A map \( f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^m \) is polynomial if its components \( f_i \) are polynomials. Analogously, \( f \) is regular if its components can be represented as quotients \( f_i = \frac{g_i}{h_i} \) of two polynomials \( g_i, h_i \) such that \( h_i \) never vanishes on \( \mathbb{R}^n \). By Tarski-Seidenberg’s principle the image of an either polynomial or regular map is a semialgebraic set, that is, it has a description by a finite boolean combination of polynomial equalities and inequalities. In 1990 Oberwolfach talk algebraische Geometrie week Gamboa proposed:

**Main Problem.** Characterize the semialgebraic sets in \( \mathbb{R}^n \) which are either polynomial or regular images of some \( \mathbb{R}^m \).

Two approaches to this problem: (1) **Explicit construction** of polynomial and regular representations for large families of semialgebraic sets, so far with piecewise linear boundary, and (2) **Search for obstructions** to be polynomial/regular images of \( \mathbb{R}^n \). Potential applications. Optimization. Positivstellens"atze or parametrizations of semialgebraic sets.

### The Open Quadrant Problem

Is the set \( Q = \{ x > 0, y > 0 \} \subset \mathbb{R}^2 \) a polynomial image of \( \mathbb{R}^2 \)? Answer: **YES**

**First solution.** The initial answer was presented in 2002 Oberwolfach talk algebraische Geometrie week. Required computer assistance for Sturm’s algorithm.

**Second solution.** The shortest proof (sketched below).

\[ f(x,y) = (x^2 - y^2 + x^2 - y^2 + x^2 - y^2) \]

**Third solution.** The sparsest (known) polynomial map. A topological argument shows that the image of the map below is \( Q \).

\[ f(x,y) = ((x^2 + x^2 - y^2 - 1)^2 + x^2)^{2n} + (x^2)^{2n} - x^2 - 1)^2 + x^2 \]

### On Convex Polyhedra

**Theorem 1.** An \( n \)-dimensional convex polyhedron and its interior are regular images of \( \mathbb{R}^n \) (\( n \geq 2 \)).

**Definition.** Let \( \mathcal{X} \subset \mathbb{R}^n \) be a convex polyhedron. Its recession cone is 
\[ \mathcal{R}(\mathcal{X}) = \{ f \in \mathbb{R}^n : p + \lambda \vec{e} \in \mathcal{X}, \ \forall p \in \mathcal{X}, \ \lambda > 0 \}. \]

**Theorem 2.** Let \( \mathcal{X} \subset \mathbb{R}^n \) be an unbounded, \( n \)-dimensional convex polyhedron whose recession cone \( \mathcal{R}(\mathcal{X}) \) is \( n \)-dimensional. Then \( \mathcal{X} \) is a polynomial image of \( \mathbb{R}^n \). In addition, if \( \mathcal{X} \) has not bounded faces, then \( \text{Int}(\mathcal{X}) \) is also a polynomial image of \( \mathbb{R}^n \).

**Theorem 3.** Let \( \mathcal{X} \subset \mathbb{R}^n \) be an \( n \)-dimensional convex polyhedron that is not affinely equivalent to a layer \([a, a] \times \mathbb{R}^{n-1}\). Then the semialgebraic sets \( \mathcal{X} \setminus \text{Int}(\mathcal{X}) \) and \( \text{Int}(\mathcal{X}) \) are polynomial images of \( \mathbb{R}^n \).

**Full picture for convex polyhedra**

**Definition of \( p \) and \( t \) invariants:**
\[ p(\mathcal{X}) = \min\{n \in \mathbb{N} : \mathcal{X} = f(\mathbb{R}^n) \}, \ f(\text{polynomial}) \]
\[ t(\mathcal{X}) = \min\{n \in \mathbb{N} : \mathcal{X} = f(\mathbb{R}^n), \ f(\text{regular}) \} \]

### Related Problems

A map \( f : \mathbb{R}^n \to \mathbb{R}^m \) is **Nash** if each component of \( f \) is a Nash function. that is, a smooth function with semialgebraic graph. Let \( \mathcal{S} \subset \mathbb{R}^n \) be a semialgebraic set of dimension \( d \).

**Shiota’s conjecture.** \( \mathcal{S} \) is a Nash image of \( \mathbb{R}^d \) if and only if \( \mathcal{S} \) is pure dimensional and there exists an analytic path \( \alpha : [0, 1] \to \mathcal{S} \) whose image meets all connected components of the set of regular points of \( \mathcal{S} \).

**Corollary 8.** Assume \( \mathcal{S} \) is pure dimensional, irreducible and with arc-regularized closure. Then \( \mathcal{S} \) is a Nash image of \( \mathbb{R}^d \).

**Corollary 9.** Assume \( \mathcal{S} \) is Nash path connected. Then \( \mathcal{S} \) is the projection of an irreducible algebraic set \( X \subset \mathbb{R}^n \) whose connected components are Nash diffeomorphic to \( \mathbb{R}^d \). In addition, each connected component of \( X \) maps onto \( \mathcal{S} \).

### Characterization for the 1-Dimensional Case

Let \( \mathcal{S} \subset \mathbb{R}^n \) be a 1-dimensional semialgebraic set.

**Theorem 6.** The following assertions are equivalent:
(i) \( \mathcal{S} \) is a polynomial image of \( \mathbb{R}^n \) for some \( n \geq 1 \).
(ii) \( \mathcal{S} \) is irreducible, unbounded and \( Cl_{\text{zar}}(\mathcal{S}) \) is an invariant rational curve such that \( Cl_{\text{zar}}(\mathcal{S}) \cap H_0(\mathcal{S}) = \{ p \} \) and the germ \( Cl_{\text{zar}}(\mathcal{S}) \) is irreducible.

If that is the case, \( p(\mathcal{S}) \leq 2 \). In addition, \( p(\mathcal{S}) = 1 \iff \mathcal{S} \) is closed in \( \mathbb{R}^n \).

**Theorem 7.** The following assertions are equivalent:
(i) \( \mathcal{S} \) is a regular image of \( \mathbb{R}^n \) for some \( n \geq 1 \).
(ii) \( \mathcal{S} \) is irreducible and \( Cl_{\text{zar}}(\mathcal{S}) \) is a rational curve.

If that is the case, then \( t(\mathcal{S}) \leq 2 \). In addition, \( t(\mathcal{S}) = 1 \iff \text{either } Cl_{\text{zar}}(\mathcal{S}) = \mathcal{S} \) or \( Cl_{\text{zar}}(\mathcal{S}) \setminus \mathcal{S} = \{ p \} \) and the analytic closure of the germ \( \mathcal{S} \) is irreducible.

**Tables:**

<table>
<thead>
<tr>
<th>( \mathcal{S} )</th>
<th>( R ) or ( 0, +\infty )</th>
<th>( \mathbb{R} )</th>
<th>( [0, +\infty) )</th>
<th>( (0, +\infty) )</th>
<th>( (0, 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(\mathcal{S}) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( t(\mathcal{S}) )</td>
<td>2</td>
<td>+\infty</td>
<td>2</td>
<td>+\infty</td>
<td>+\infty</td>
</tr>
</tbody>
</table>

### Relevant Papers

Selected References


**MEGA 2015**

**Title:** Polynomial and Regular Images of \( \mathbb{R}^n \)

**Authors:** José F. Fernández, Carlos Ueno

**Affiliation:** Universidad Complutense de Madrid • Università di Pisa