From the Boltzmann equation to the incompressible Navier-Stokes equations

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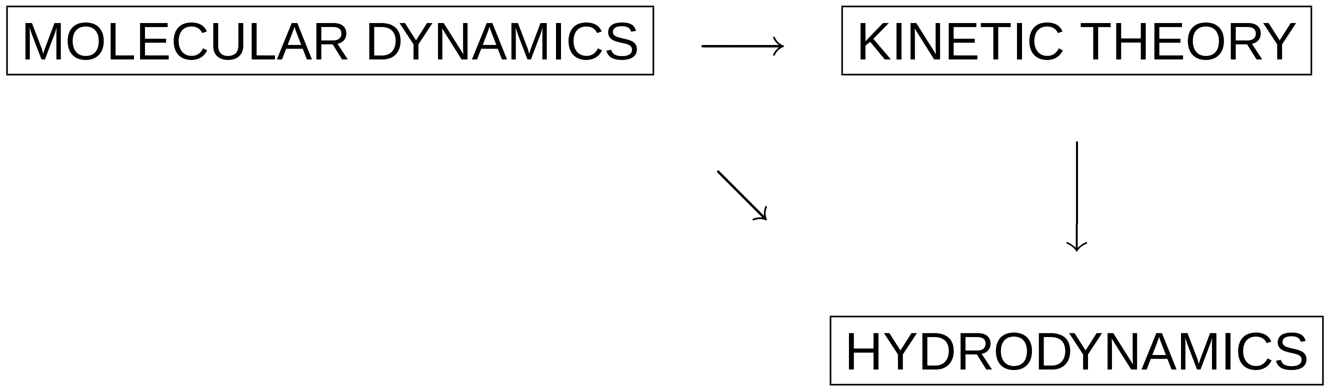
Hydrodynamic limits: origins of the problem

•In his 1866 paper on the kinetic theory of gases, Maxwell explained how the viscosity of a monatomic gas can be computed in terms of data at the molecular scale (scattering cross-section and diameter of the molecules) as well as macroscopic data (the pressure and temperature in the gas).

•Hilbert's 6th problem (1900):" [. . .] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of *continua*"



- Derivation of the Boltzmann equation from molecular dynamics on short time intervals by O.E. Lanford (1975)

- "Formal" derivations of hydrodynamics from molecular dynamics by C.B. Morrey (1951)

- Rigorous results for stochastic models of molecular dynamics on short time intervals by S. Olla, S.R.S. Varadhan and H.T. Yau (1993)

In this talk, we discuss the derivation of the Navier-Stokes equations for incompressible flows from the Boltzmann equation

•In the case of the Boltzmann and the 3D Navier-Stokes equations, only weak solutions are known to exist globally, and for initial data of arbitrary size

$\Rightarrow$ seek weak stability argument — i.e. a compactness result in the weak topology of some appropriate function space.

The Boltzmann equation for a hard sphere gas

•Unknown: the number density $F \equiv F(t, x, v) \geq 0$ in the 1-particle phase space

•In the absence of external forces (electromagnetic force, gravity...) the number density F satisfies

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F)$$

where $\mathcal{C}(F)$ is the Boltzmann collision integral

•Collisions other than binary are neglected; besides, these collisions are viewed as instantaneous and purely local (molecular radius $\simeq 0$)

$\mathcal{C}$ is a bilinear operator acting only on the v variable in F

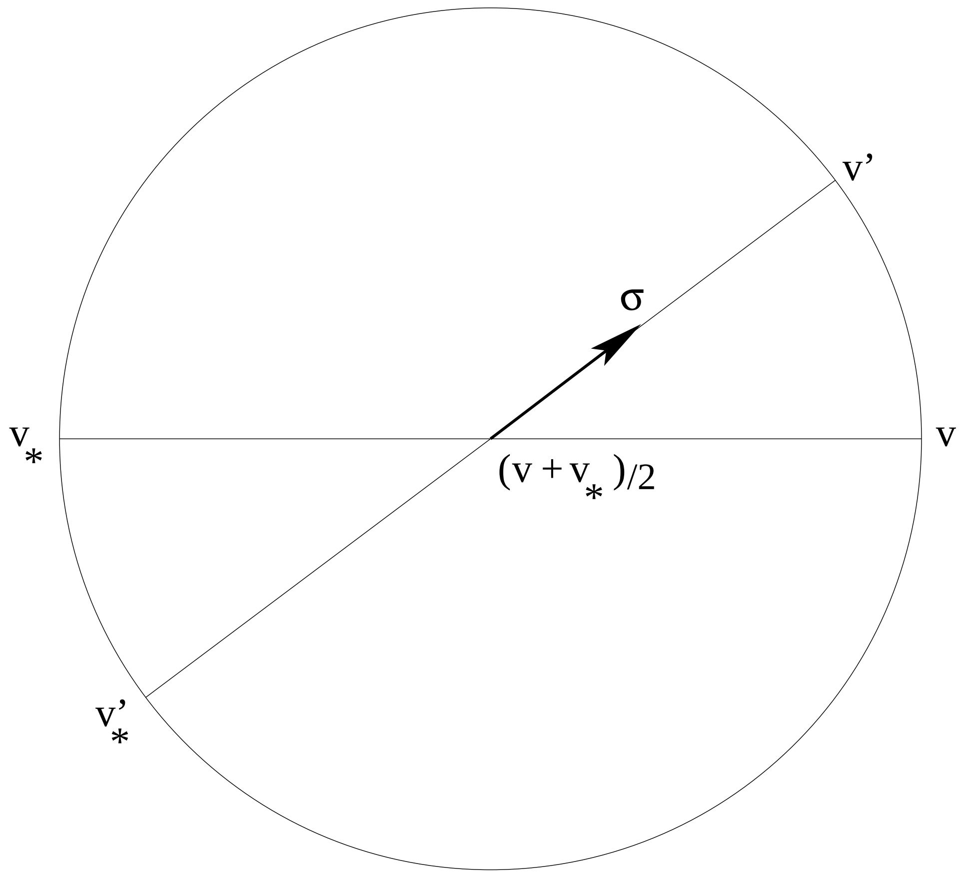
The Boltzmann collision integral

- For a hard sphere gas, the collision integral is

$$\mathcal{C}(F)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (F(v')F(v'_*) - F(v)F(v_*))|v - v_*| dv_* d\sigma$$

where the velocities v' and v'_* are defined in terms of $v, v_* \in \mathbf{R}^3$ and $\sigma \in \mathbf{S}^2$ by

$$v' \equiv v'(v, v_*, \sigma) = \tfrac{1}{2}(v + v_*) + \tfrac{1}{2}|v - v_*|\sigma$$
$$v'_* \equiv v'_*(v, v_*, \sigma) = \tfrac{1}{2}(v + v_*) - \tfrac{1}{2}|v - v_*|\sigma$$



Geometric interpretation of binary elastic collisions
in the center of mass reference frame

Local conservation laws

•Assume that $F \equiv F(v) \geq 0$ a.e. is rapidly decaying; then

$$\int_{\mathbf{R}^3} \mathcal{C}(F) \begin{pmatrix} 1 \\ v_k \\ |v|^2 \end{pmatrix} dv = 0 \,, \qquad k = 1, 2, 3$$

In particular, any weak solution F of the Boltzmann equation that is rapidly decaying in v satisfies the following local conservation laws

$$\partial_t \int_{\mathbf{R}^3} F dv + \text{div}_x \int_{\mathbf{R}^3} F v dv = 0 \,, \qquad \text{(mass)}$$

$$\partial_t \int_{\mathbf{R}^3} F v dv + \text{div}_x \int_{\mathbf{R}^3} F v \otimes v dv = 0 \,, \qquad \text{(momentum)}$$

$$\partial_t \int_{\mathbf{R}^3} F \tfrac{1}{2}|v|^2 dv + \text{div}_x \int_{\mathbf{R}^3} F v \tfrac{1}{2}|v|^2 dv = 0 \,, \qquad \text{(energy)}$$

Boltzmann's H Theorem

•Assume that $F \equiv F(v) > 0$ a.e. is rapidly decaying and such that $\ln F$ has polynomial growth at infinity. Then, the local entropy production rate

$$R(F) = -\int_{\mathbf{R}^3} \mathcal{C}(F) \ln F dv \geq 0$$

•The following conditions are equivalent:

$$R(F) = 0 \;\Leftrightarrow\; \mathcal{C}(F) = 0 \text{ a.e. } \;\Leftrightarrow\; F \text{ is a Maxwellian}$$

i.e. there exists $\rho, \theta > 0$ and $u \in \mathbf{R}^3$ such that

$$F(v) = \mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \quad \text{a.e. in } v \in \mathbf{R}^3$$

Weakly nonlinear regimes

•In all the cases considered here, the nonlinearity at the kinetic level is weak — but not necessarily so at the hydrodynamic level. The number density is sought as a perturbation of some global equilibrium, which, by rescaling and Galilean invariance, is chosen to be

$$M = \mathcal{M}_{(1,0,1)} \qquad \text{(the centered, reduced Gaussian distribution)}$$

•The size of the number density fluctuations around the equilibrium state M is measured in terms of the relative entropy defined as

$$H(F|M) = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \left[F \ln\left(\frac{F}{M}\right) - F + M \right] dx dv \quad (\geq 0)$$

for each measurable $F \geq 0$ a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$

Global existence theory for the Boltzmann equation

•Consider the Boltzmann equation in $\mathbf{R}^3$ with Maxwellian equilibrium M at infinity:

$$\begin{aligned} \partial_t F + v \cdot \nabla_x F &= \mathcal{C}(F)\,, && (t,x,v) \in \mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3 \\ F(t,x,v) &\to M\,, && \text{as } |x| \to +\infty\,, \\ F\Big|_{t=0} &= F^{in}\,. \end{aligned}$$

•The convergence of F to M at infinity is supposed tight enough so that

$$H(F(t)|M) < +\infty \qquad \text{for all } t \geq 0\,.$$

•Observation: for each $r > 0$, one has

$$\iint_{|x|+|v|\leq r} \frac{\mathcal{C}(F)}{\sqrt{1+F}} dv dx \leq C_r \int_{|x|\leq r} \left(R(F) + \int_{\mathbf{R}^3} (1+|v|^2) F dv \right) dx$$

Definition. *A renormalized solution relative to M of the Boltzmann equation is a nonnegative $F \in C(\mathbf{R}_+, L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3))$ such that*

$$H(F(t)|M) < +\infty \quad \textit{and} \quad \Gamma'\left(\frac{F}{M}\right)\mathcal{C}(F) \in L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$$

and that satisfies

$$M(\partial_t + v \cdot \nabla_x)\Gamma\left(\frac{F}{M}\right) = \Gamma'\left(\frac{F}{M}\right)\mathcal{C}(F)$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3 \times \mathbf{R}^3$, for each $0 \le \Gamma \in C^1(\mathbf{R}_+)$ such that

$$\Gamma'(Z) \le \frac{1}{\sqrt{1+Z}}\,, \qquad Z \ge 0\,.$$

Theorem. (R. DiPerna - P.-L. Lions, Ann. Math. 1990) *For $F^{in} \geq 0$ a.e. measurable and s.t. $H(F^{in}|M) < +\infty$, there is a renormalized solution relative to M of the Boltzmann equation with initial data F^{in}. This solution satisfies the continuity equation and the local conservation of momentum up to the divergence of a Radon measure m with values in the cone of nonnegative symmetric matrices*

$$\partial_t \int_{\mathbf{R}^3} F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF dv = 0 \,,$$
$$\partial_t \int_{\mathbf{R}^3} vF dv + \mathrm{div}_x \int_{\mathbf{R}^3} v \otimes vF dv + \mathrm{div}_x m = 0 \,,$$

together with the following entropy inequality for each $t > 0$:

$$H(F(t)|M) + \int_{\mathbf{R}^3} \mathrm{trace}\, m(t) + \int_0^t \int_{\mathbf{R}^3} R(F)(s,x) dx ds \leq H(F^{in}|M) \,.$$

The incompressible Navier-Stokes limit

Theorem. (G.-St-Raymond, Invent. Math. 2004) *Let* $u^{in} \in L^2(\mathbf{R}^3)$ *and* $\theta^{in} \in L^\infty(\mathbf{R}^3)$ *be s.t.* $\operatorname{div}_x u^{in} = 0$*. Let* F_ϵ *be a renormalized solution relative to* M *of the Boltzmann equation with initial data*

$$F_\epsilon^{in}(x,v) = \mathcal{M}_{(1-\epsilon\theta^{in}(\epsilon x),\epsilon u^{in}(\epsilon x),1+\epsilon\theta^{in}(\epsilon x))}(v)$$

Then, in the limit as $\epsilon \to 0$

$$\frac{1}{\epsilon}\int_{\mathbf{R}^3}\left(F_\epsilon\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right) - M\right)\begin{pmatrix} v \\ (\frac{1}{3}|v|^2 - 1)\end{pmatrix} dv \to \begin{pmatrix} u(t,x) \\ \theta(t,x)\end{pmatrix}$$

weakly in $L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3)$ *modulo extraction of a subsequence, where* (u,θ) *is a "Leray solution" of the Navier-Stokes-Fourier system*

$$\partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu\Delta_x u\,, \qquad \operatorname{div}_x u = 0\,,$$
$$\tfrac{5}{2}(\partial_t\theta + \operatorname{div}_x(u\theta)) = \kappa\Delta_x\theta\,,$$

with initial data (u^{in},θ^{in})*.*

•The viscosity and heat conductivity are given by the formulas

$$\nu = \tfrac{1}{5}\mathcal{D}^*(v \otimes v - \tfrac{1}{3}|v|^2 I)\,, \qquad \kappa = \tfrac{2}{3}\mathcal{D}^*(\tfrac{1}{2}(|v|^2 - 5)v)$$

where $\mathcal{D}$ is the Dirichlet form of the collision integral linearized at M, i.e.

$$\mathcal{D}(\Phi) = \tfrac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi + \Phi_* - \Phi' - \Phi'_*|^2 |v - v_*| M M_* dv dv_* d\sigma\,.$$

•Recall the formula for the Legendre dual:

$$\mathcal{D}^*(\xi) = \sup_x (\xi \cdot x - \mathcal{D}(x))$$

•P.-L. Lions and N. Masmoudi (ARMA 2000) proved a version of the above theorem without deriving the heat equation for θ.

•A “Leray solution" of the Navier-Stokes-Fourier system is an element (u,θ) of $C(\mathbf{R}_+; w-L^2(\mathbf{R}^3))$ that solves the system in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$, satisfies the initial condition at $t=0$, and verifies the “energy inequality" for each $t>0$:

$$\frac{1}{2}\int_{\mathbf{R}^3}(|u|^2+\tfrac{5}{2}|\theta|^2)(t,x)dx+\int_0^t\int_{\mathbf{R}^3}(\nu|\nabla_x u|^2+\kappa|\nabla_x\theta|^2)dxds \\ \le \frac{1}{2}\int_{\mathbf{R}^3}(|u^{in}|^2+\tfrac{5}{2}|\theta^{in}|^2)(x)dx$$

•Program started by Bardos-G.-Levermore (CPAM 1993); partial results by Lions-Masmoudi (ARMA 2001); complete solution for the BGK model by L. Saint-Raymond (Ann. Scient. ENS 2003). For small data in the Navier-Stokes equations, convergence proof by Bardos-Ukai (M3AS 1991); short time convergence by DeMasi, Esposito and Lebowitz (CPAM 1990).

Main ideas in the proof

•Introduce the relative number density fluctuation g_ϵ:

$$g_\epsilon(t,x,v) = \frac{F_\epsilon\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}, v\right) - M(v)}{\epsilon M(v)}, \quad \text{where } M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}$$

•In terms of g_ϵ, the Boltzmann equation becomes

$$\epsilon\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L}g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon)$$

where the linearized collision operator $\mathcal{L}$ and $\mathcal{Q}$ are defined by

$$\mathcal{L}g = -M^{-1}D\mathcal{C}[M](Mg)\,, \qquad \mathcal{Q}(g,g) = \tfrac{1}{2}M^{-1}D^2\mathcal{C}[M](Mg, Mg)$$

Lemma. (Hilbert, Math. Ann. 1912) *The operator $\mathcal{L}$ is self-adjoint, Fredholm, unbounded on $L^2(\mathbf{R}^3; M dv)$ with* $\ker\mathcal{L} = \mathrm{span}\{1, v_1, v_2, v_3, |v|^2\}$

1. Asymptotic fluctuations

•Multiplying the Boltzmann equation by ϵ and letting $\epsilon \to 0$ suggests that

$$g_\epsilon \to g \quad \text{with } \mathcal{L}g = 0$$

By Hilbert's lemma, g is an infinitesimal Maxwellian, i.e. is of the form

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \tfrac{1}{2}\theta(t,x)(|v|^2 - 3)$$

Notice that g is parametrized by its own moments, since

$$\rho = \langle g \rangle\,, \quad u = \langle vg \rangle\,, \quad \text{and } \theta = \langle (\tfrac{1}{3}|v|^2 - 1)g \rangle$$

•<u>NOTATION:</u>

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M(v)\, dv$$

2. Local conservation laws

•We start from the local conservation of mass

$$\epsilon\partial_t\langle g_\epsilon\rangle + \mathrm{div}_x\langle v g_\epsilon\rangle = \frac{1}{\epsilon^2}\int_{\mathbf{R}^3}\mathcal{C}(F_\epsilon) = 0\,,$$
$$\text{and thus }\ \mathrm{div}_x\langle v g\rangle = \mathrm{div}_x\, u = 0$$

which is the incompressibility condition in the Navier-Stokes equations.

•Likewise, the local conservation of momentum together with entropy production controls entails

$$\partial_t\langle v g_\epsilon\rangle + \mathrm{div}_x\left(\langle v g_\epsilon\rangle \otimes \langle v g_\epsilon\rangle\right) - \nu\Delta_x\langle v g_\epsilon\rangle \to 0 \text{ modulo gradients}$$

which gives the Navier-Stokes motion equation in the limit as $\epsilon \to 0$.

•Indeed, denoting $A(v) = v \otimes v - \frac{1}{3}|v|^2 I$ (the traceless part of $v \otimes v$)

$$\partial_t \langle v g_\epsilon \rangle + \mathrm{div}_x \frac{1}{\epsilon}\langle A(v) g_\epsilon \rangle + \nabla_x \frac{1}{\epsilon}\langle \tfrac{1}{3}|v|^2 g_\epsilon \rangle = 0$$

•Fredholm's alternative $\Rightarrow A = \mathcal{L}\hat{A}$ for some $\hat{A} \perp \ker \mathcal{L}$; thus

$$\frac{1}{\epsilon}\langle A(v) g_\epsilon \rangle = \left\langle \hat{A}(v) \frac{1}{\epsilon}\mathcal{L} g_\epsilon \right\rangle = \langle \hat{A}\mathcal{Q}(g_\epsilon, g_\epsilon)\rangle - \langle \hat{A}(\epsilon\partial_t + v \cdot \nabla_x) g_\epsilon \rangle$$
$$\simeq A(\langle v g_\epsilon \rangle) - \nu D(\langle v g_\epsilon \rangle)$$

where $D(u) = \nabla u + (\nabla_x u)^T - \frac{2}{3}(\mathrm{div}_x\, u) I$ (the deformation tensor of u)

•The approximation $\simeq$ above comes from vanishing entropy production

•Difficulty: renormalized solutions of the Boltzmann equation are not known to satisfy the local conservation laws above.

•Therefore, we give up the exact local conservation laws and use instead conservation laws modulo defect terms, that are satisfied by renormalized solutions of the Boltzmann equation

$$\partial_t \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\epsilon}{M}\right) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} M dv + \operatorname{div}_x \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\epsilon}{M}\right) v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} M dv$$
$$= \int_{\mathbf{R}^3} \Gamma'\left(\frac{F_\epsilon}{M}\right) \mathcal{C}(F_\epsilon) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv \,.$$

In the weakly nonlinear hydrodynamic limits considered here, $F_\epsilon/M \longrightarrow 1$ at a speed that is controlled by the entropy production rate and the size of the number density fluctuations: hence one expects that the usual symmetries of the collision integral that imply the local conservation laws are recovered in the rhs. of the above equalities as the $\epsilon \longrightarrow 0$.

3. Compactness arguments

•The DiPerna-Lions entropy inequality gives *a priori* bounds on the number density fluctuations that are uniform in ϵ; therefore

$$(1+|v|^2)g_\epsilon \text{ is relatively compact in weak-}L^1_{loc}(\mathbf{R}_+; L^1(\mathbf{R}^3\times\mathbf{R}^3))$$

•Modulo extracting subsequences, for each $\phi = O(|v|^2)$ at infinity

$$\phi g_\epsilon \to \phi g \text{ weakly in } L^1_{loc}(\mathbf{R}_+; L^1(\mathbf{R}^3\times\mathbf{R}^3))$$

and this justifies passing to the limit in expressions that are linear in g_ϵ.

•It remains to pass to the limit in the nonlinear term, i.e. to justify that

$$\langle vg_\epsilon\rangle \otimes \langle vg_\epsilon\rangle \to \langle vg\rangle \otimes \langle vg\rangle \text{ as } \epsilon \to 0$$

and this requires a.e. pointwise, instead of weak convergence.

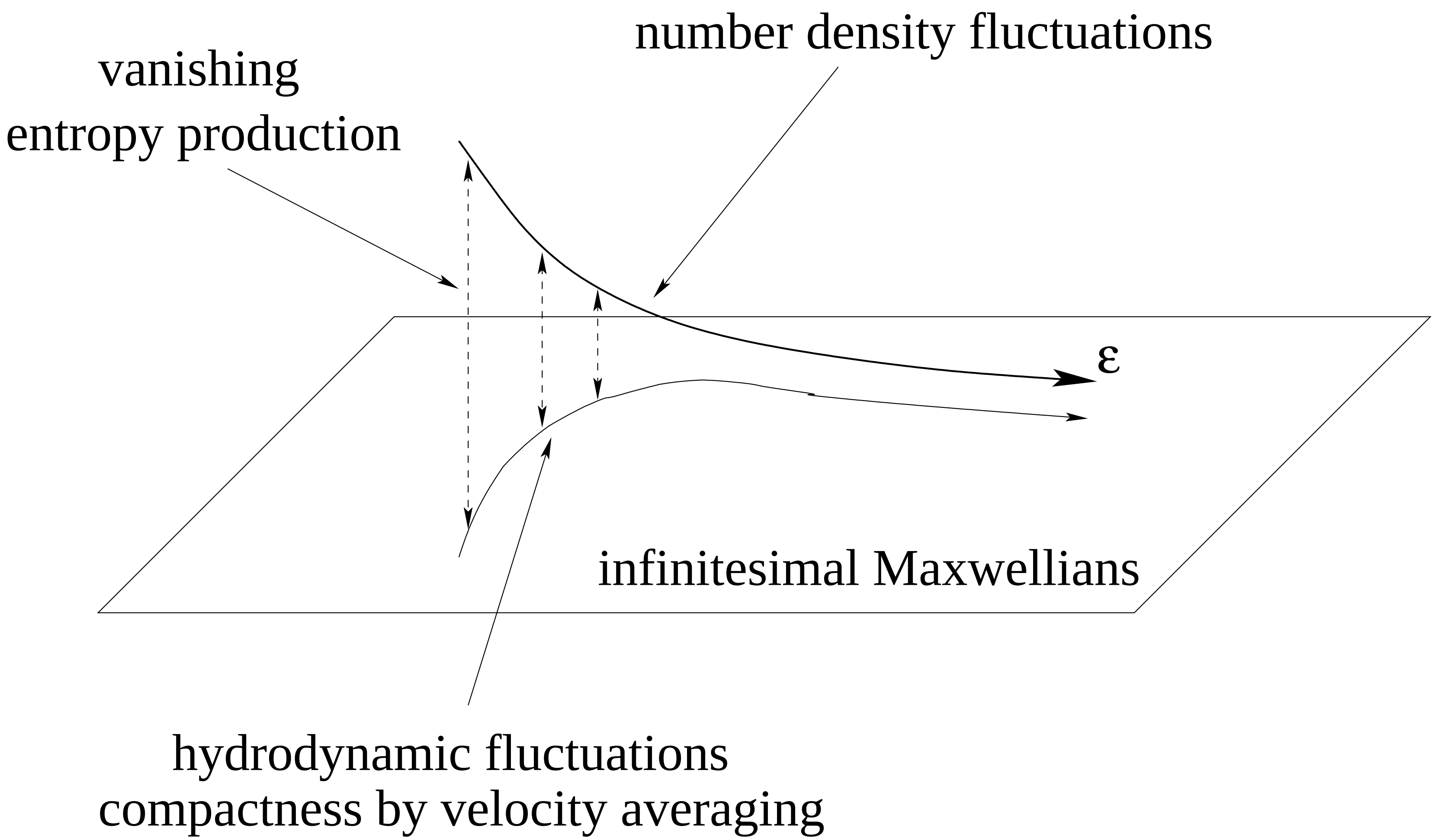
•This is done by using a “velocity averaging" lemma, a typical example of which (in a time-independent situation) is as follows:

Lemma. (F.G.-L. Saint-Raymond, C.R. Acad. Sci. 2002) *Let* $f_n \equiv f(x,v)$ *be a bounded sequence in* $L^1(\mathbf{R}^D_x; L^p(\mathbf{R}^D_v))$ *for some* $p > 1$ *such that the sequence* $v \cdot \nabla_x f_n$ *is bounded in* $L^1(\mathbf{R}^D \times \mathbf{R}^D)$*. Then*

•*the sequence* f_n *is weakly relatively compact in* $L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D)$*; and*

•*for each* $\phi \in C_c(\mathbf{R}^D)$*, the sequence of moments*

$$\int_{\mathbf{R}^D} f_n(x,v)\phi(v)dv \text{ is strongly relatively compact in } L^1_{loc}(\mathbf{R}^D)$$



Strategy for compactness

REMARKS ON VELOCITY AVERAGING:

• L^2-variant proved with Fourier techniques (small divisors involving the symbol of $v \cdot \nabla_x$) by F.G.-B. Perthame-R. Sentis (C.R. Acad. Sci. 1985)

• L^2-based Sobolev regularity of moments by F.G. - P.-L. Lions - B.P. - R.S. (J. Funct. Anal. 1988)

• $L^1_x(L^p_v)$ case: in physical space (instead of Fourier space), one sees that the group generated by $v \cdot \nabla_x$ exchanges x- and v- regularity for $t \neq 0$

$$e^{tv\cdot\nabla_x}\phi(x,v) = \phi(x+tv,v)$$

$\Rightarrow$ dispersion estimates “à la Strichartz”; conclude by interpolation using $t > 0$ as parameter.

Other hydrodynamic limits

•From the Boltzmann equation to the Euler equations for compressible flows: analogous to an infinite relaxation system

a) for smooth solutions, before onset of shock waves: see Nishida (Comm. Math. Phys. 1978), and Caflisch (Comm. Pure and Appl. Math. 1980)

b) acoustic limit, under sub-optimal scaling assumptions, done by F.G. - D. Levermore (Comm. Pure Appl. Math. 2002)

c) small BV solutions in the 1D case, “à la Glimm/Bressan”? major open problem, partial results obtained by T.P. Liu, H.S. Yu & T. Yang for the Riemann problem