How often should Louis and Peter be **simultaneously happy?**

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There is an obvious answer: they should be happy **all the time**.

I want to start by showing another mathematical paper with a title that is at least as odd as mine. I hope you agree that my title appears to be a bit more positive.

How Often Should You Beat Your Kids?

DON ZAGIER
University of Maryland
College Park, MD 20742

A result is proved which shows, roughly speaking, that one should beat one's kids every day except Sunday.

This note is a follow-up to the note "How to Beat Your Kids at Their Own Game," by K. Levasseur [1], in which the author proposes the following game to be played against one's two-year-old children: Starting with a deck consisting of n red cards and n black cards (in typical applications, $n = 26$), the cards are turned up one at a time, each player at each stage predicting the color of the card which is about to appear. The kid is supposed to guess "Red" or "Black" randomly with equal probability (this solves the problem of constructing a perfect random number generator), while you play what is obviously the optimal strategy—guessing randomly (or, if you prefer, always saying "Black") whenever equal numbers of cards of both colors remain in the deck and otherwise predicting the color which is currently in the majority. Levasseur analyzes the game and shows that on the average you will have a score of $n + (\sqrt{\pi n} - 1)/2 + O(n^{-1/2})$, while the kid, of course, will have an average score of exactly n.

We, however, maintain that only the most degenerate parent would play against a two-year-old for money, and that our concern must therefore be, not *by how much* you can expect to win, but with what probability you will win *at all*. Our principal

Back to the title of my talk.

How often should Louis and Peter be **simultaneously happy?**

We already know: they should be happy **all the time**.

However my talk deals with a probability question and many of you may consider this out of place in a **PDE** meeting.

I hope to convince you that historically as well as in this specific case, many problems dealing with **stochastic processes** are intimately related to problems in **PDE**.

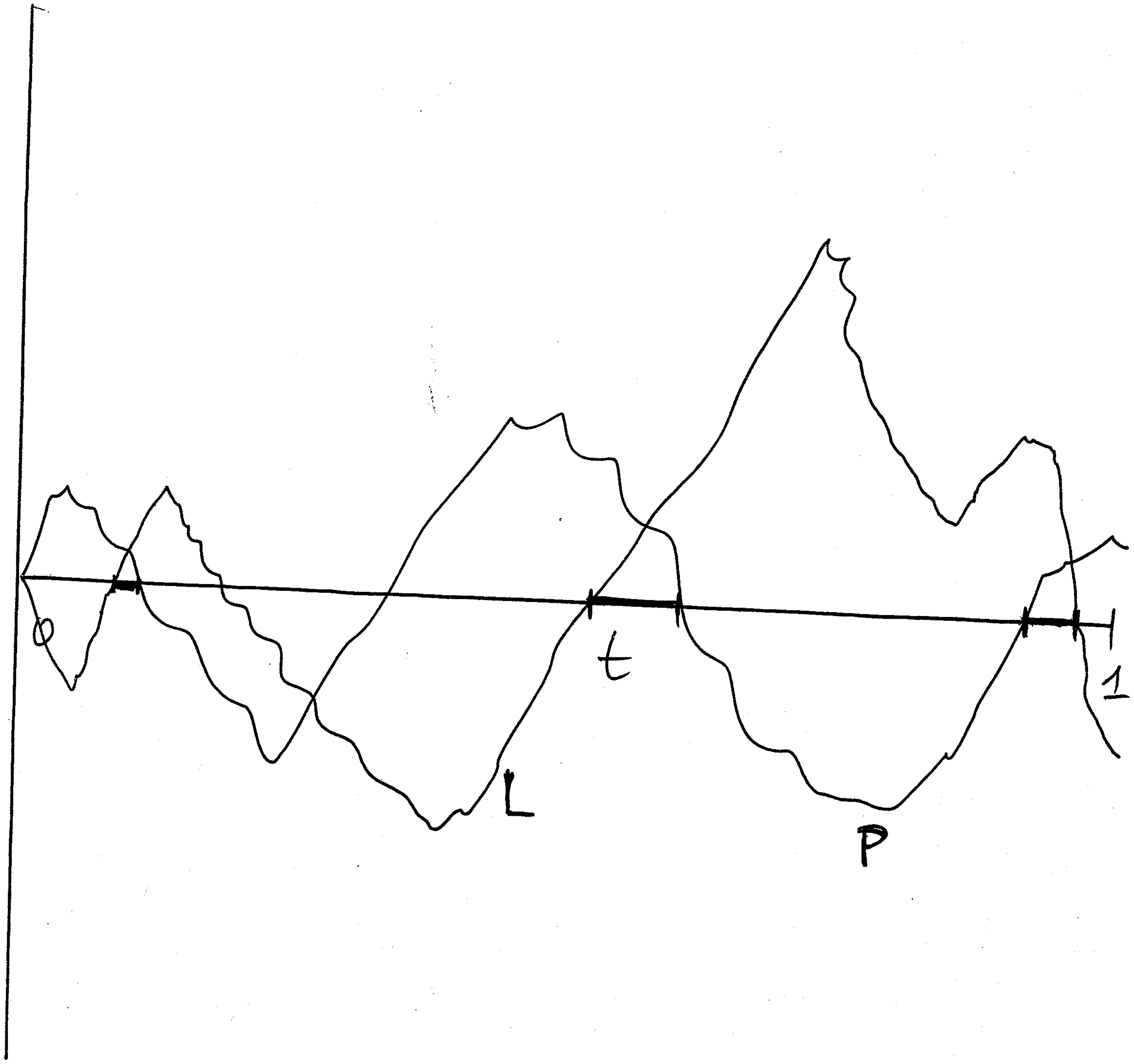
This observation is certainly now new....

But....you have to wait, or wake up at the right momemt.

Louis and Peter go to the casino and spend a long night playing repeteadly a coin game not against each other but rather they play **independent games of HEADS or TAILS** against the house.

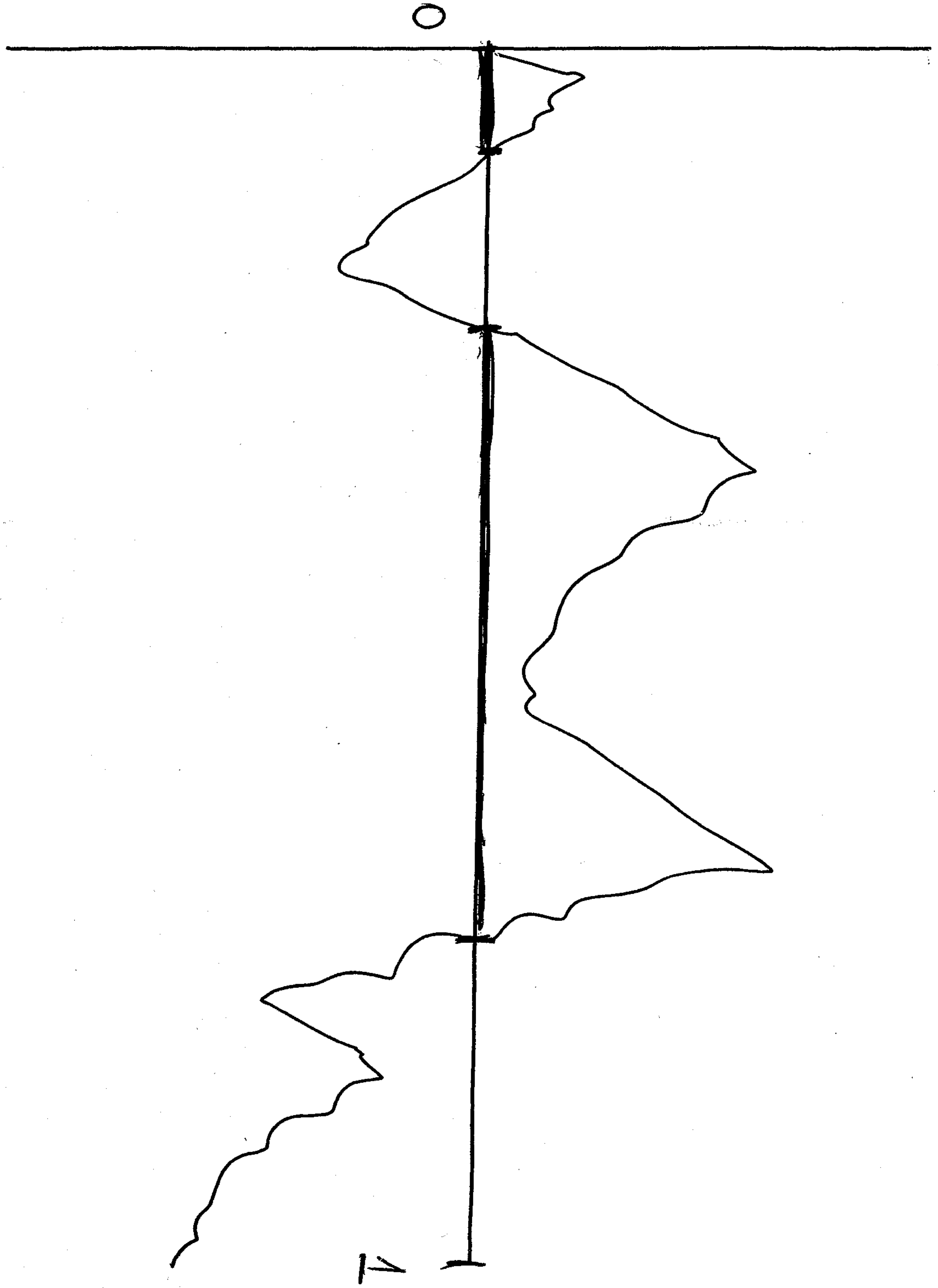
How often should they be **simultaneoulsy ahead of the casino?**

If you plot their net fortunes as functions of time you will get something like this


0
t
1
L
P

If we were asking the question for just **one player**, let us say Peter, it turns out that the answer to the question is both known and interesting.

In this case Peter's fortune as a function of time may look like


0
17

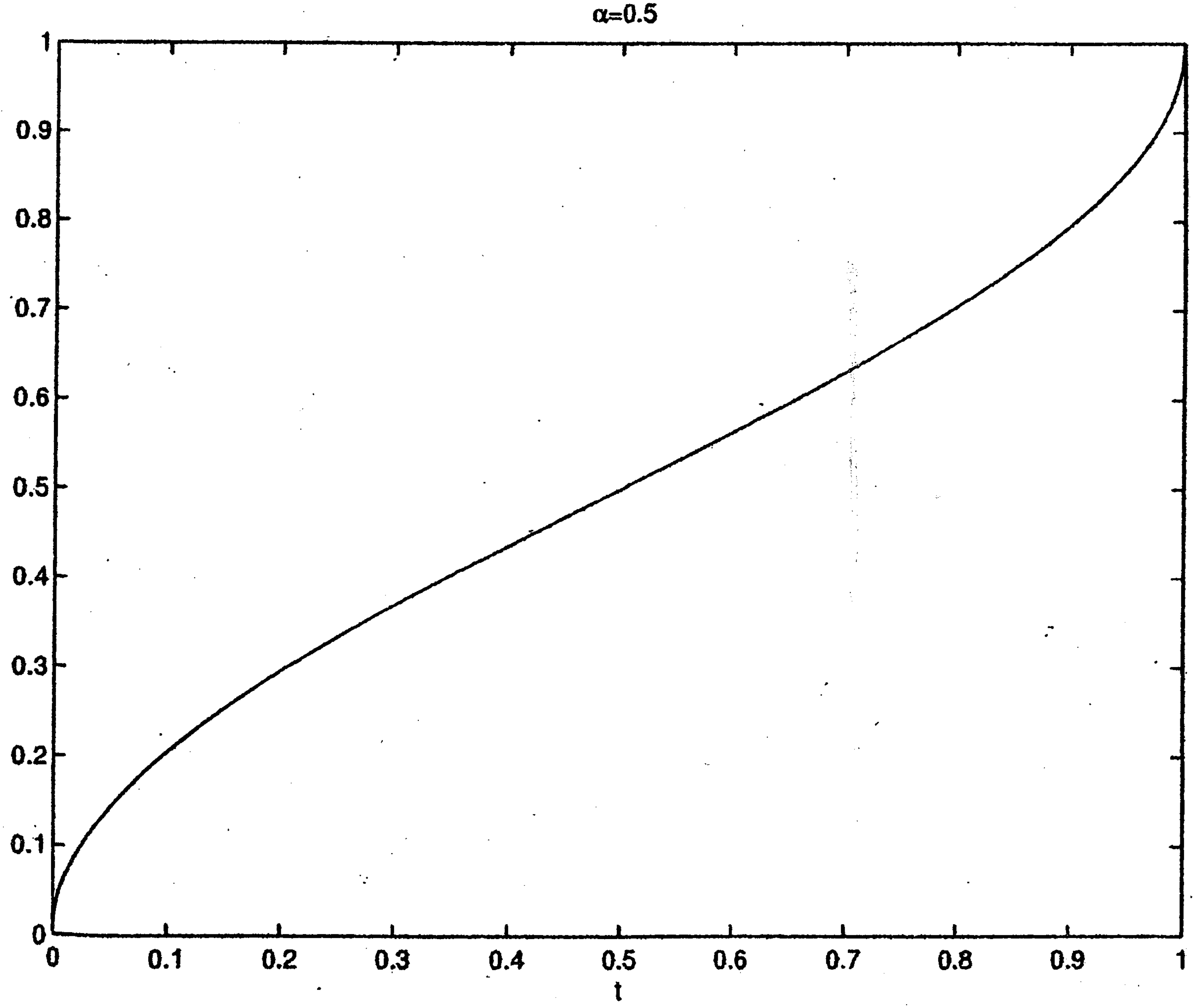
The case of one player.

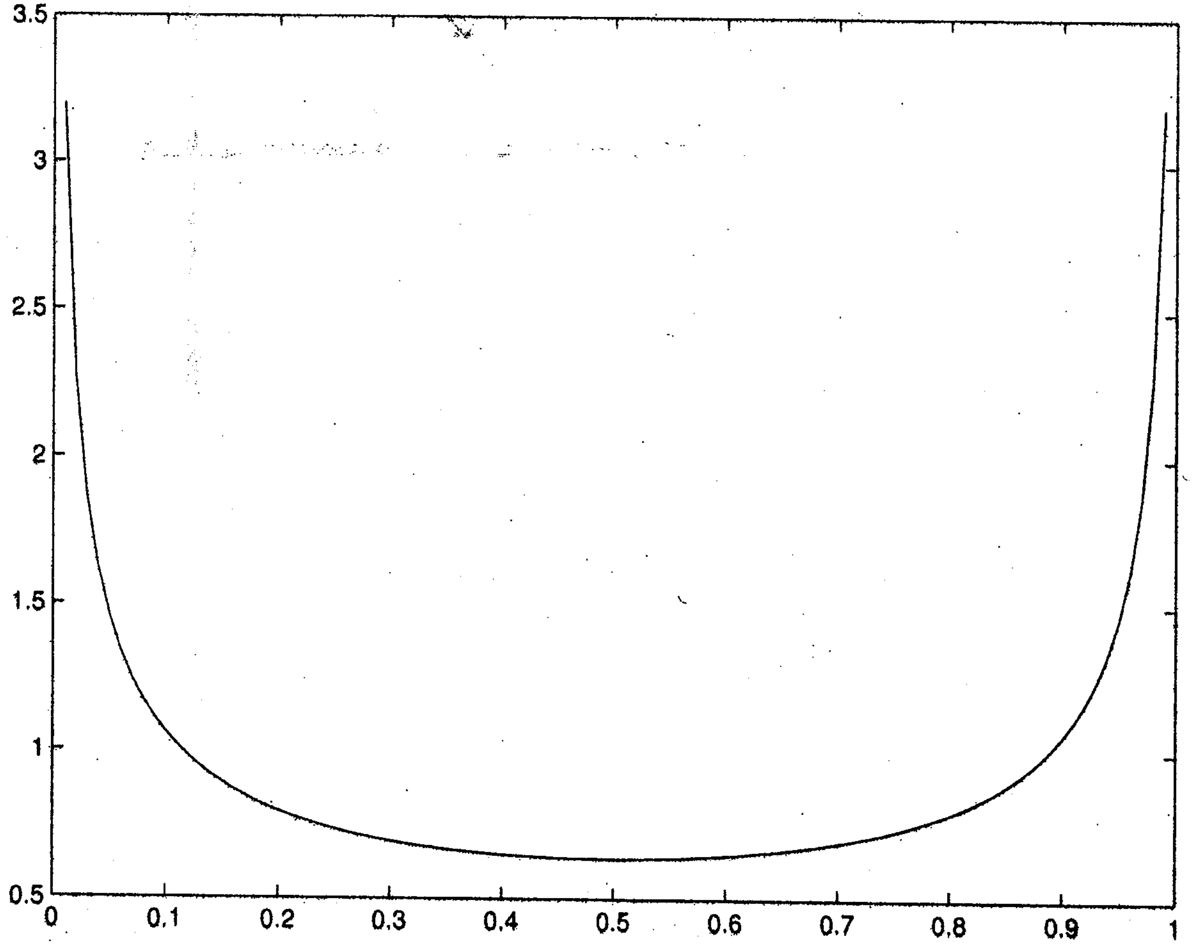
To deal with the case of a coin game is not so simple, and it is easier to deal with the limiting case when you replace the corresponding one dimensional **random walk** with one dimensional **Brownian motion**.

Peter (or Louis) plays for a finite night but he plays so fast that we can approximate random walk by Brownian motion. We normalize the length of the night to be 1.

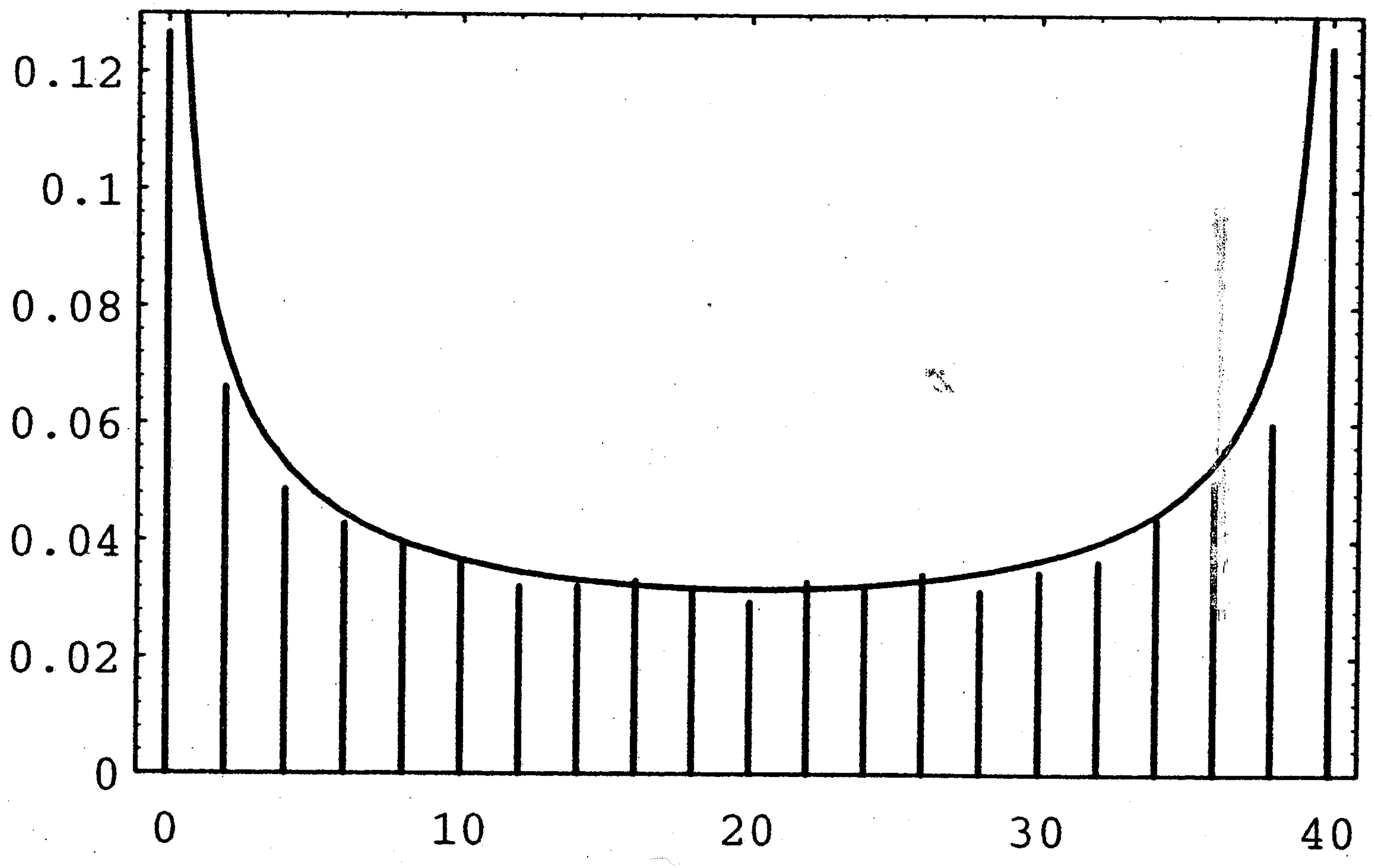
If we denote by $\tau(w)$ the length of time, in the interval $(0, 1)$, that a Brownian particle spends on the positive side, then Paul Levy proved a long time ago that

$$
\begin{aligned}
Pr(\omega, \tau(\omega) \leq x) &= \frac{1}{\pi} \int_0^x \frac{dt}{\sqrt{t(1-t)}} \\
&= \frac{2}{\pi} \arcsin \sqrt{x}
\end{aligned}
$$


α=0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t


3.5
3
2.5
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

In the case of an actual coin here is a histogram illustrating 10,000 repetitions of a game that consists of tossing a coin forty time.



10

This result shocks most people, and it should. It proves, for instance, that the probability of having a **well balanced night** is much smaller that that of having a **wild night**. A well balanced night is one when Peter is ahead more or less half of the time and a wild night is one when he is either ahead or behind almost all the time.

Notice that I have not defined happiness very carefully, so here it is: Peter is happy when he is **ahead**, and when Louis and Peter are both playing they are **simultaneously happy** when they are **BOTH AHEAD.**

Being a bit more quantitative in the case of **one player**:

it can be seen from the expression above that it is about **FOUR TIMES** more likely that Peter will be happy either less that 5 percent of the time or more than 95 percent of the time than he will be happy somewhere between 45 and 55 percent of the time.

Explicitly:

$$\begin{aligned} Pr(\omega, .45 \leq \tau(\omega) \leq .55) &\cong .06377 \\ Pr(\omega, \tau(\omega) \leq .5 \text{ or } \tau(\omega) \geq .95) &\cong .2871 \\ \text{ratio} &\cong 4.5 \end{aligned}$$

In each case we have a 10 percent window, but extreme behaviour is about four times **more likely** than very balanced behaviour on the part of the coin.

Needless to say, all the above holds if Louis is the lonely player.

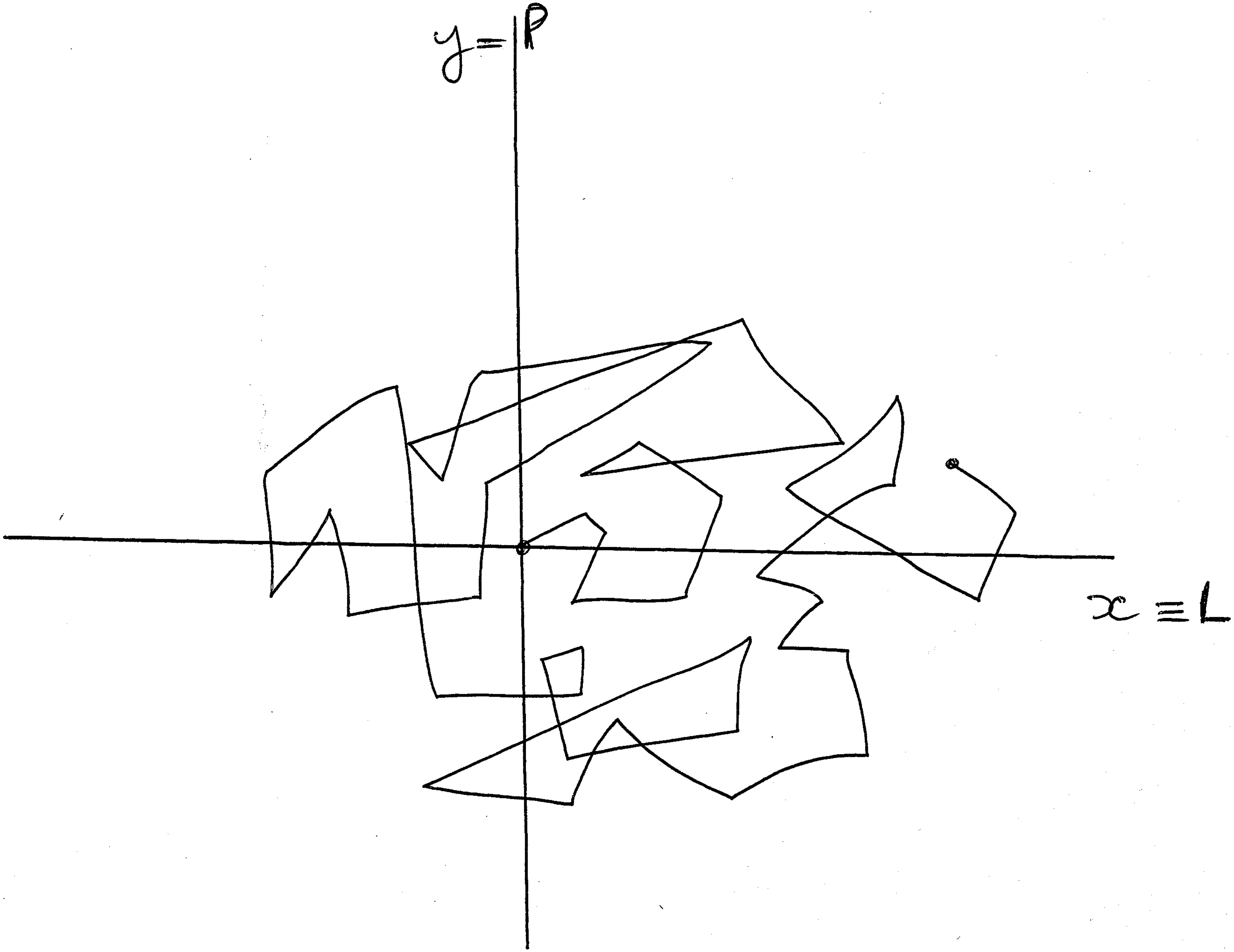
MORAL: either Peter or Louis have a larger chance of having a **wild night** than a **well balanced night.**

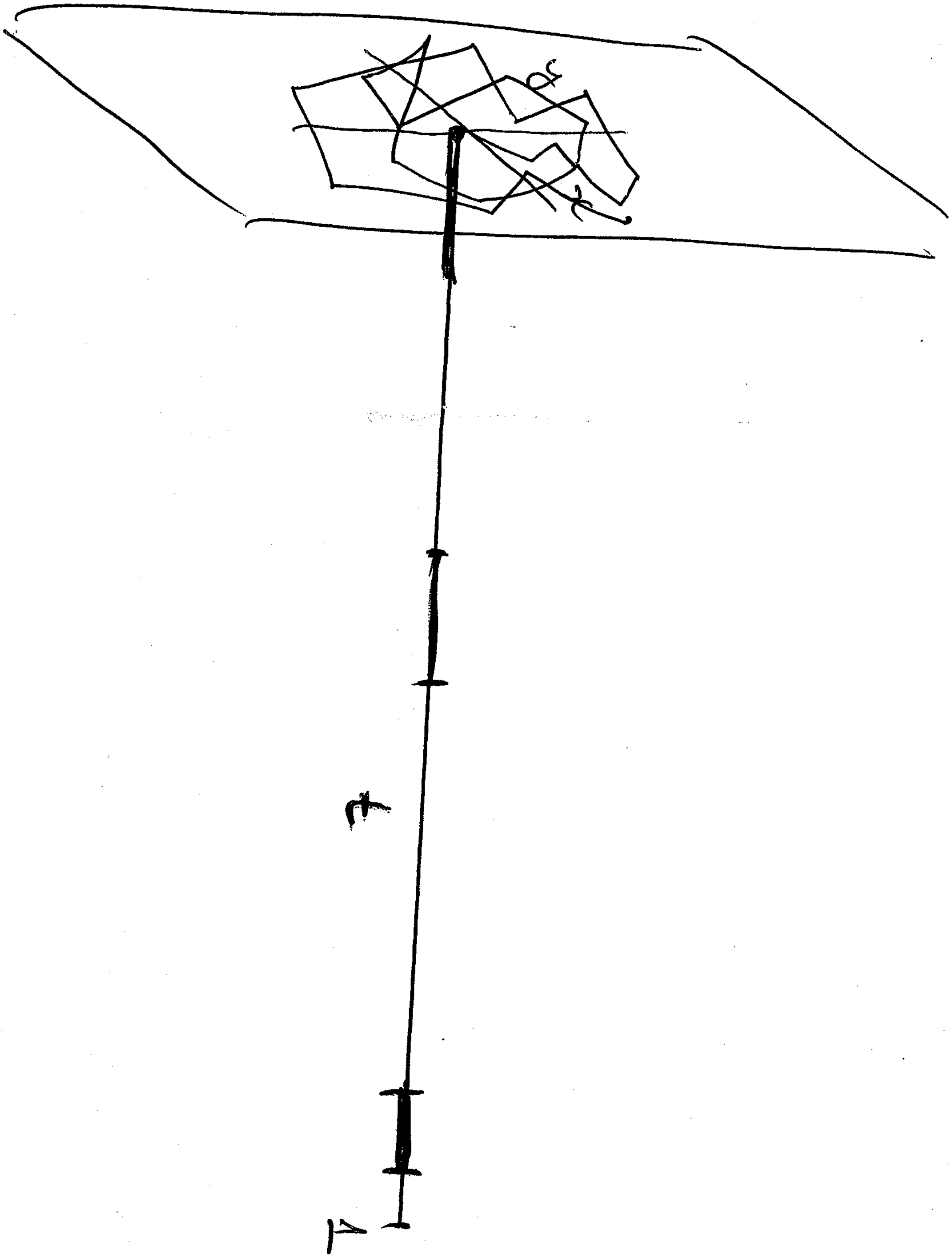
Most people find this very surprising and a contradiction to common sense.

Returning to the case of **two players**:

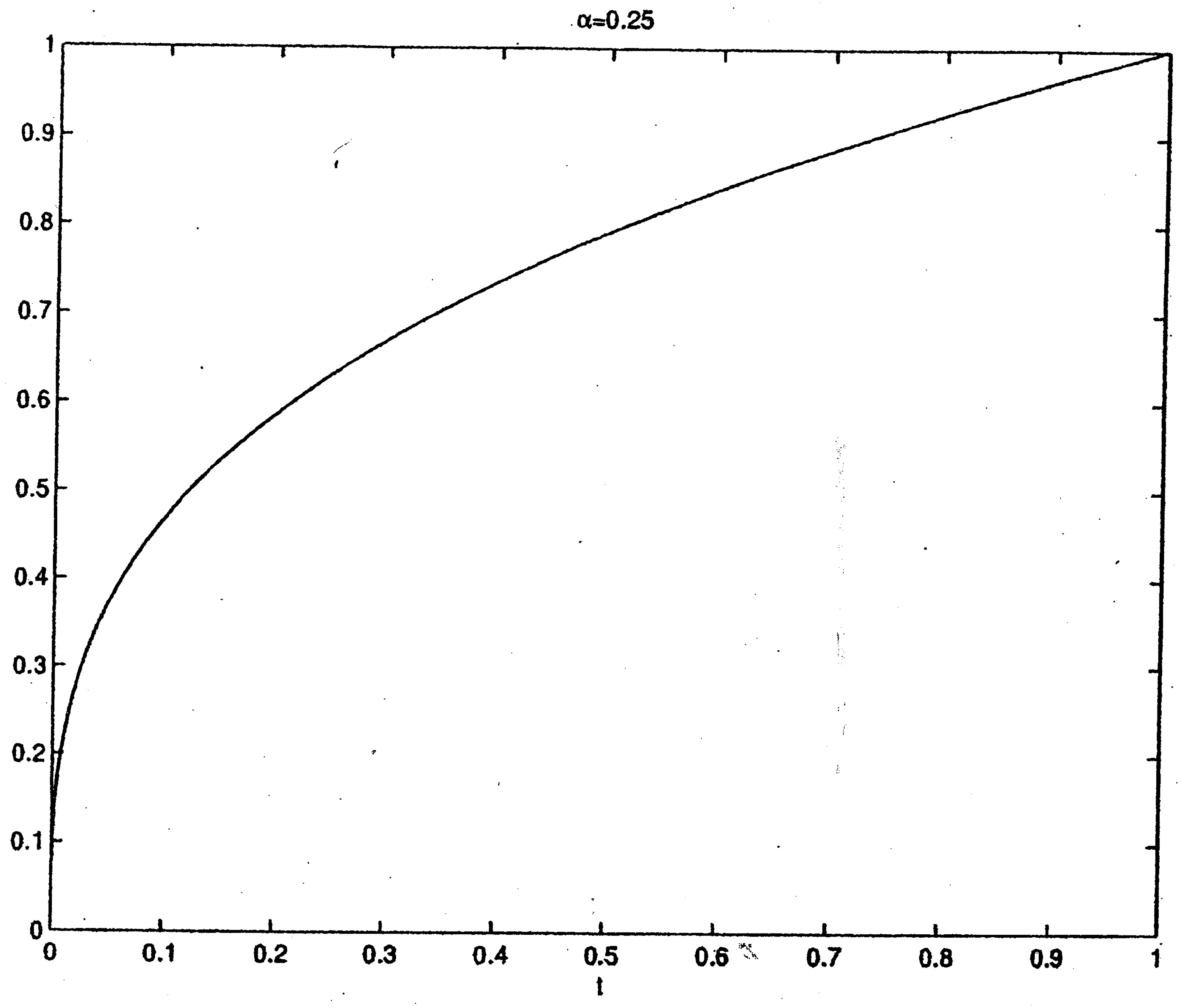
We will be interested only in the total time , in the interval $(0, 1)$, when **BOTH** Louis' and Peter' fortunes, which are two independent one dimensional Brownian motions, are **positive**. It is rather natural to describe the **joint state** of their fortunes at time t, by a **TWO DIMENSIONAL BROWNIAN MOTION**.

We are clearly interested in the length of time , in $(0, 1)$, that this Brownian particle spends on the **positive quadrant**.


y = P
x ≡ L

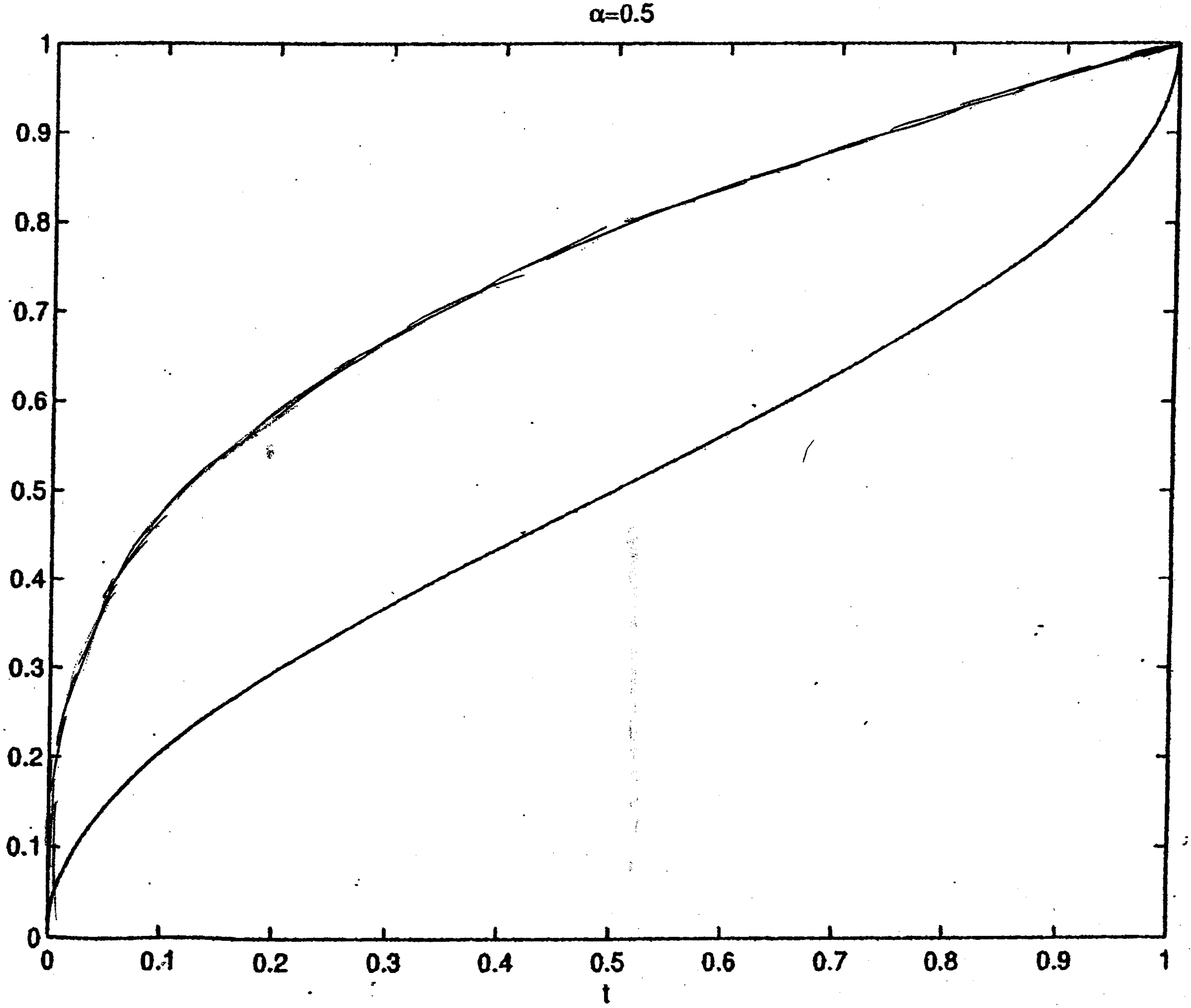

α
x
t
T

A detailed, but **not explicit**, solution to the question at hand is contained in the following picture that plots the (cumulative) distribution function for the length of time, in (0,1), that a two dimensional Brownian particle spends on the positive quadrant.


α=0.25
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

This already shows that as long as we are interested in the time that Louis and Peter are **simultaneously happy** we can trust our common sense. By looking at both graphs together we see that now nothing dramatic happens.

If Louis and Peter spend a very long night at the casino and they keep track of the time when they are **both ahead in their independent games** they are not going to be too pleased. Unfortunately common sense prevails.


α=0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t

The figure shown earlier is the result of rather careful numerical simulation of Brownian motion, based on random Fourier series in the fashion of N. Wiener and random wavelet expansions in the fashion of P. Levy.

Wiener's construction of Brownian motion.

$$X(t,\omega) \quad = \quad \sum_{i=1}^{\infty} \sin\left(i+\frac{1}{2}\right)\pi t\; X_i(\omega)$$

$X_i(\omega)$ independent $(0,1)$ Gaussians

The quality of this simulation has been validated by checking a few of the lower order moments of the **unknown** distribution. The results agree typically to order between $10^{-}4$ and $10^{-}6$.

The explict expression for this distribution is not known but it is in principle possible to compute the moments of its density, i.e. the **expected valued** of powers of the random variable

$\tau(w)$ = **Proportion of time that Louis and Peter are simultaneously happy.**

We have, for instance,

$$\begin{aligned} E(\tau(\omega)) &= \frac{1}{4} \\ E(\tau^2(\omega)) &= \frac{5}{32} - \frac{1}{8\pi^2} \cong 0.14358 \\ E(\tau^3(\omega)) &= \frac{53}{512} + \frac{7}{18\pi} - \frac{347}{288\pi^2} \cong .10522 \end{aligned}$$

The first one above is very natural, whereas the computation of each one of the other two is extremely laborious. I have never seen the values for higher order moments.

Hopefully all of these computations could be obviated if one had an explicit form of the distribution or its density.

These results are due to N. Bingham and R. Doney (1988), while the numerical simulations shown earlier were carried out by Caroline McGrouther as part of a Master's Thesis with me at Berkeley (2002).

The conjectures alluded to later are given in work of T. Meyre and W. Werner (1995).

This careful numerical study allows us to refine and correct some conjectures that have been made in the literature as to the asymptotic behaviour of the distribution close to the end points of the interval.

For example while in the case of **one player** one has

$$Pr(\omega, \tau(\omega) < x) \sim \frac{2}{\pi} x^{1/2} \qquad x \sim 0$$

in the case of **two players** we surmise

$$Pr(\omega, \tau(\omega) < x) \sim k x^{1/3} \qquad x \sim 0$$

Some historical comments

Polya

Courant, Friedrichs, Lewy.

Kakutani

von Neumann

All previous attemps to determine the explict expression for the distribution in the case of two players have failed. As far as I know these attempts have not tried to exploit any connection to **PDE's**.

I want to describe some work in progress that could lead to an explicit expression.

The problem consists of looking at a nice Poisson type equation in the plane and finding its value at the origin explicitly in terms of two parameters.

How can we connect the problem of determining the distribution for the random variable $\tau(w)$ introduced above, to a **PDE's** problem?

For this is I need to remind you of what is called the **FEYNMAN-KAC formula**, i.e. a way of relating the solution of certain PDEs to the evaluation of certain integrals in function space.

The historical motivation of this piece of mathematics done by Mark Kac is the attempt by Richard Feynman (in his thesis, 1948) to produce a **Lagrangian** formulation of quantum mechanics. You can consider the Schrödinger equation as the **Hamiltonian** version of the same story.

I will make no attempt to prove this formula here but I will give you an "after the fact" way of making it believable.

How do you solve the initial value problem

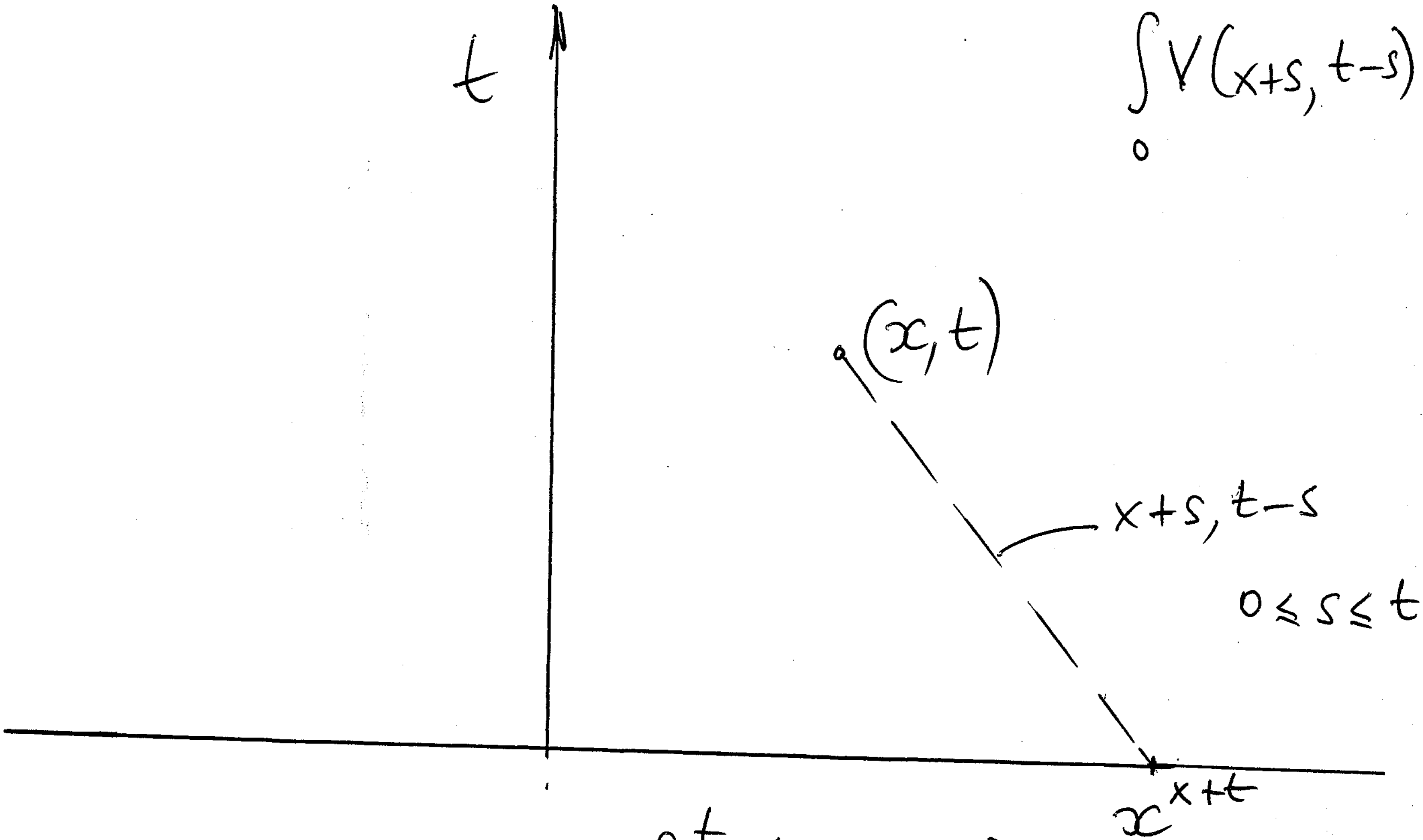
$$\begin{aligned} u_t &= u_x + V(x,t)u \qquad x \in R,\ t \geq 0 \\ u(x,0) &= f(x) \end{aligned}$$

The explicit formula is (of course)

$$u(x,t) = e^{\int_0^t V(x+s,t-s)ds} f(x+t)$$

I want to interpret this formula by means of the picture

$$\int_0^t V(x+s, t-s)\, ds$$



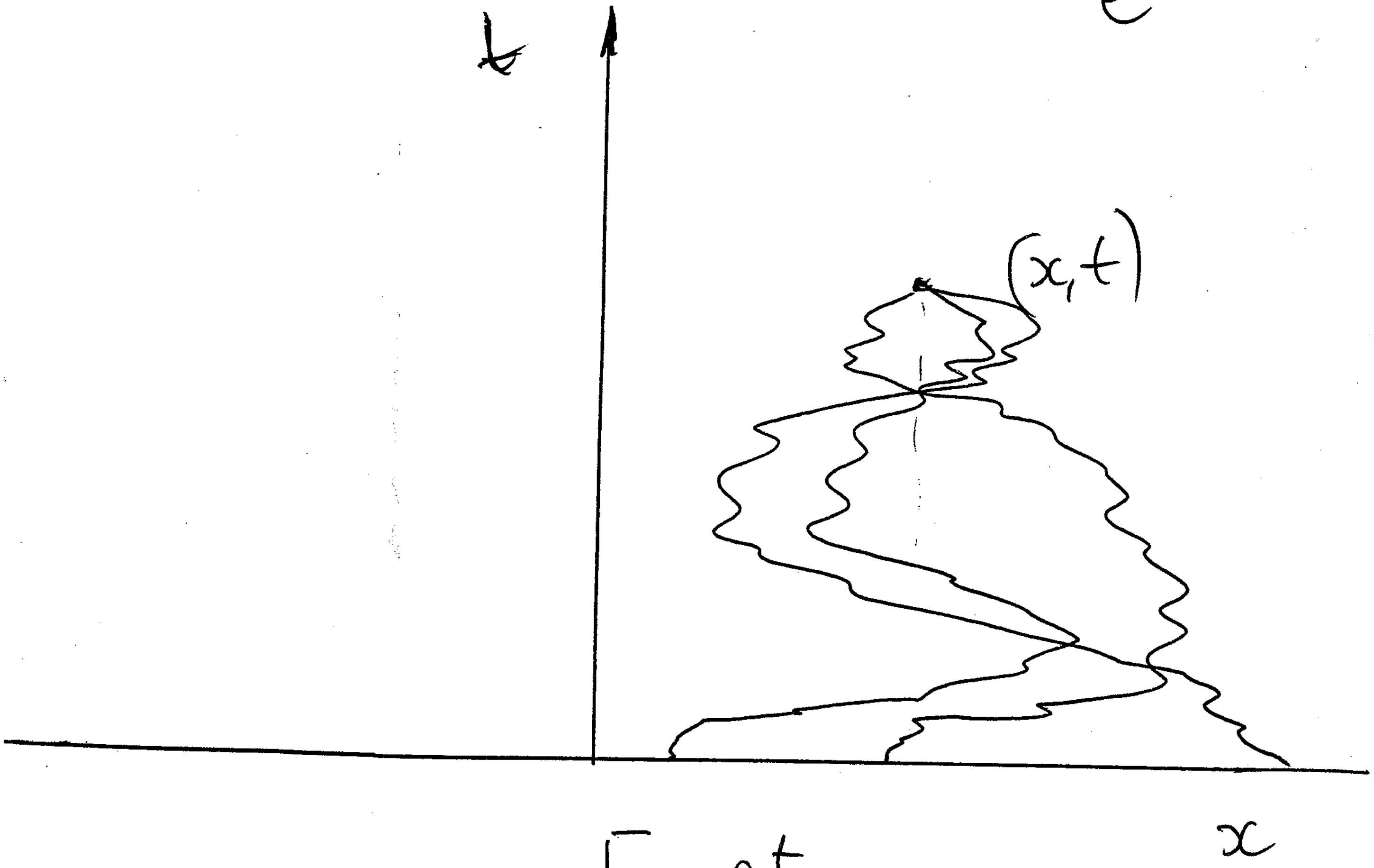
$$U(x,t) = e^{\int_0^t V(x+s, t-s)\, ds} f(x+t)$$

So, what do you do if you want to solve

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + V(x,t)u \qquad x \in R,\ t \geq 0 \\ u(x,0) &= f(x) \end{aligned}$$

The answer is contained in the following picture:

$$e^{\int_0^t V(x+w(s),\, t-s)\, d}$$



$$u(x,t) = E_w\left[e^{\int_0^t V(x+w(s),\, t-s)\, ds} f(x+w(t)) \right]$$

Stochastic methods have can also be used to solve problems that can be solved by others means. This is not the case of the Feynman-Kac formula where there is no alternative way to solve the problem, but here is a different example.

$$
\begin{aligned}
X(t,y,\omega) &= \sum_{i=1}^{n} e^{(4\xi_i^2 t - \xi_i y)}\sqrt{m_i}X_i(\omega) \\
&\; X_i(\omega),\ 1 \le i \le n,\ \text{independent } (0,1) \text{ Gaussians} \\
&\; \xi_i, m_i \text{ positive} \\
\tau(x,t) &\equiv E_w\left(e^{-\frac{1}{2}\int_x^\infty X^2(t,y,\omega)dy}\right) \\
u(x,t) &= 2\frac{\partial^2}{\partial x^2}\log\tau(x,t)
\end{aligned}
$$

solves KdV.

Back to the Feynman-Kac formula.

Using it M. Kac showed that a way of reproving the result of Paul Levy is given by finding the bounded solution of the **ODE** (both A and B below are positive and $B > A$.)

$$\frac{1}{2}u_{xx} - V(x)u = -1 \qquad -\infty < x < \infty$$

$$V(x) = \begin{cases} B & \text{if } x \geq 0 \\ A & \text{if } x < 0 \end{cases}$$

evaluating it at $x = 0$, finding its value to be

$$1/\sqrt{AB}$$

and then noticing that this is a **DOUBLE Laplace transform**

$$\frac{1}{\sqrt{AB}} = \int_0^\infty e^{-At} dt \int_0^t \left(\frac{1}{\pi} \frac{e^{-(B-A)s}}{\sqrt{(t-s)s}} \right) ds$$

Here is the **PDE** that one needs to solve in the case of two players

$$\frac{1}{2}\nabla^2 u - V(x,y)u = -1 \qquad (x,y) \in R^2$$

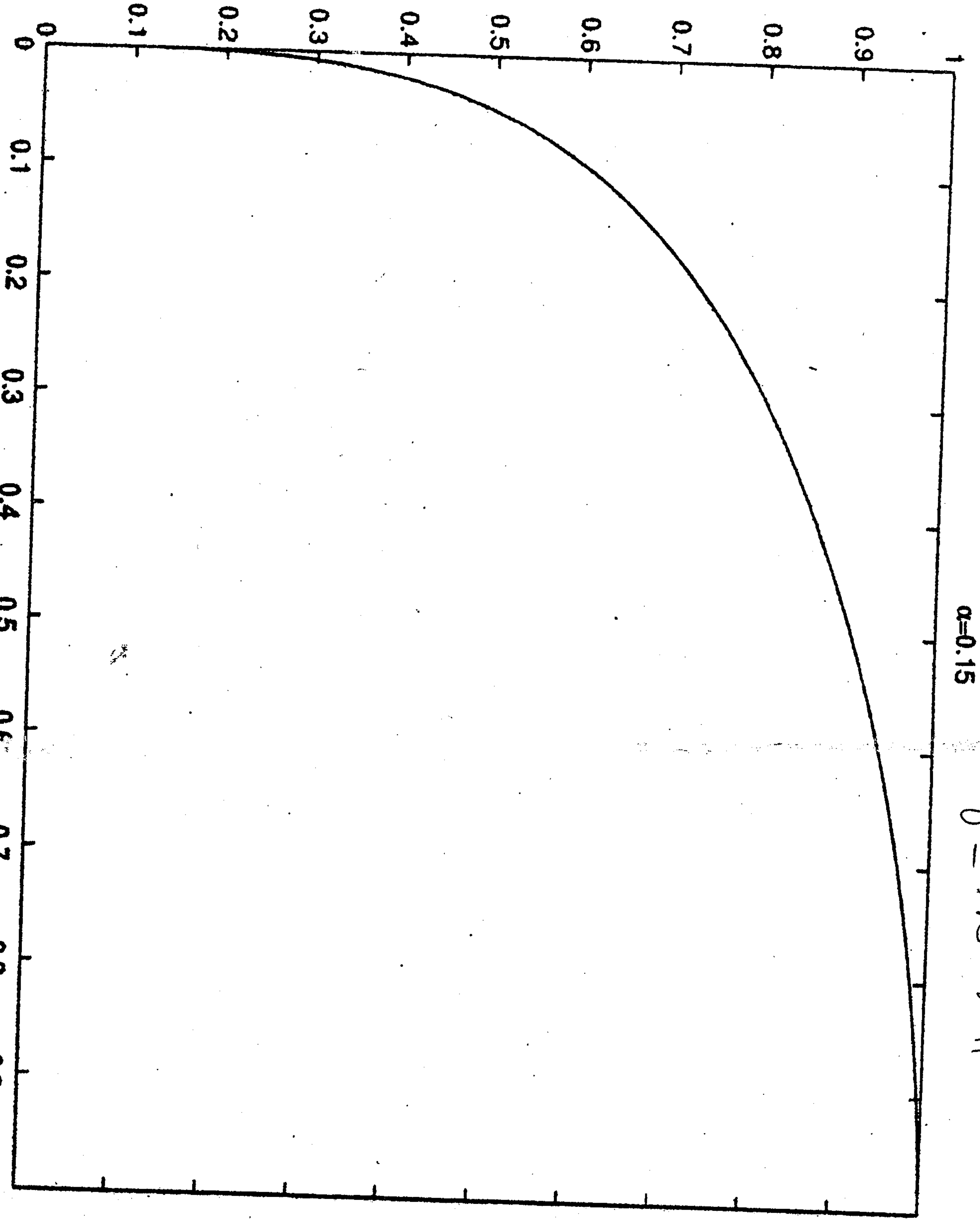
$$V(x,y) = \begin{cases} B & \text{if } x,y \geq 0 \\ A & \text{otherwise} \end{cases}$$

One only needs to evaluate the (bounded) solution at the origin, and then pray for some Laplace transform miracles.

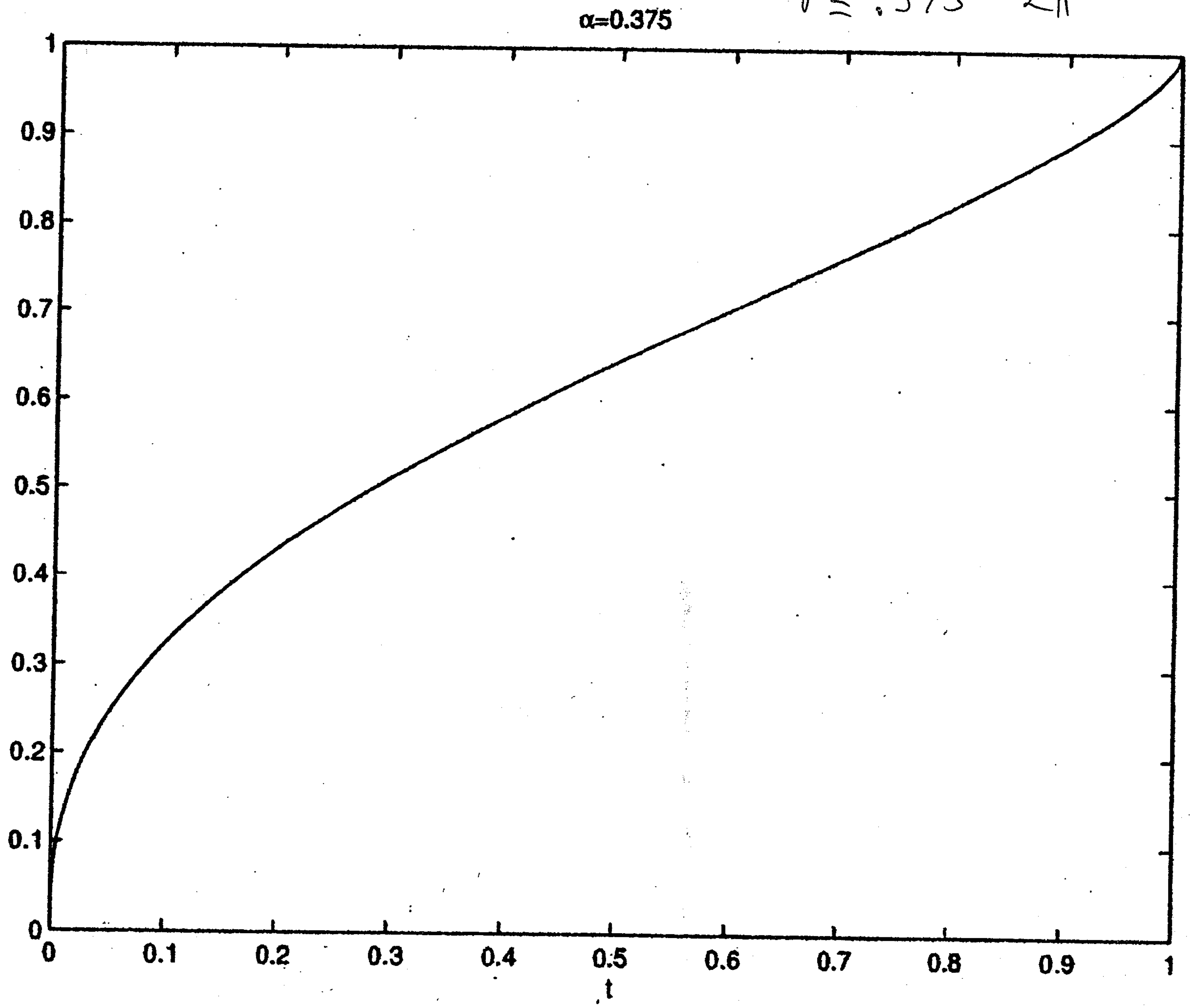
In the case of two players, Louis and Peter, we are dealing with a wedge in the plane of angle $\pi/2$.

It is very natural to embed this problem in the family of problems obtained when the opening of the wedge has magnitude γ.

One could even dream of obtaining a **compatibility condition** for our function u as a function of all these different parameters, and maybe consider the corresponding **nonlinear equation** .....



γ = .15 2π



$\gamma = .375 \quad 2\pi$

A very natural guess for the value of the solution of our **PDE** at the origin is given by a formula of the form

$$u(0,0) = \frac{1}{A^{\nu_1(\gamma)} B^{\nu_2(\gamma)}}$$

where the exponents are somehow related (maybe proportional to) the two angles in question, γ and its complement.

This is actually the case for $\gamma = \pi$ and the extreme case γ=0 or 2π.

Unfortunately, one can give a **rigorous argument** showing that this cannot be true for the case of interest, namely

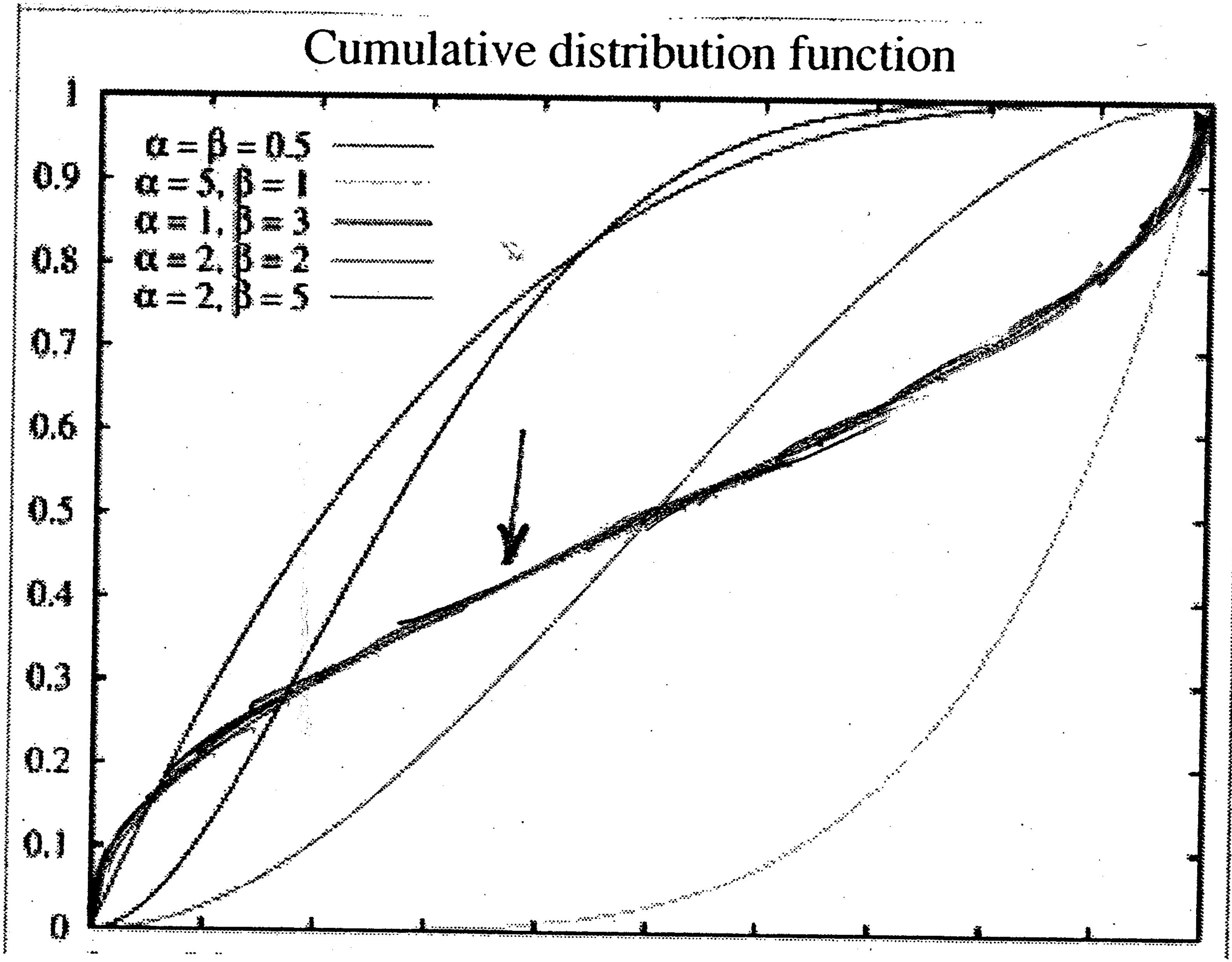
$$\gamma = \pi/2.$$

If the value of the function $u(x,y)$ at $(0,0)$ were as suggested above one can see that by doing a **DOUBLE -inverse- Laplace transform** as before, that the density of our random variable $\tau(w)$ should be given by a **BETA distribution**, i.e. the density would be

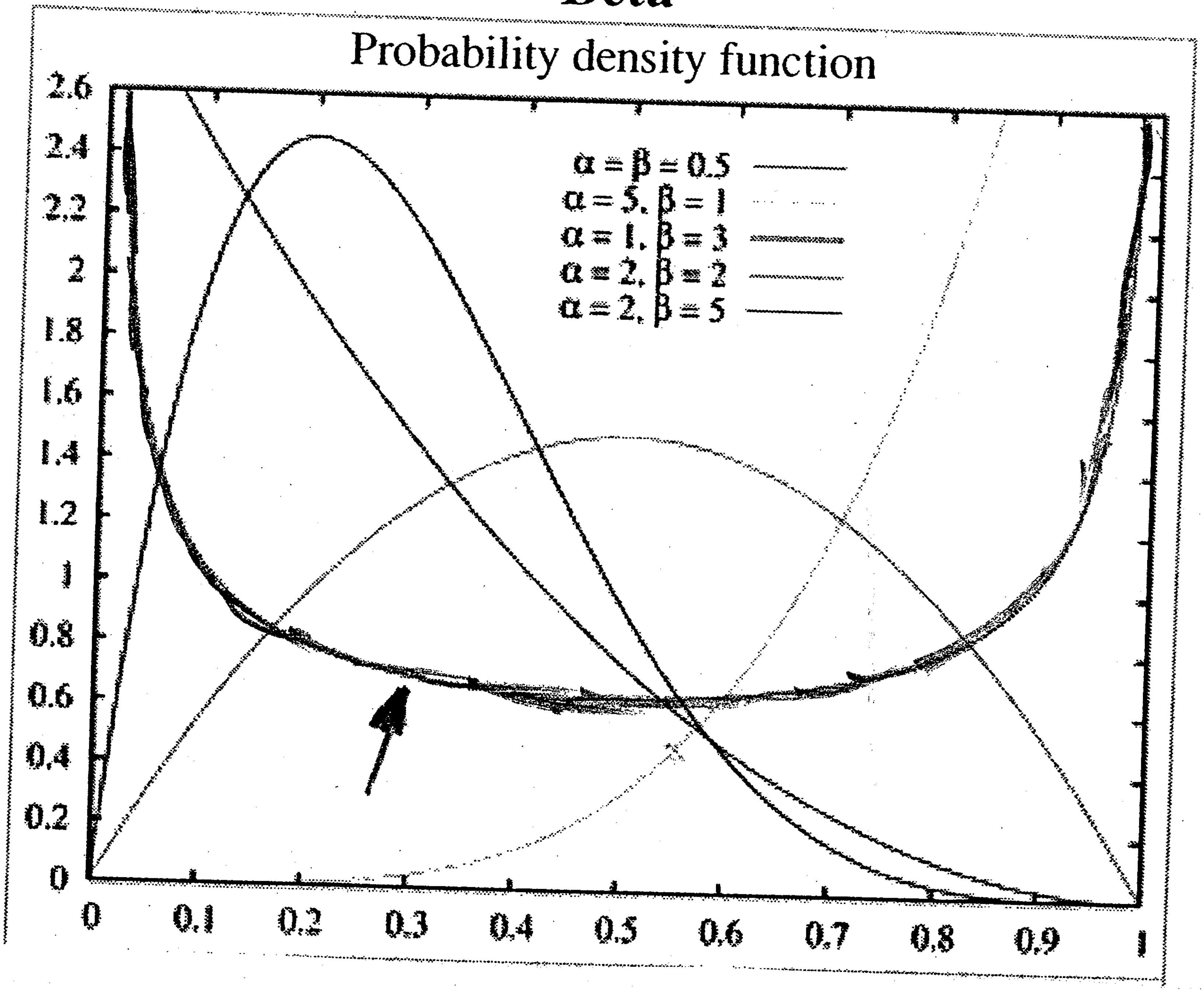
$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1} \qquad 0 < t < 1$$

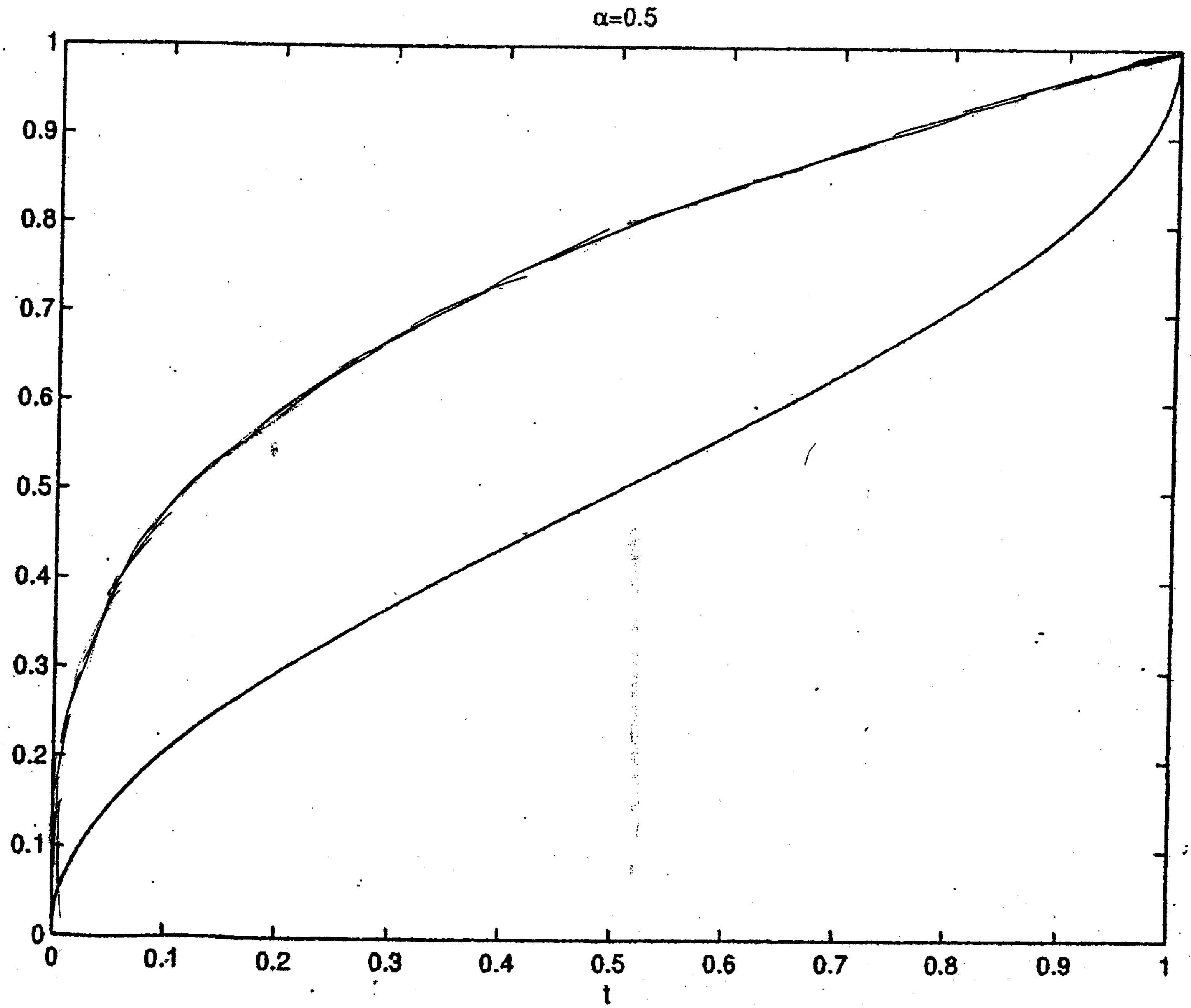
$$\alpha, \beta > 0$$

This can be ruled out from the explicit computation of the first few moments reported earlier.


Cumulative distribution function
α = β = 0.5
α = 5, β = 1
α = 1, β = 3
α = 2, β = 2
α = 2, β = 5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Beta


Probability density function
α = β = 0.5
α = 5, β = 1
α = 1, β = 3
α = 2, β = 2
α = 2, β = 5
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


α=0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t