A MATHEMATICAL TRIBUTE

to Professor

José María Montesinos Amilibia

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
FACULTAD DE CIENCIAS MATEMÁTICAS – UCM
A MATHEMATICAL TRIBUTE
to Professor José María Montesinos Amilibia

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Oh my knots!

This volume contains the contributions presented by several colleagues as a tribute to the mathematical and human qualities of José María Montesinos Amilibia on the occasion of his seventieth birthday. The editors would like to express their thanks to the contributors and their very especial gratitude to José María for his example through many years of scientific and personal contact.

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Contents

Lectures addressed on the 8 of September, 2015

Morphismes analytiques finis et revêtements ramifiés................................. 1

Claude Weber

Algunas contribuciones matemáticas del Profesor José María Montesinos Amilibia .............................................................. 23

Maite Lozano

José María Montesinos Amilibia. Biographical Sketches......................... 33

Sebastià Xambó

Contributions on the occasion of JMMA’s 70 anniversary

Proof of the rigidity of model nilpotent Lie algebras by means of the Internal Set Theory................................................................. 63

J. M. Ancochea Bermúdez, R. Campoamor Stursberg

Una caracterización de las proyecciones de Lagrange............................. 75

Jorge L. Andrés, Jesús Otero

The dodecahedron: from intersections of quadrics to Borromean rings....... 85

Enrique Artal Bartolo, Santiago López de Medrano, María Teresa Lozano

\(p\)-pairs in graphs representing surfaces........................................ 105

P. Bandieri

Fixed points, bounded orbits and attractors of planar flows.................... 125

Héctor Barge, José M. R. Sanjurjo

The contact structure in the space of light rays .................................. 133

Alfredo Bautista, Alberto Ibort, Javier Lafuente

Universal groups and super regular tessellations................................. 161


Groups of automorphisms of bordered 2-tori..................................... 181

Emilio Bujalance, José Javier Etayo, Ernesto Martínez
Classifying PL 4-manifolds via crystallizations: results and open problems

Maria Rita Casali, Paola Cristofori, Carlo Gagliardi

On certain classes of closed 3-manifolds with different geometric structures

Alberto Cavicchioli, Fulvia Spaggiari

Algunas exploraciones matemáticas del mundo

Capi Corrales Rodríguez

A note on regular branched foldings

Antonio F. Costa

Partially flat surfaces solving $k$-Hessian perturbed equations

Gregorio Díaz, Jesús Ildefonso Díaz

Hyperbolic surfaces of genus 3 with symmetry $S_4$

Raquel Díaz, Victor González Aguilera

Modelo plano conforme del Plano Proyectivo

Fernando Etayo, Ujué R. Trías

Toroidal Dehn Surgeries

Mario Eudave-Muñoz, Enrique Ramírez-Losada

The open quadrant problem: A topological proof

José F. Fernando, J. M. Gamboa, Carlos Ueno

A la búsqueda de la espiritualidad perdida. Meditar itinerante acerca del número, el tacto, la duración y el Arte so pretexto de las Matemáticas y la matematización del mundo

Jesús Fortea Pérez

Persistencia uniforme de atractores

Antonio Giraldo, Victor Fernández Laguna, José M. Rodríguez Sanjurjo

2-dimensional stratifolds

J. C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil

Surface knot groups and 3-manifold groups

Francisco González-Acuña, Arturo Ramírez

Some geometric properties of variable exponent Lebesgue spaces

Francisco L. Hernández, Cesar Ruiz

Carousel wild knots are ambient homogeneous

Gabriela Hinjojosa, Cynthia Verjovsky Marcotte, Alberto Verjovsky

Global bifurcation for Fredholm operators

Julián López-Gómez
Pascal’s triangle, Stirling numbers and the Euler characteristic
Ana Luzón, Manuel A. Morón, Felipe Prieto-Martínez

Probabilidades en espacios topológicos
Juan Margalef-Roig, Salvador Miret-Artés, Enrique Outerelo-Domínguez

Compact Hausdorff group topologies for the additive group of real numbers
Elena Martín-Peinador, Monserrat Bruguera Padró

Unraveling the Dogbone Space
Diego Mondéjar Ruiz

SL(3, C)-character variety of torus knots
Vicente Muñoz, Jonathan Sánchez

Spectral limits of semiclassical commuting self-adjoint operators
Álvaro Pelayo, S. Vű Ngoc

Nontrivial twisted Alexander polynomials
Joan Porti

Matrices de rotaciones, simetrías y roto-simetrías
María Jesús de la Puente

La Biblioteca de El Escorial. Un culto a la matemática
María Concepción Romo Santos

Geometría en el siglo XIV: los trabajos de Thomas Bradwardine
Juan Tarrés Freixenet
The open quadrant problem: A topological proof

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Dedicated to José María Montesinos on the occasion of his 70th birthday.

ABSTRACT

In this work we present a new polynomial map \( f := (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) whose image is the open quadrant \( \Omega := \{ x > 0, y > 0 \} \subset \mathbb{R}^2 \). The proof of this fact involves arguments of topological nature that avoid hard computer calculations. In addition each polynomial \( f_i \in \mathbb{R}[x, y] \) has degree \( \leq 16 \) and only 11 monomials, becoming the simplest known map solving the open quadrant problem.

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Key words: Polynomial map, polynomial image, semialgebraic set, open quadrant, total degree, total number of monomials.

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1. Introduction

Although it is usually said that the first work in Real Geometry is due to Harnack [13], who obtained an upper bound for the number of connected components of a non-singular real algebraic curve in terms of its genus, modern Real Algebraic Geometry was born with Tarski’s article [15], where it is proved that the image of a semialgebraic set under a polynomial map is a semialgebraic set. We are interested in studying what might be called the ‘inverse problem’. In the 1990 Oberwolfach Reelle algebraische Geometrie week [12] the second author proposed:

Problem 1.1 Characterize the (semialgebraic) subsets of $\mathbb{R}^m$ that are either polynomial or regular images of $\mathbb{R}^n$.

A map $f := (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial map if its components $f_k \in \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ are polynomials. Analogously, $f$ is a regular map if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials $g_k, h_k \in \mathbb{R}[x]$ such that $h_k$ never vanishes on $\mathbb{R}^n$. A subset $S \subset \mathbb{R}^n$ is semialgebraic when it admits a description by a finite boolean combination of polynomial equalities and inequalities.

Open semialgebraic sets deserve a special attention in connection with the real Jacobian Conjecture [14]. In particular the second author stated in [12] the ‘open quadrant problem’:

Problem 1.2 Determine whether the open quadrant $\Omega := \{x > 0, y > 0\}$ of $\mathbb{R}^2$ is a polynomial image of $\mathbb{R}^2$.

This problem stimulated the interest of many specialists in the field. However, only after twelve years a first solution was found in [4] and presented by the first author in the 2002 Oberwolfach Reelle algebraische Geometrie week [2].

The open quadrant problem was the germ of a more systematic study of ‘Polynomial and regular images of Euclidean spaces’ developed by the authors during the last decade and which was the topic of the Ph.D. Thesis of the third author [16]. Since then we have worked on this issue with two main objectives:

- Finding obstructions to be an either polynomial or regular image.
- Proving (constructively) that large families of semialgebraic sets with piecewise linear boundary (convex polyhedra, their interiors, complements and the interiors of their complements) are either polynomial or regular images of some Euclidean space. The positive answer to the open quadrant problem has been a recurrent starting point for this approach.

In [4, 5] we presented the first steps to approach Problem 1.1. A complete solution to Problem 1.1 for the one-dimensional case appears in [3], whereas in [6, 8, 9, 17, 18] we approached constructive results concerning the representation as either polynomial or regular images of the semialgebraic sets with piecewise linear boundary commented
above. Articles [7, 10] are of different nature because we find in them new obstructions for a subset of $\mathbb{R}^n$ to be either a polynomial or a regular image of $\mathbb{R}^n$. In the first one we found some properties of the difference $\text{Cl}(S) \setminus S$ while in the second it is shown that the set of points at infinite of a polynomial image of $\mathbb{R}^n$ is a connected set.

The constructive solution to the open quadrant problem provided in [4] involves quite complicated computer calculations that the third author never liked. In fact he provided in his Ph.D. Thesis a different topological proof for the map proposed in [4], together with an algebraic proof involving a different polynomial map. This map has inspired the first and third authors for a short algebraic proof of the open quadrant problem involving a new polynomial map [11] and has led us to look for a polynomial map with optimal algebraic structure whose image is the open quadrant. It is important to establish clearly the meaning of ‘optimal algebraic structure’ [11, §3(A)]. It is natural to wonder how a polynomial map looks like when completely expanded and how it compares with other polynomial maps. We care about the total degree of the involved polynomial map (the sum of the degrees of its components) and its total number of (non-zero) monomials. We would like to find a polynomial map with the less possible total degree and the less possible number of monomials. The example in [4] has total degree 56 and its total number of monomials is 168. The polynomial map in [11] has total degree 72 and its total number of monomials is 350. In this work we will prove:

**Theorem 1.3** The open quadrant $Q$ is the image of the polynomial map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \ (x, y) \mapsto ((x^2y^4 + x^4y^2 - y^2 - 1)^2 + x^6y^4, (x^6y^2 + x^2y^2 - x^2 - 1)^2 + x^6y^2).$$

This polynomial map has total degree 28 and its total number of monomials is 22, which certainly improves the already known explicit solutions to the open quadrant problem. It has been constructed following a similar strategy to that in [4, §3]. Our experience approaching this problem suggests us that this map is surely close to have the optimal desired algebraic structure.

The article is organized as follows. In Section 2 we present all basic notions and topological preliminaries used in Section 3 to prove Theorem 1.3.

### 2. Topological preliminaries

Denote the closed disc of center the origin and radius $A > 0$ of the plane $\mathbb{R}^2$ with $D_A$. A warped disc is a subset $D_{A, \xi} := \{z = \xi(x, y), \ x^2 + y^2 \leq A^2\} \subset \mathbb{R}^3$ where $\xi : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Consider the homeomorphism

$$\zeta : \mathbb{R}^3 \to \mathbb{R}^2, \ (x, y, z) \mapsto (x, y, z - \xi(x, y))$$

that maps $D_{A, \xi}$ onto $D_A \times \{0\}$. The image of $D_{A, \xi}$ under a permutation of the variables of $\mathbb{R}^3$ will be also called a warped disc.
J. F. Fernando, J. M. Gamboa, C. Ueno

The open quadrant problem

Figure 1: The homeomorphism $\zeta(x, y) := \sqrt{B^2 - \min(y^2, B^2)}$ acting on $\mathbb{R}^3$.

For each $\varepsilon > 0$ consider the open neighborhood

$$
\mathcal{D}_A(\varepsilon) := \{x^2 + y^2 < (A + \varepsilon)^2\} \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3
$$

of $\mathcal{D}_A$. Clearly, $\mathcal{D}_{A, \varepsilon}(\varepsilon) := \zeta^{-1}(\mathcal{D}_A(\varepsilon))$ is an open neighborhood of $\mathcal{D}_{A, \varepsilon}$ in $\mathbb{R}^3$.

**Definition 2.1** A (continuous) path $\alpha : [a, b] \to \mathbb{R}^3$ meets transversally once the warped disc $\mathcal{D}_{A, \varepsilon}$ if there exist $s_0 \in (a, b)$ and $\varepsilon > 0$ such that $J := \alpha^{-1}(\mathcal{D}_{A, \varepsilon}(\varepsilon)) = (s_0 - \varepsilon, s_0 + \varepsilon)$ is an open subinterval of $[a, b]$ and $(\zeta \circ \alpha)|_J(t) = (0, 0, t - s_0)$.

**Remark 2.2** If the path $\alpha : [a, b] \to \mathbb{R}^3$ meets transversally once the warped disc $\mathcal{D}_{A, \varepsilon}$, then $\alpha([a, b]) \cap \partial \mathcal{D}_{A, \varepsilon} = \emptyset$.

Let $C$ be a topological space homeomorphic to a closed disc and let $\phi : C \to \mathbb{R}^3$ be a continuous map. The restriction $\partial \phi := \phi|_{\partial C}$ is called the boundary map of $\phi$. We say that the boundary map $\partial \phi$ meets transversally once a warped disc $\mathcal{D}_{A, \varepsilon} \subset \mathbb{R}^3$ if there exists a parameterization $\beta$ of $\partial C$ such that $\alpha := \phi \circ \beta$ meets transversally once the warped disc $\mathcal{D}_{A, \varepsilon}$.
Given a path-connected topological space \( X \) and a point \( x_0 \in X \) we denote the fundamental group of \( X \) at the base point \( x_0 \) with \( \pi_1(X, x_0) \). Each path \( \alpha \) starting and ending at \( x_0 \) is called a loop with base point \( x_0 \) and represents an element of \( \pi_1(X, x_0) \), that we denote with \([\alpha]\).

**Lemma 2.3** Let \( D_{A, \varepsilon} \) be a warped disc of \( \mathbb{R}^3 \) and let \( X := \mathbb{R}^3 \setminus \partial D_{A, \varepsilon} \). Let \( \alpha : [a, b] \to X \) be a loop with base point \( x_0 \in X \) that meets transversally once \( D_{A, \varepsilon} \). Then \( [\alpha] \) is a generator of \( \pi_1(X, x_0) \cong \mathbb{Z} \).

**Proof.** Keep the notations introduced above. Let \( s_0 \in (a, b) \) and \( \varepsilon > 0 \) be such that

\[
J := \alpha^{-1}(D_{A, \varepsilon}(\varepsilon)) = (s_0 - \varepsilon, s_0 + \varepsilon)
\]

is an open subinterval of \([a, b]\) and \((\zeta \circ \alpha)|_J(t) = (0, 0, t - s_0)\). After a reparameterization of \( \alpha \) we may assume \( s_0 = 0 \).

As \( \zeta \) is a homeomorphism of \( \mathbb{R}^3 \), we will prove the statement for \( \beta := \zeta \circ \alpha \), \( Y := \mathbb{R}^3 \setminus \partial D_A \) and the base point \( y_0 := \beta(-\varepsilon) = (0, 0, -\varepsilon) \). Consider the path \( \gamma : [0, 1] \to \mathbb{R}^3 \) given by

\[
\gamma(t) := \begin{cases} 
(3(A + \varepsilon)t, 0, \varepsilon) & \text{if } 0 \leq t \leq \frac{1}{3}, \\
(A + \varepsilon, 0, \varepsilon - (t - \frac{1}{3})6\varepsilon) & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\
(A + \varepsilon - 3(A + \varepsilon)(t - \frac{2}{3}), 0, -\varepsilon) & \text{if } \frac{2}{3} < t \leq 1.
\end{cases}
\]

![Figure 2: The path \( \beta \) meets transversally once the disk \( D_A \).](image-url)
Write $\beta_0 := \beta|_{I}$ and $\beta_1 := \beta|_{[a, b]} \ast \beta|_{[a, -\varepsilon]}$. We claim:

$$[\beta] = [\beta_0 \ast \beta_1] = [\beta_0 \ast \gamma] \cdot [\gamma^{-1} \ast \beta_1] = g \cdot e = g,$$

where $e$ and $g$ are respectively the identity element and a generator of $\pi_1(Y, y_0) \cong \mathbb{Z}$.

The loop $\gamma^{-1} \ast \beta_1$ with base point $y_0$ is contained in $\mathbb{R}^3 \setminus \mathbb{D}_A$, which is a simply connected space. Consequently, $[\gamma^{-1} \ast \beta_1] = e$ in $\pi_1(Y, y_0)$.

The class $[\beta_0 \ast \gamma]$ generates $\pi_1(Y, y_0)$. Indeed, $Y$ has as deformation retract the set $Z := \partial \mathbb{D}_A(\varepsilon) \cup I_\varepsilon$ where $I_\varepsilon := \{(0,0)\} \times \{-\varepsilon \leq z \leq \varepsilon\}$. It is an exercise of algebraic topology to show that $[\beta_0 \ast \gamma]$ is a generator of $\pi_1(Z, y_0) \cong \pi_1(Y, y_0) \cong \mathbb{Z}$, as required.

\begin{lemma}
Let $\phi : C \to X$ be a continuous map and assume that $C$ is homeomorphic to a closed disc. Let $\beta : [a, b] \to \partial C$ be a parameterization starting and ending at $z_0 \in \partial C$. Then $[\phi \circ \beta]$ is the identity element of $\pi_1(X, \phi(z_0))$.
\end{lemma}

\begin{proof}
Let $\psi : C \to \{x^2 + y^2 \leq 1\}$ be a homoeomorphism. The continuous map

$$H : [0, 1] \times [a, b] \to X, \quad (\rho, t) \mapsto (\phi \circ \psi^{-1})(\rho \cdot (\psi \circ \beta)(t) + (1 - \rho) \cdot \psi(z_0))$$

is a homotopy map between $\phi \circ \beta$ and the constant path, as required.
\end{proof}

\begin{proposition}
Let $C$ be a topological space homeomorphic to a closed disc and $\phi : C \to \mathbb{R}^3$ a continuous map. Assume $\partial \phi : \partial C \to \mathbb{R}^3$ meets transversally once a warped disc $D \subset \mathbb{R}^3$. Then $\partial D \cap \phi(\text{Int}(C)) \neq \emptyset$.
\end{proposition}

\begin{proof}
Assume by contradiction $\partial D \cap \phi(\text{Int}(C)) = \emptyset$. As $\partial \phi$ meets transversally once $D$, the image $\phi(\partial C)$ does not intersect $\partial D$ by Remark 2.2. Thus, $\phi(C) \subset X := \mathbb{R}^3 \setminus \partial D$. Let $\beta : [a, b] \to \partial C$ be a parameterization starting and ending at $z_0 \in \partial C$ such that $\phi \circ \beta$ meets transversally once $D$. By Lemma 2.4 the class $[\phi \circ \beta]$ is the identity element of $\pi_1(X, \phi(z_0))$. However, by Lemma 2.3 the class $[\phi \circ \beta]$ is a generator of $\pi_1(X, \phi(z_0)) \cong \mathbb{Z}$, which is a contradiction. Consequently, $\partial D \cap \phi(\text{Int}(C)) \neq \emptyset$, as required.
\end{proof}

3. Proof of Theorem 1.3

Observe first that the map $f$ in the statement of Theorem 1.3 is the composition $f_2 \circ f_1$ of the polynomial maps

\begin{align*}
f_1 : \mathbb{R}^2 &\to \mathbb{R}^2, \quad (x, y) \mapsto (x^2, y^2), \\
f_2 : \mathbb{R}^2 &\to \mathbb{R}^2, \quad (x, y) \mapsto ((xy^2 + x^2 y - y - 1)^2 + x^3 y^2, (x^3 y + xy - x - 1)^2 + x^3 y^2).
\end{align*}

As $f_1(\mathbb{R}^2)$ is the closed quadrant $\overline{Q} := \{x \geq 0, y \geq 0\}$, we have to prove the equality

$$f_2(\overline{Q}) = \Omega. \quad (3.1)$$


The inclusion $f_2(\overline{Q}) \subset Q$ is straightforward because both components of $f_2$ are strictly positive on $\overline{Q}$. It only remains to show the inclusion
\[ Q \subset f_2(\overline{Q}). \] (3.2)

### 3.1. Reduction of the proof of inclusion (3.2)

Consider the (continuous) semialgebraic maps
\[
g : \overline{Q} \to \mathbb{R}^3, \quad (x, y) \mapsto (xy^2 + x^2y - y - 1, x^{3/2}y, x^3y + xy - x - 1),
\]
\[
h : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x^2 + y^2, y^2 + z^2).
\]

As $f_2 = h \circ g$, we have to show that for each tuple $(A^2, B^2) \in Q$ there exists $(x_0, y_0) \in \overline{Q}$ such that $(h \circ g)(x_0, y_0) = (A^2, B^2)$. This is equivalent to check that the intersection $h^{-1}((\{A^2, B^2\})) \cap g(\overline{Q})$ is non-empty.

Denote $S := g(\overline{Q})$ and fix values $B \geq A > 0$. It holds that sets
\[
h^{-1}(\{A^2, B^2\}) = \{x^2 + y^2 = A^2, y^2 + z^2 = B^2\},
\]
\[
h^{-1}(\{B^2, A^2\}) = \{y^2 + z^2 = A^2, x^2 + y^2 = B^2\}
\]
contain respectively the boundaries of the warped discs
\[
D_1 : z = \xi_1(x, y), \quad x^2 + y^2 \leq A^2, \tag{3.3}
\]
\[
D_2 : x = \xi_2(y, z), \quad y^2 + z^2 \leq A^2, \tag{3.4}
\]
for the (continuous) semialgebraic functions
\[
\xi_1 : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \sqrt{B^2 - \min\{y^2, B^2\}}, \tag{3.5}
\]
\[
\xi_2 : \mathbb{R}^2 \to \mathbb{R}, \quad (y, z) \mapsto \sqrt{B^2 - \min\{y^2, B^2\}}. \tag{3.6}
\]

Consequently, we are reduced to prove:

#### 3.1.1. For fixed values $B \geq A > 0$ the intersections $\partial D_1 \cap S$ and $\partial D_2 \cap S$ are non-empty.

### 3.2. Proof of Statement 3.1.1

Write $\mathcal{R} := [0, +\infty) \times (0, \frac{\pi}{2})$ and $\overline{\mathcal{R}} := [0, +\infty) \times [0, \frac{\pi}{2}]$. Consider the map $\phi := (\phi_1, \phi_2, \phi_3) : \mathbb{R}^2 \to \mathbb{R}^3$ where
\[
\phi_1(\rho, \theta) := \cos \theta \sin \theta (\cos \theta - \sin \theta)^2 + \rho(2 \cos^4 \theta \sin \theta + \cos \theta \sin^4 \theta + \cos^5 \theta) + \rho^2 \cos^5 \theta \sin \theta,
\]
\[
\phi_2(\rho, \theta) := \sqrt{\cos \theta \sin \theta (\cos \theta + \sin \theta + \rho \cos \theta \sin \theta)},
\]
\[
\phi_3(\rho, \theta) := \rho \sin \theta.
\]

Let us prove now some properties of the map $\phi$ and the sets $\mathcal{R}$ and $\overline{\mathcal{R}}$:
\[3.2.1. \phi(\mathbb{R}) \subset S.\]

Proof. The analytic map
\[
\psi : \mathbb{R} \rightarrow \Omega, \quad (\rho, \theta) \mapsto \left( \frac{\sin \theta}{\cos \theta}, \frac{(\cos \theta + \sin \theta + \rho \cos \theta \sin \theta) \cos^2 \theta}{\sin \theta} \right),
\]
satisfies \(\psi(\mathbb{R}) \subseteq \mathcal{R}\) and \(g \circ \psi = \phi|_{\mathbb{R}}\). Consequently, \(\phi(\mathbb{R}) \subset S\), as required.

\[3.2.2. \text{The inequality } \phi_1^2(\rho, \theta) + \phi_3^2(\rho, \theta) \geq \frac{\rho^2}{4} \text{ holds for each } (\rho, \theta) \in \mathcal{R}. \text{ Consequently, } \]
\[
\text{dist}(\phi(\rho, \theta), 0) \geq \frac{\rho}{2} \tag{3.7}
\]
for each \((\rho, \theta) \in \mathcal{R}\).

Proof. As \(\rho, \cos \theta, \sin \theta \geq 0\) on \(\mathcal{R}\), we have
\[
\phi_1(\rho, \theta) = \rho \cos \theta(\cos^4 \theta + \sin^4 \theta) = \rho \cos \theta(1 - 2 \cos^2 \theta \sin^2 \theta)
\]
\[
= \rho \cos \theta \left(1 - \frac{\sin^2(2\theta)}{2}\right) \geq \frac{\rho}{2} \cos \theta.
\]
In addition, \(\phi_3(\rho, \theta) = \rho \sin \theta \geq \frac{\rho}{2} \sin \theta\), so
\[
\phi_1^2(\rho, \theta) + \phi_3^2(\rho, \theta) \geq \frac{\rho^2}{4} \cos^2 \theta + \frac{\rho^2}{4} \sin^2 \theta = \frac{\rho^2}{4},
\]
as required.

\[3.2.3. \text{The map } \phi \text{ satisfies } \phi(0, \theta) = \phi(0, \frac{\pi}{2} - \theta) \text{ for } \theta \in [0, \frac{\pi}{2}]. \text{ Fix } M > 0 \text{ and consider the rectangle } \mathcal{R}_M := [0, M] \times [0, \frac{\pi}{2}]. \text{ Denote } \phi_M := \phi|_{\mathcal{R}_M}. \text{ Identify the points } (0, \theta) \text{ and } (0, \frac{\pi}{2} - \theta) \text{ for } \theta \in [0, \frac{\pi}{2}] \text{ and endow the quotient space } \mathcal{R}_M \text{ with the quotient topology. Observe that the interior } \text{Int}(\mathcal{R}_M) \text{ of } \mathcal{R}_M \text{ as a topological manifold with boundary is the quotient space } \tilde{\mathcal{R}}_M \text{ obtained identifying the points } (0, \theta) \text{ and } (0, \frac{\pi}{2} - \theta) \text{ of } \mathcal{R}_M := [0, M] \times (0, \frac{\pi}{2}), \text{ where } \theta \in (0, \frac{\pi}{2}).
\]

The canonical projection \(\pi_M : \tilde{\mathcal{R}}_M \rightarrow \mathcal{R}_M\) is continuous. As \(\phi_M\) is compatible with \(\pi_M\), there exists a continuous map \(\phi_M : \mathcal{R}_M \rightarrow \mathbb{R}^3\) such that the following diagram is commutative. In addition, \(\phi_M(\mathcal{R}_M) = \phi(\mathcal{R}_M) \subset S\).
3.2.4. $\overline{R}_M$ is homeomorphic to a disc and its boundary is the set

$$\pi_M(\{\rho = M\} \cup \{\theta = 0\} \cup \{\theta = \frac{\pi}{2}\}).$$

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$ (interchanging the order of the variables $(\rho, \theta) \rightarrow (\theta, \rho)$) and consider the continuous map

$$\mu : \mathbb{C} \rightarrow \mathbb{C}, \ z := \theta + \sqrt{-1}\rho \mapsto w := u + \sqrt{-1}v = (\frac{1}{2}z - 1)^2.$$  

The restriction $\mu|_{\{\rho > 0\}} : \{\rho > 0\} \rightarrow \mathbb{C} \setminus ([0, +\infty) \times \{0\})$ is a homeomorphism and the image of $\overline{R}_M \setminus \{\rho = 0\}$ is

$$\mathcal{T}_M := \{(u, v) \in \mathbb{R}^2 : (\frac{uv}{\pi M})^2 - (\frac{4M}{\pi})^2 \leq u \leq 1 - (\frac{v}{2})^2 \} \setminus ([0, 1] \times \{0\}).$$

The closure $\overline{\mathcal{T}}_M$ of $\mathcal{T}_M$ is a compact convex set (as it is a closed bounded intersection of two convex sets). By [1, Cor.11.3.4] $\overline{\mathcal{T}}_M$ is homeomorphic to a closed disc. In addition

$$\mu|_{\{\rho = 0\}} : \{\rho = 0\} \rightarrow [0, +\infty) \times \{0\}, \ \theta \mapsto (\frac{\pi}{2}\theta - 1)^2$$

transforms the segment $[0, \frac{\pi}{2}] \times \{0\}$ onto the interval $[0, 1]$. The preimage of $t_0 \in [0, 1]$ under $\mu|_{\{\rho = 0\}}$ is

$$\{\theta_1 := \frac{\pi}{4}(1 + \sqrt{t_0}), \theta_2 := \frac{\pi}{4}(1 - \sqrt{t_0})\}.$$  

As $\theta_1 = \frac{\pi}{2} - \theta_2$, the map $\lambda := \mu|_{\overline{R}_M} : \overline{R}_M \rightarrow \overline{\mathcal{T}}_M$ factors through $\overline{\mathcal{T}}_M$ and there exists a continuous map $\overline{\lambda} : \overline{R}_M \rightarrow \overline{\mathcal{T}}_M$ such that the following diagram is commutative.

Figure 3: Left and right views of $\phi_M(\mathcal{R}_M) \subset S$.  

345
The map $\tilde{\lambda}$ is continuous and bijective and it maps the compact set $\overline{R}_M$ onto the Hausdorff space $\overline{T}_M$, so it is a homeomorphism. Consequently, $\overline{R}_M$ is homeomorphic to a disc and its boundary is $\pi_M(\{\rho = M\} \cup \{\theta = 0\} \cup \{\theta = \frac{\pi}{2}\})$, as required.

3.2.5. Fix $B \geq A > 0$ and consider the warped discs $D_1$ and $D_2$ introduced in (3.3) and (3.4). Then there exists $M > 0$ such that the boundary map $\partial\Phi_M : \partial\overline{R}_M \to \mathbb{R}^3$ meets transversally once both discs $D_1$ and $D_2$.

Proof. As $D_1$ and $D_2$ are bounded set, there exists $M_0 > 0$ such that $D_1 \cup D_2 \subset \{\|(x, y, z)\| < M_0\}$. Take $M := 4M_0$ and consider the set $\overline{R}_M$ and the continuous map $\tilde{\Phi}_M$ introduced in paragraph 3.2.3.

We claim: the boundary map $\partial\tilde{\Phi}_M : \partial\overline{R}_M \to \mathbb{R}^3$ meets transversally once $D_1$. 

Figure 4: Behavior of the map $\mu : \overline{R}_M \to \overline{T}_M$. 

\[ \begin{align*}
\overline{R}_M &\xrightarrow{\pi_M|_{\overline{R}_M}} \overline{R}_M \\
\overline{R}_M &\xrightarrow{\lambda} \overline{S}_M
\end{align*} \]
Consider the parameterization of $\partial \tilde{R}_M$ given by
\[
\beta_1(t) := \begin{cases} 
\pi_M(t, \frac{M}{2}), & \text{if } 0 \leq t \leq M, \\
\pi_M(M, M + \frac{M}{2} - t), & \text{if } M < t \leq M + \frac{M}{2}, \\
\pi_M(2M + \frac{M}{2} - t, 0), & \text{if } M + \frac{M}{2} < t \leq 2M + \frac{M}{2}.
\end{cases}
\]
We have
\[
\alpha_1(t) := \tilde{\phi}_M \circ \beta_1(t) = \begin{cases} 
\phi(t, \frac{M}{2}), & \text{if } 0 \leq t \leq M, \\
\phi(M, M + \frac{M}{2} - t), & \text{if } M < t \leq M + \frac{M}{2}, \\
\phi(2M + \frac{M}{2} - t, 0), & \text{if } M + \frac{M}{2} < t \leq 2M + \frac{M}{2}.
\end{cases}
\]
Choose $0 < \varepsilon < \min\{B, M_0 - B\}$ and consider the homeomorphism
\[
\zeta_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \ (x, y, z) \mapsto (x, y, z - \xi_1(x, y)),
\]
where $\xi_1$ is the (continuous) semialgebraic function introduced in (3.5). Denote $D_1(\varepsilon) := \zeta_1^{-1}(D_A(\varepsilon))$. It is enough to check:
\[
\alpha_1^{-1}(D_1(\varepsilon)) = (B - \varepsilon, B + \varepsilon).
\]
Pick $p_0 := \alpha_1(t_0) \in \text{Im}(\alpha_1)$. We distinguish three cases:

(i) If $0 \leq t_0 \leq M$, then $\zeta_1(p_0) = (\zeta_1 \circ \phi)(t_0, 0) = (0, 0, t_0 - B)$. Consequently, $\zeta_1(p_0) \in D_A(\varepsilon)$ if and only if $-B < -\varepsilon < t_0 - B < \varepsilon < M - B$. 

Figure 5: The boundary map $\partial \tilde{R}_M : \partial \tilde{R}_M \rightarrow \mathbb{R}^3$ meets transversally once $D_1$. 

(ii) If \( M < t_0 \leq M + \frac{\pi}{2} \), we have by (3.7)

\[
\text{dist}(p_0, 0) \geq \frac{M}{2} = 2M_0 > \sqrt{2}M_0 > \text{dist}(q, 0)
\]

for each \( q \in D_1(\varepsilon) \). Therefore \( p_0 \not\in D_1(\varepsilon) \).

(iii) If \( M + \frac{\pi}{2} < t_0 \leq 2M + \frac{\pi}{2} \), then

\[
p_0 = \alpha_1(t_0) = \phi(2M + \frac{\pi}{2} - t_0, 0) = (2M + \frac{\pi}{2} - t_0, 0, 0),
\]

so \( \zeta_1(p_0) = (2M + \frac{\pi}{2} - t_0, 0, -B) \). As \( \varepsilon < B \), it holds \( \zeta_1(p_0) \not\in D_A(\varepsilon) \), so \( p_0 \not\in D_1(\varepsilon) \).

We conclude \( \alpha_{-1}^{-1}(D_1(\varepsilon)) = (B - \varepsilon, B + \varepsilon) \), so \( \alpha_1 \) meets transversally once \( D_1 \).

Analogously one shows: the boundary map \( \partial\bar{\phi}_M : \partial\bar{\mathbb{R}}M \to \mathbb{R}^3 \) meets transversally once \( D_2 \).

Consider in this case the parameterization of \( \partial\bar{\mathbb{R}}M \) given by

\[
\beta_2(t) := \begin{cases} 
\pi_M(t, 0), & \text{if } 0 \leq t \leq M, \\
\pi_M(M, t - M), & \text{if } M < t \leq M + \frac{\pi}{2}, \\
\pi_M(2M + \frac{\pi}{2} - t, \frac{\pi}{2}), & \text{if } M + \frac{\pi}{2} < t \leq 2M + \frac{\pi}{2}.
\end{cases}
\]

We have

\[
\alpha_2(t) := \bar{\phi}_M \circ \beta_2(t) = \begin{cases} 
\phi(t, 0), & \text{if } 0 \leq t \leq M, \\
\phi(M, t - M), & \text{if } M < t \leq M + \frac{\pi}{2}, \\
\phi(2M + \frac{\pi}{2} - t, \frac{\pi}{2}), & \text{if } M + \frac{\pi}{2} < t \leq 2M + \frac{\pi}{2}.
\end{cases}
\]
Proceed as above keeping the same values for $A$ and $\varepsilon$ and using in this case the homeomorphism

$$\xi_2 : \mathbb{R}^3 \to \mathbb{R}^3, \ (x,y,z) \mapsto (z,y,x - \xi_2(z,y)),$$

where $\xi_2$ is the (continuous) semialgebraic function introduced in (3.6), to prove that $\alpha_2$ meets transversally once the warped disk $D_2$.

3.2.6. By 3.2.4 $\overline{M}$ is homeomorphic to a closed disc. By Proposition 2.5 applied to the continuous map $\tilde{\phi}_M : \overline{M} \to \mathbb{R}^3$ and 3.2.5, we deduce that the boundaries of both warped discs $D_1$ and $D_2$ meet $\phi_M(\mathbb{R}^3) \cap S$. Thus, 3.1.1 holds, as required. □

References


