show the first inclusion, we work with cylindrical coordinates. Then f_2 becomes $\bar{f}_2(\rho, \alpha, z) = (\bar{h}(\rho, z)\rho, \alpha, z)$, where $\bar{h}(\rho, z) = 1 + (\rho^2(1 + z^2) - (2 + z^2))^2$. We have

$$\mathcal{A} = \{\rho \geq 1\} \quad \text{and} \quad \mathcal{B} = \{(1 + z^2)\rho^2 - (2 + z^2) \geq 0\}.$$

Set $\mathcal{A}_{(\alpha,z)} := \{(\rho, \alpha, z) : \rho \geq 1\}$ and $\mathcal{B}_{(\alpha,z)} := \{(\rho, \alpha, z) : \rho \geq r_z := \sqrt{\frac{2+z^2}{1+z^2}}\}$ for fixed $(\alpha, z) \in \mathbb{R}^2$. Notice that $\mathcal{B}_{(\alpha,z)} \subset \mathcal{A}_{(\alpha,z)}$ for each $(\alpha, z) \in \mathbb{R}^2$,

$$\mathcal{A} = \bigcup_{(\alpha,z)\in\mathbb{R}^2} \mathcal{A}_{(\alpha,z)} \quad \text{and} \quad \mathcal{B} = \bigcup_{(\alpha,z)\in\mathbb{R}^2} \mathcal{B}_{(\alpha,z)}.$$

In addition, $\bar{f}_2(r_z, \alpha, z) = (r_z, \alpha, z)$ and $\lim_{\rho\to+\infty} \bar{h}(\rho, \alpha, z)\rho = +\infty$. Having these properties of $\bar{f}_2$ in mind, the reader checks that $\bar{f}_2$ satisfies

$$\mathcal{B}_{(\alpha,z)} \subset \bar{f}_2(\mathcal{B}_{(\alpha,z)}) \subset \bar{f}_2(\mathcal{A}_{(\alpha,z)}) \quad \text{for each } (\alpha, z) \in \mathbb{R}^2$$

and this readily implies $\mathcal{B} \subset f_2(\mathcal{A})$.

4.B.3. Set $\psi(x, y, z) := z(1 - g^2(x, y, z)z^2)$ and consider the polynomial map

$$f_3 : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (x, y, \psi(x, y, z)).$$

We claim: $\mathcal{S} \subset f_3(\mathcal{B}) \subset f_3(\mathcal{S}) \subset \mathcal{S}$ (see Fig. 1).

The middle inclusion is clear since $\mathcal{B} \subset \mathcal{S}$. The inclusion $f_3(\mathcal{S}) \subset \mathcal{S}$ holds because f_3 leaves invariant vertical lines. Let us check the inclusion $\mathcal{S} \subset f_3(\mathcal{B})$. Fix $(a, b) \in \mathbb{R}^2$ so that $a^2 + b^2 = r^2 > 1$ and denote $\ell_{(a,b)}$ the vertical line $x = a$, $y = b$. For each $r \in \mathbb{R}$ with $1 < r^2 < 2$ define $c_r := +\sqrt{\frac{2-r^2}{r^2-1}}$. Then

$$\ell_{(a,b)} \cap \mathcal{B} = \Big\{(a, b, z) :\ z^2 \geq \frac{2 - r^2}{r^2 - 1}\Big\} = \begin{cases} \{(a, b)\} \times \mathbb{R} & \text{if } r^2 \geq 2, \\ \{(a, b)\} \times I_r & \text{if } 1 < r^2 < 2, \end{cases}$$

where $I_r :=]-\infty, -c_r] \cup [c_r, +\infty[$. Notice that for $a^2 + b^2 > 1$ the last coordinate $\psi(a, b, z)$ of $f_3(a, b, z)$ is a polynomial of odd degree whose leading coefficient is negative, so $\lim_{z\to\pm\infty} \psi(a, b, z) = \mp\infty$. In particular, $f_3(\ell_{(a,b)} \cap \mathcal{B}) = \ell_{(a,b)}$ if $r^2 \geq 2$. Assume next $1 < r^2 < 2$. We have $f_3(a, b, \pm c_r) = (a, b, \pm c_r)$. Thus,

$$[-c_r, +\infty[\subset \psi(\{(a, b)\}\times]-\infty, -c_r]) \quad \text{and} \quad]-\infty, c_r] \subset \psi(\{(a, b)\} \times [c_r, +\infty[),$$

so $\psi(\{(a, b)\} \times I_r) = \mathbb{R}$. Therefore $f_3(\ell_{(a,b)} \cap \mathcal{B}) = \ell_{(a,b)}$ for each (a, b) with $a^2 + b^2 > 1$. Consequently, $\mathcal{S} \subset f_3(\mathcal{B})$, as claimed.

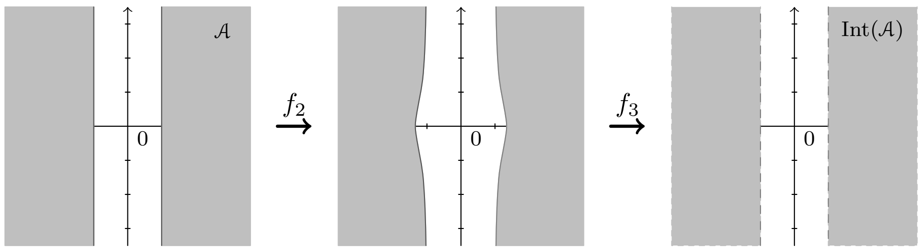

FIGURE 1. Effect of $f_3 \circ f_2$ on each plane vertical section of $\mathcal{A}$ containing the line $\{x = 0, y = 0\}$.

4.B.4. By 4.B.2 and 4.B.3 we have $\mathcal{S} \subset f_3(\mathcal{B}) \subset f_3(f_2(\mathcal{A})) \subset f_3(\mathcal{S}) \subset \mathcal{S}$, so we conclude $f_3(f_2(f_1(\mathbb{R}^3))) = f_3(f_2(\mathcal{A})) = \mathcal{S}$, as required. □

4.C. Connexion by polynomial and regular paths. It is difficult in general to determine precisely if an arbitrary semialgebraic set $\mathcal{S}$ whether satisfies or not the property: *given a finite set $\mathcal{F} \subset \mathcal{S}$, there exists a regular semiline $\mathcal{L}$ such that $\mathcal{F} \subset \mathcal{L} \subset \mathcal{S}$,* even if we restrict to finite sets $\mathcal{F}$ with only two points. This type of semialgebraic sets may be called *rationally connected* as a generalization of rationally connected complex algebraic sets [**K**, Ch. IV]. Analogously, a semialgebraic set $\mathcal{S}$ such that given any pair of points $p, q \in \mathcal{S}$ there exists a parametric semiline $\mathcal{L} \subset \mathcal{S}$ through p, q will be called *polynomially connected.* The reader can check that every polynomially connected semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ is irreducible, pure dimensional, and its image under any polynomial map $h : \mathbb{R}^m \to \mathbb{R}$ is either unbounded or a singleton. In fact, polynomial images of an Euclidean space are polynomially connected, whereas regular images of an Euclidean space are rationally connected.

LEMMA 4.4. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set. We have:*

(i) *Assume* $\mathrm{p}(\mathcal{S})$ *is finite. Then, given a finite set $\mathcal{F} \subset \mathcal{S}$, there exists a parametric semiline $\mathcal{L}$ such that $\mathcal{F} \subset \mathcal{L} \subset \mathcal{S}$.*

(ii) *Assume* $\mathrm{r}(\mathcal{S})$ *is finite. Then, given a finite set $\mathcal{F} \subset \mathcal{S}$, there exists a regular semiline $\mathcal{L}$ such that $\mathcal{F} \subset \mathcal{L} \subset \mathcal{S}$.*

PROOF. We prove only (i) because (ii) is analogous. Write $\mathcal{F} := \{q_1, \dots, q_r\} \subset \mathcal{S}$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map such that $f(\mathbb{R}^n) = \mathcal{S}$. Let $\mathcal{G} := \{p_1, \dots, p_r\} \subset \mathbb{R}^n$ be such that $f(p_i) = q_i$ for $i = 1, \dots, r$. Let us construct a polynomial involution $h : \mathbb{R}^n \to \mathbb{R}^n$ that maps each p_i to a point $(a_{1i}, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

After a linear change of coordinates in $\mathbb{R}^n$ we may assume that the first coordinates of the points p_i are pairwise different, that is, if $p_i = (a_{1i}, \dots, a_{ni})$ then $a_{1i} \neq a_{1j}$ if $i \neq j$. Let $P_j \in \mathbb{R}[\mathtt{t}]$ be an interpolating polynomial such that $P_j(a_{1i}) = a_{ji}$ for $j = 2, \dots, n$, so

$$h : \mathbb{R}^n \to \mathbb{R}^n,\ (x_1, x_2, \dots, x_n) \mapsto (x_1, P_2(x_1) - x_2, \dots, P_n(x_1) - x_n)$$

is a polynomial involution of $\mathbb{R}^n$ that satisfies $h(a_{1i}, 0) = p_i$ for $i = 1, \dots, r$.

The parametric semiline $\mathcal{L} := \mathrm{im}(f \circ h \circ \alpha)$ where $\alpha : \mathbb{R} \to \mathbb{R}^n,\ t \to (t, 0, \dots, 0)$ satisfies $\mathcal{F} \subset \mathcal{L} \subset \mathcal{S}$ as required. □

EXAMPLES 4.5. (i) The reader can check that the semialgebraic set

$$\mathcal{S} := \{0 < x < 1, y \geq 0\} \cup \{0 \leq x \leq y\} \subset \mathbb{R}^2$$

is polynomially connected and $\delta\mathcal{S}$ is a finite union of parametric semilines. By Thm. 4.6 below $\mathrm{p}(\mathcal{S}) = +\infty$ because its "set of points at infinity" is not connected.

(ii) A rationally connected semialgebraic set is irreducible and pure dimensional. An open connected semialgebraic set $\mathcal{A} \subset \mathbb{R}^m$ is rationally connected. Use Stone-Weierstrass approximation Theorem and recall that an interval $[a, b]$ is by Thm. 2.6 a regular image of $\mathbb{R}$.

(iii) The situation is different for general semialgebraic sets. It is proved in [**C**, **V**] that for every $m \geq 4$ and $d \geq 2m-2$ a generic complex algebraic hypersurface in $\mathbb{CP}^m$ of degree d contains no rational projective algebraic curves and the same holds if $m = 2, 3$ and $d \geq 2m - 1$. Thus, if the Zariski closure of a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ is a generic complex algebraic hypersurface of big enough degree it follows from Cor. 4.4 that $\mathrm{r}(\mathcal{S}) = +\infty$.

(iv) A semialgebraic set $\mathcal{T} \subset \mathbb{R}^m$ is *generically uniruled* if there is a dense open semialgebraic subset $\mathcal{U} \subset \mathcal{T}$ such that for each point $x \in \mathcal{U}$ there exists a regular semiline through x contained in $\mathcal{T}$. If $\mathcal{S} \subset \mathbb{R}^m$ is an open semialgebraic set such that the Zariski closure of $\delta\mathcal{S}$ in $\mathbb{RP}^m$ is a generic algebraic hypersurface of $\mathbb{RP}^m$ of high degree, then its Zariski closure $\mathcal{Z}$ in $\mathbb{CP}^m$ does not contain rational projective algebraic curves, so $(\delta\mathcal{S})_{(m-1)}$ is not generically uniruled and by Prop. 4.1 $\mathrm{r}(\mathcal{S}) > m$.

4.D. Set of points at infinity. The most general obstruction we have found for a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ with $\mathrm{p}(\mathcal{S}) < +\infty$ is the following result from [**FU2**]. Its proof is long and involved and it requires deep knowledge of resolution of singularities, complex algebraic geometry and algebraic topology. Thm. 4.6 provides a new evidence of the differences between regular and polynomial images of $\mathbb{R}^n$ (see Ex. 4.9).

THEOREM 4.6. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set such that $\mathrm{p}(\mathcal{S}) < +\infty$ and it is not a singleton. Then, the set $\mathcal{S}_\infty := \mathrm{Cl}_{\mathbb{RP}^m}(\mathcal{S}) \cap \mathsf{H}_\infty(\mathbb{R})$ of* points at infinity *of $\mathcal{S}$ is nonempty and connected.*

Once Thm. 4.6 is known the next question arises naturally.

QUESTION 4.7. *Let S_0 be a connected closed semialgebraic subset of $\mathsf{H}_\infty(\mathbb{R})$.* Is there a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\mathbb{R}^n)_\infty = \mathsf{S}_0$?

For $m = 2$ the answer is affirmative, as we show in the next example, but we have no further information for higher dimension.

EXAMPLE 4.8. Let us denote by $\ell_\infty(\mathbb{R})$ the line of infinite of the real projective plane $\mathbb{RP}^2$. Then, for every connected closed semialgebraic subset $\mathsf{S}_0 \subset \ell_\infty(\mathbb{R})$ there exists a polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\dim(f(\mathbb{R}^2)) = 2$ and $(f(\mathbb{R}^2))_\infty = \mathsf{S}_0$.

If S_0 is a singleton it suffices to observe that $f(\mathbb{R}^2)_\infty = \{[0 : 1 : 0]\}$ where

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto (x, y^2 + x^2),$$

and if S_0 is not a singleton the result follows straightforwardly from Cor. 1.5.

The behavior at infinity of regular images of $\mathbb{R}^n$ is not so rigid as in the polynomial case. Let us see this in some elementary examples.

EXAMPLES 4.9. (i) The 1-dimensional semialgebraic set $\mathcal{S} := \{x > 0,\ xy = 1\}$ is the image of the regular map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \Big((xy-1)^2 + x^2, \frac{1}{(xy-1)^2 + x^2}\Big).$$

Thus $\mathcal{S}_\infty = \{[0:1:0], [0:0:1]\}$ is disconnected, $\mathrm{r}(\mathcal{S}) = 2$ and $\mathrm{p}(\mathcal{S}) = +\infty$.

(ii) The image of the regular map

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \Big(\frac{x^2}{1+y^2}, \frac{y^2}{1+x^2}\Big)$$

is $\mathcal{S} := \{x \geq 0,\ y \geq 0,\ xy < 1\}$, so $\mathcal{S}_\infty = \{[0:1:0], [0:0:1]\}$ is disconnected too. Thus, $\mathrm{r}(\mathcal{S}) = 2$ and $\mathrm{p}(\mathcal{S}) = +\infty$.

(iii) In [**FU2**, Prop. 4.3] we construct for each finite set $\mathcal{F} \subset \ell_\infty(\mathbb{R})$ a regular map $f : \mathbb{R}^2 \to \mathbb{R}^2$ whose 2-dimensional image $\mathcal{S}$ satisfies $\mathcal{S}_\infty = \mathcal{F}$. Consequently, $\mathrm{r}(\mathcal{S}) = 2$ and $\mathrm{p}(\mathcal{S}) = +\infty$.

(iv) Consider the regular map

$$g : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \Big(\frac{x^2+1}{1+x^2y^2}, \frac{y^2+1}{1+x^2y^2}\Big)$$

whose image is contained in the open quadrant $\mathcal{Q} := \{x > 0,\ y > 0\}$. Define

$$h_0(\mathtt{x},\mathtt{y}) := 1 + \mathtt{x}(\mathtt{x}-1)^2\mathtt{y}(\mathtt{y}-1)^2, \quad h_1(\mathtt{x},\mathtt{y}) := \mathtt{x}((\mathtt{y}-1)^2\mathtt{y}^2 + 2(\mathtt{x}-1)^2\mathtt{x}),$$
$$h_2(\mathtt{x},\mathtt{y}) := \mathtt{y}((\mathtt{x}-1)^2\mathtt{x}^2 + 2(\mathtt{y}-1)^2\mathtt{y}).$$

Notice that h_0 does not vanish in $\mathcal{Q}$, so

$$h : \mathcal{Q} \to \mathbb{R}^2,\ (x, y) \mapsto \Big(\frac{h_1(x,y)}{h_0(x,y)}, \frac{h_2(x,y)}{h_0(x,y)}\Big)$$

is a regular map and the composition $f := h \circ g : \mathbb{R}^2 \to \mathbb{R}^2$ is a regular map too. It is proved in [**FU2**, 4.4] that the set $\mathcal{S}_\infty$ of points at infinity of $\mathcal{S} := f(\mathbb{R}^2)$ satisfies

$$\mathcal{S}_\infty = \{[0:x:1] \in \mathbb{RP}^2 :\ 0 \leq 2x \leq 1\} \cup \{[0:1:y] \in \mathbb{RP}^2 :\ 0 \leq 2y \leq 1\},$$

that has two connected components of dimension 1. Thus, $\mathrm{r}(\mathcal{S}) = 2$ and $\mathrm{p}(\mathcal{S}) = +\infty$.

QUESTION 4.10. Let S_0 be a closed semialgebraic subset of the hyperplane at infinity $\mathsf{H}_\infty(\mathbb{R})$. Is there a regular map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $(f(\mathbb{R}^n))_\infty = \mathsf{S}_0$?

5. Piecewise linear semialgebraic sets as polynomial and regular images

In this section we provide a complete answer to the problem of calculating $\mathrm{p}(\mathcal{S})$ and $\mathrm{r}(\mathcal{S})$ in case $\mathcal{S}$ is either a polyhedron $\mathcal{K} \subset \mathbb{R}^m$, its interior $\mathrm{Int}(\mathcal{K})$, its complementary set $\mathbb{R}^m \setminus \mathcal{K}$ and the complementary $\mathbb{R}^m \setminus \mathrm{Int}(\mathcal{K})$ of its interior.

5.A. Convex polyhedra. After the results in the 2-dimensional case in Section 3, we approached in [**FGU1**] the regular case for arbitrary dimension:

THEOREM 5.1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex polyhedron of dimension $d \geq 2$. Then,* $\mathrm{r}(\mathcal{K}) = \mathrm{r}(\mathrm{Int}(\mathcal{K})) = d$.

As the image of a non constant polynomial map is an unbounded semialgebraic set, it is not possible to represent an arbitrary convex polyhedron as a polynomial image of an Euclidean space. This is not the unique obstruction and we need to recall the concept of *recession cone* $\vec{\mathcal{C}}(\mathcal{K})$ of a convex polyhedron $\mathcal{K}$, see [**R**, II.§8] and [**Z**, Ch.1]. Fix a point $p \in \mathcal{K}$ and let $\vec{\mathcal{C}}(\mathcal{K}) := \{\vec{v} \in \mathbb{R}^m :\ \{p + \lambda\vec{v},\ \forall\, \lambda \geq 0\} \subset \mathcal{K}\}$.

Then $\vec{\mathcal{C}}(\mathcal{K})$ is a convex cone and it does not depend on the choice of p. The *facets* of a d-dimensional polyhedron are its faces of dimension $d-1$. The main results in [**FGU4**] can be summarized as follows:

THEOREM 5.2. *Let $\mathcal{K}$ be a convex polyhedron of dimension $d \geq 2$. We have*

(i) $\mathrm{p}(\mathcal{K}) = d$ *if and only if* $\dim(\vec{\mathcal{C}}(\mathcal{K})) = d$. *Otherwise,* $\mathrm{p}(\mathcal{K}) = +\infty$.

(ii) $\mathrm{p}(\mathrm{Int}(\mathcal{K}))$ *is finite if and only if* $\dim(\vec{\mathcal{C}}(\mathcal{K})) = d$. *In addition,* $\mathrm{p}(\mathrm{Int}(\mathcal{K})) \leq d+1$ *and* $\mathrm{p}(\mathrm{Int}(\mathcal{K})) = d$ *if and only if* $\mathcal{K}$ *has no bounded facets.*

We collect in the Table below the values of the invariants $\mathrm{r}(\mathcal{K})$, $\mathrm{r}(\mathrm{Int}(\mathcal{K}))$, $\mathrm{p}(\mathcal{K})$ and $\mathrm{p}(\mathrm{Int}(\mathcal{K}))$ for a convex polyhedron $\mathcal{K}$ of dimension $d \geq 1$.

<table>
<tr><th rowspan="2"></th><th colspan="2">$\mathcal{K}$ bounded</th><th colspan="2">$\mathcal{K}$ unbounded</th></tr>
<tr><th>$d=1$</th><th>$d\geq 2$</th><th>$d=1$</th><th>$d\geq 2$</th></tr>
<tr><td>$\mathrm{r}(\mathcal{K})$</td><td>1</td><td rowspan="2">d</td><td>1</td><td rowspan="2">d</td></tr>
<tr><td>$\mathrm{r}(\mathrm{Int}(\mathcal{K}))$</td><td>2</td><td>2</td></tr>
<tr><td>$\mathrm{p}(\mathcal{K})$</td><td colspan="2" rowspan="2">$+\infty$</td><td>1</td><td>d, $+\infty$</td></tr>
<tr><td>$\mathrm{p}(\mathrm{Int}(\mathcal{K}))$</td><td>2</td><td>$d$, $d+1$, $+\infty$</td></tr>
</table>

5.B. Open and closed balls. A d-dimensional closed ball $\overline{\mathcal{B}}$ and its relative interior $\mathcal{B}$ can be understood as 'limits of bounded convex polyhedra and their interiors, when the number of facets tends to infinity. We proved in [**FGU1**] that both are regular images of $\mathbb{R}^m$ and in fact $\mathrm{r}(\mathcal{B}) = \mathrm{r}(\overline{\mathcal{B}}) = d$.

5.C. Complements of convex polyhedra. Next we are concerned with the complements $\mathcal{S} := \mathbb{R}^m \setminus \mathcal{K}$ and $\overline{\mathcal{S}} := \mathbb{R}^m \setminus \mathrm{Int}(\mathcal{K})$ of a convex polyhedron $\mathcal{K} \subsetneq \mathbb{R}^m$. These semialgebraic sets are unbounded, so it is reasonable to wonder it they are polynomial images of Euclidean spaces. We see below that the unique obstruction for the finiteness of $\mathrm{p}(\mathcal{S})$ and $\mathrm{p}(\overline{\mathcal{S}})$ is that $\mathcal{S}$ and/or $\overline{\mathcal{S}}$ are connected. A *layer in* $\mathbb{R}^m$ is a convex polyhedron $\mathcal{K} \subset \mathbb{R}^m$ affinely equivalent to $[-a, a] \times \mathbb{R}^{m-1}$ for some $a \geq 0$. Layers are the unique convex polyhedra of $\mathbb{R}^m$ that disconnect $\mathbb{R}^m$. The main results in this direction are collected in [**FU3**, **FU4**, **FU5**] and can be summarized as follows. We point out that we proved first in [**FU3**] that $\mathrm{r}(\mathbb{R}^m \setminus \mathcal{K}) = \mathrm{r}(\mathbb{R}^m \setminus \mathrm{Int}(\mathcal{K})) = m$ whenever $\mathcal{K} \subset \mathbb{R}^m$ is a convex polyhedron that is not a layer. Recently, we have improved that result in [**FU5**] showing the following.

THEOREM 5.3. *Let $m \geq 2$ and let $\mathcal{K} \subset \mathbb{R}^m$ be a convex polyhedron that is not a layer. Then* $\mathrm{p}(\mathbb{R}^m \setminus \mathcal{K}) = \mathrm{p}(\mathbb{R}^m \setminus \mathrm{Int}(\mathcal{K})) = m$.

We summarize in the Table below the previous information concerning $\mathrm{p}(\mathcal{S})$, $\mathrm{p}(\overline{\mathcal{S}})$, $\mathrm{r}(\mathcal{S})$ and $\mathrm{r}(\overline{\mathcal{S}})$ when $\dim(\mathcal{K}) = m$ and we include what happens with the 1-dimensional case. For simplicity we exclude the case when $\mathcal{K}$ is a layer if $m \geq 2$.

5.D. Complements of open and closed balls. Again we deal with an m-dimensional closed ball $\overline{\mathcal{B}}_m$ in $\mathbb{R}^m$ and its interior $\mathcal{B}_m$. In [**FU4**, Prop. 8.1] we have proved that $\mathrm{p}(\mathbb{R}^m \setminus \mathcal{B}_m) = m$. Although $\mathbb{R}^m \setminus \overline{\mathcal{B}}_m$ can be understood as the limit of the complements of suitable unions of simplices of a triangulation of the closed ball $\overline{\mathcal{B}}_m$, it is not a polynomial image of $\mathbb{R}^m$ by Prop. 4.1 (i). However, we proved in [**FU4**, Cor. 8.2] that $\mathrm{p}(\mathbb{R}^m \setminus \overline{\mathcal{B}}) = m+1$.

	$\mathcal{K}$ bounded		$\mathcal{K}$ unbounded	
	$m=1$	$m\geq 2$	$m=1$	$m\geq 2$
$\mathrm{r}(\mathcal{S})$	$+\infty$	m	2	m
$\mathrm{r}(\overline{\mathcal{S}})$			1	
$\mathrm{p}(\mathcal{S})$			2	
$\mathrm{p}(\overline{\mathcal{S}})$			1	

6. Nash images of $\mathbb{R}^n$

The rigidity of polynomial and regular maps makes really difficult to obtain a satisfactory geometric characterization of those semialgebraic sets that are either polynomial or regular images of some $\mathbb{R}^n$. Shiota suggested in 1990 the following variant of the problem stated above concerning the representability of semialgebraic sets as polynomial and/or regular images of Euclidean spaces:

PROBLEM 6.1. To characterize the subsets of $\mathbb{R}^m$ that are Nash images of $\mathbb{R}^n$.

A *Nash map* on $\mathcal{S}$ with values in $\mathbb{R}^n$ is a map $f := (f_1, \dots, f_n) : \mathcal{S} \to \mathbb{R}^n$ such that each f_i is a Nash function on $\mathcal{S}$ (see Def. 2.3). Images of semialgebraic sets under Nash maps are semialgebraic sets. Shiota outlined a vague schedule that sustains the following conjecture (wrongly announced in [**G**, **FG1**] as proved by Shiota) in order to provide a satisfactory answer to Problem 6.1.

CONJECTURE 6.2 (Shiota). *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d. Then $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$ if and only if $\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ whose image meets all connected components of the set* $\operatorname{Reg}(\mathcal{S})$ *of regular points of* $\mathcal{S}$.

The set $\operatorname{Reg}(\mathcal{S})$ of regular points of a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ is defined as follows. Let X be the Zariski closure of $\mathcal{S}$ in $\mathbb{R}^m$ and let $\widetilde{X}$ be the complexification of X, that is, the smallest complex algebraic subset of $\mathbb{C}^m$ that contains X. Define $\operatorname{Reg}(X) := X \setminus \operatorname{Sing}(\widetilde{X})$ and let $\operatorname{Reg}(\mathcal{S})$ be the interior of $\mathcal{S} \setminus \operatorname{Sing}(\widetilde{X})$ in $\operatorname{Reg}(X)$.

In 2004 we met again with Shiota and discussed about possible ways to attack his conjecture. It was not clear how to follow certain parts of his 1990 schedule. However, that fruitful meeting was the starting point for the achievement by the first author of this article of a positive answer to the conjecture in [**Fe**] and some related results [**BFR**, **FGR**]. The latter include useful tools concerning:

(i) Extension of Nash functions on a Nash manifold with boundary to a Nash manifold of its same dimension that contains it as a closed subset [**FGR**].
(ii) Approximation results on a Nash manifold relative to a Nash subset with monomial singularities [**BFR**].
(iii) Equivalence of Nash classification and $\mathcal{C}^2$-semialgebraic classification for Nash manifolds with boundary [**BFR**].

We will state next the main result in [**Fe**] and some of its consequences. A *Nash manifold* is a pure dimensional semialgebraic subset M of some affine space $\mathbb{R}^m$ that is a smooth submanifold with or without boundary of an open subset of $\mathbb{R}^m$. In addition, when we refer to a Nash manifold with boundary, we assume that the boundary is a Nash submanifold.

6.A. Main results. The main result concerning Nash images of Euclidean spaces is Thm. 6.3, that includes a positive solution to Shiota's Conjecture. Its statement requires some preliminary definitions. Let $\alpha : [0,1] \to \mathbb{R}^m$ be a semialgebraic path, that is, a continuous map whose graph is semialgebraic. Let $A \subset (0,1)$ be the smallest subset of $(0,1)$ such that the restriction $\alpha|_{(0,1)\setminus A}$ is a Nash map. The set A is finite and we write $\eta(\alpha) := \alpha(A)$. A semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ is *well-welded* if it is pure dimensional and for each pair of points $x, y \in \mathcal{S}$ there exists a semialgebraic path $\alpha : [0,1] \to \mathcal{S}$ such that $\alpha(0) = x$, $\alpha(1) = y$ and $\eta(\alpha) \subset \operatorname{Reg}(\mathcal{S})$.

THEOREM 6.3 (Nash images of Euclidean spaces). *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d. The following assertions are equivalent:*

(i) *$\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*
(ii) *$\mathcal{S}$ is a Nash image of $\mathbb{R}^m$ for some $m \geq d$.*
(iii) *$\mathcal{S}$ is connected by Nash paths.*
(iv) *$\mathcal{S}$ is connected by analytic paths.*
(v) *$\mathcal{S}$ is pure dimensional and there exists a Nash path $\alpha : [0,1] \to \mathcal{S}$ that meets all the connected components of* $\operatorname{Reg}(\mathcal{S})$.
(vi) *$\mathcal{S}$ is pure dimensional and there exists an analytic path $\alpha : [0,1] \to \mathcal{S}$ that meets all the connected components of* $\operatorname{Reg}(\mathcal{S})$.
(vii) *$\mathcal{S}$ is well-welded.*

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) and (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) are straightforward. It requires more work to show that a semialgebraic set $\mathcal{S}$ satisfying either condition (iii), (iv), (v) or (vi) is well-welded but the really demanding part of the proof is (vii) $\Longrightarrow$ (i). An important milestone for the proof of Thm. 6.3 is the following result, which has its own interest.

THEOREM 6.4. *Let $N \subset \mathbb{R}^m$ be a connected d-dimensional Nash manifold with boundary. Then N is a Nash image of $\mathbb{R}^d$.*

Compare Prop. 6.5 below with the restrictive characterizations for 1-dimensional polynomial and regular images of Euclidean spaces (see Thms. 2.4 and 2.6).

PROPOSITION 6.5 (The 1-dimensional case). *Let $\mathcal{S} \subset \mathbb{R}^m$ be a 1-dimensional semialgebraic set. Then $\mathcal{S}$ is a Nash image of some $\mathbb{R}^m$ if and only if $\mathcal{S}$ is irreducible. In addition, if such is the case $\mathcal{S}$ is a Nash image of $\mathbb{R}$.*

6.B. Consequences. We present three remarkable consequences of Thm. 6.3.

6.B.1. *Arc-symmetric semialgebraic sets.* Arc-symmetric semialgebraic sets were introduced by Kurdyka in [**Ku**] and subsequently studied by many authors. Recall that a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m$ is *arc-symmetric* if $\gamma((-1,1)) \subset \mathcal{S}$ for each analytic arc $\gamma : (-1,1) \to \mathbb{R}^m$ with $\gamma((-1,0)) \subset \mathcal{S}$. In particular, arc-symmetric semialgebraic sets are closed subsets of $\mathbb{R}^m$. An arc-symmetric semialgebraic set $\mathcal{S} \subset \mathbb{R}^n$ is *irreducible* (as an arc-symmetric semialgebraic set) if it cannot be written as the union of two proper arc-symmetric semialgebraic subsets [**Ku**, §2]. This is equivalent to the following fact: *$\mathcal{S}$ is irreducible if and only if the ring $\mathcal{N}(\mathcal{S})$ is an integral domain* (see Def. 2.3). It follows from Thm. 6.3 and [**Ku**, Cor.2.8] that a pure d-dimensional irreducible arc-symmetric semialgebraic set is a Nash image of $\mathbb{R}^d$. In addition:

COROLLARY 6.6. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a pure d-dimensional irreducible semialgebraic set whose closure $\operatorname{Cl}(\mathcal{S})$ is arc-symmetric. Then $\mathcal{S}$ is a Nash image of $\mathbb{R}^d$.*

6.B.2. *Elimination of inequalities.* Tarski-Seidenberg principle on elimination of quantifiers can be restated geometrically by saying that the projection of a semialgebraic set is again semialgebraic. A converse problem, to find an algebraic set in $\mathbb{R}^{m+k}$ whose projection is a given semialgebraic subset of $\mathbb{R}^m$, is known as the *problem of eliminating inequalities.* Motzkin proved in [**Mo**] that this problem always has a solution for $k = 1$. However, his solution is rather complicated and is generally a reducible algebraic set. In another direction Andradas–Gamboa proved in [**AG1**, **AG2**] that if $\mathcal{S} \subset \mathbb{R}^m$ is a closed semialgebraic set whose Zariski closure is irreducible, then $\mathcal{S}$ is the projection of an irreducible algebraic set in some $\mathbb{R}^{m+k}$. In [**P**] Pecker obtained some improvements on both results: for the first by finding a construction of an algebraic set in $\mathbb{R}^{m+1}$ that projects onto the given semialgebraic subset of $\mathbb{R}^m$, far simpler than the original construction of Motzkin; for the second by proving that if $\mathcal{S}$ is a locally closed semialgebraic subset of $\mathbb{R}^m$ with non-empty interior, then $\mathcal{S}$ is the projection of an irreducible algebraic subset of $\mathbb{R}^{m+1}$. Pecker's construction plays an important role in [**FGU4**].

In [**Fe**] it is proved the following result that looks for a non-singular algebraic set with the simplest possible topology that projects onto a semialgebraic set.

Corollary 6.7. *Let $\mathcal{S} \subset \mathbb{R}^m$ be a semialgebraic set of dimension d. We have:*

(i) *If $\mathcal{S}$ is Nash path-connected there exist a non negative integer k and an irreducible non-singular algebraic set $X \subset \mathbb{R}^{m+k}$ whose connected components are Nash diffeomorphic to $\mathbb{R}^d$ such that $\mathcal{S} = \pi_k(X)$ where*

$$\pi_k : \mathbb{R}^{m+k} \to \mathbb{R}^m,\ (x_1, \dots, x_{m+k}) \mapsto (x_1, \dots, x_m).$$

In addition, each connected component of X projects onto $\mathcal{S}$ and given any two of the connected components of X there exists an automorphism of X that swaps them.

(ii) *In general, there exist a nonnegative integer k and an algebraic set $X \subset \mathbb{R}^{m+k}$ that is Nash diffeomorphic to a finite pairwise disjoint union of affine subspaces of $\mathbb{R}^{d+1}$ such that $\mathcal{S} = \pi_k(X)$.*

Even for dimension 1, it is not possible to require the connectedness of X:

Example 6.8. Let $X \subset \mathbb{R}^n$ be a real algebraic curve that is Nash diffeomorphic to $\mathbb{R}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a linear projection. Then $\pi(X)$ is not a proper open interval of $\mathbb{R}$.

Note that $Y := \mathrm{Cl}_{\mathbb{RP}^n}(X) = X \cup \{p_\infty\}$ where p_∞ is a point of the hyperplane of infinity of $\mathbb{RP}^n$. Note that π is the restriction to $\mathbb{R}^n$ of a central projection $\Pi : \mathbb{RP}^n \dashrightarrow \mathbb{RP}^1$ with center a projective $(n-2)$-dimensional subspace L of the hyperplane of infinity $\mathsf{H}_\infty(\mathbb{R})$.

If $p_\infty \notin L$, then $\Pi(Y)$ is a compact subset of $\mathbb{RP}^1$ and $\Pi(p_\infty)$ is the point at infinity of $\mathbb{RP}^1$. Thus, $\pi(X)$ is a closed semialgebraic subset of $\mathbb{R}$.

If $p_\infty \in L$, we assume by contradiction that $\pi(X)$ is a proper open interval of $\mathbb{R}$. Then Y has at least two different tangents at p_∞. However, as X is Nash diffeomorphic to $\mathbb{R}$, the analytic germ Y_{p_∞} has only one branch, which is a contradiction. Thus, $\pi(X)$ is not a proper open interval of $\mathbb{R}$.

Remark 6.9. Let $\mathcal{S} := (0,1) \subset \mathbb{R}$. By Cor. 6.7 there exist $n > 0$ and an algebraic set $X \subset \mathbb{R}^{n+1}$ whose connected components are Nash diffeomorphic to $\mathbb{R}$ and a projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\pi(X) = (0,1)$. By Ex. 6.8 we know that X is not connected.

6.B.3. *Representation of connected compact differentiable manifolds.* A classical result of Nash [**N**] states that every compact smooth manifold M is diffeomorphic to a finite union of connected and compact components of a non-singular algebraic set, that is, M is diffeomorphic to a compact Nash manifold. Later Akbulut-King proved in [**AK**, Thm.1.1] that a pair (M, N) constituted by a compact smooth manifold M and a closed smooth submanifold N is diffeomorphic to a pair (X, Z) constituted by a compact non-singular real algebraic set X and a non-singular algebraic subset Z. This combined with Thm. 6.4 provides the following.

Corollary 6.10. *Let N be a connected d-dimensional compact smooth manifold with boundary. Then N is the image of $\mathbb{R}^d$ under a smooth map.*

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